ARESTOV'S THEOREMS ON BERNSTEIN'S INEQUALITY

Tamás Erdélyi

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Abstract. Following Arestov's Λ-method we give a simple, elementary, and at least partially new proof of Arestov's famous extension of Bernstein's inequality in L_p to all $p \geq 0$. Our crucial observation is that Boyd's approach to prove Mahler's inequality for algebraic polynomials $P_n \in \mathcal{P}_n^c$ can be extended to all trigonometric polynomials $T_n \in \mathcal{F}_n^c$.

1. Introduction and Notation

Let $\mathcal{F}_n^c$ be the collection of all trigonometric polynomials T_n of the form

$$T_n(z) = \sum_{j=-n}^{n} a_j z^j \,, \qquad a_j \in \mathbb{C} \,, \quad z \in \mathbb{C} \setminus \{0\} \,.$$

Let $\mathcal{P}_n^c$ be the collection of all algebraic polynomials P_n of the form

$$P_n(z) = \sum_{j=0}^{n} a_j z^j \,, \qquad a_j \in \mathbb{C} \,, \quad z \in \mathbb{C} \,.$$

Let D denote the open unit disk of the complex plane, and ∂D denote its boundary. We define the Mahler measure (geometric mean of Q on ∂D)

$$\|Q\|_0 = M_0(Q) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|Q(e^{it})|\, dt\right)$$

for bounded measurable functions Q on ∂D. It is well known, see [HL-95], for instance, that

$$\|Q\|_0 = M_0(Q) = \lim_{p \to 0+} M_p(Q) \,,$$

where

$$\|Q\|_p = M_p(Q) := \left(\frac{1}{2\pi}\int_0^{2\pi} \left|Q(e^{it})\right|^p \, dt\right)^{1/p} \,, \qquad p > 0 \,.$$

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It is also well known that for a function Q continuous on ∂D we have

$$\|Q\|_\infty = M_\infty(Q) := \max_{t\in[0,2\pi]} |Q(e^{it})| = \lim_{p\to\infty} M_p(Q)\,.$$

It is a simple consequence of the Jensen formula that

$$M_0(Q) = |c| \prod_{k=1}^{n} \max\{1, |z_k|\}$$

for every polynomial of the form

$$Q(z) = c\prod_{k=1}^{n} (z - z_k)\,, \qquad c, z_k \in \mathbb{C}\,.$$

Bernstein's inequality

$$\|T_n'\|_\infty \le n\|T_n\|_\infty\,, \qquad T_n \in \mathcal{F}_n^c\,,$$

plays a crucial role in proving inverse theorems of approximation as well as many other results in approximation theory. See [BE-95], for instance. As far as the history of Bernstein's inequality is concerned we refer to Nevai's lovely papers [N-14] and [N-19] and the references in them. We do not repeat the full story here. In 1981 Arestov [A-81] proved that

$$\|T_n'\|_p \le n\|T_n\|_p\,, \qquad T_n \in \mathcal{F}_n^c\,, \tag{1.1}$$

for all $p \ge 0$, extending the result known only for $p \ge 1$ for a long time. Simpler proof of Bernstein's inequality in L_p for all $p \ge 0$ have been given by Golitschek and Lorentz in [GL-89] which is presented in the book [DL-93] by DeVore and Lorentz. A very elegant and even more simplified proof was published recently in [QZ-19] by Queffélec and Zarouf. A central part of their proof is to prove (1.1) for $p = 0$ first. Mahler [M-61] showed (1.1) for $p = 0$ but only for polynomials $P_n \in \mathcal{P}_n^c$, and he gave a rather involved proof. Mahler's inequality was also posed as a problem by Vaaler in the Problems section of the American Mathematical Monthly and solved by Boyd [VB-91] using an elementary theorem of Bernstein. We note that Glazyrina [G-05] proved a sharp Markov-type inequality for algebraic polynomials in L_0 on finite subintervals of the real line.

In this note, following Arestov's Λ-method in [A-81], we give a simple, elementary, and at least partially new proof of Arestov's famous extension of Bernstein's inequality in L_p to all $p \ge 0$. Our crucial observation is that Boyd's approach to prove Mahler's inequality for algebraic polynomials $P_n \in \mathcal{P}_n^c$ can be extended to all trigonometric polynomials $T_n \in \mathcal{F}_n^c$.

Theorem 1.1. *We have*

$$\|T_n'\|_0 \le n\|T_n\|_0\,, \qquad T_n \in \mathcal{F}_n^c\,.$$

Equivalently

$$\int_{\partial D} \log|T_n'(z)/n|\,|dz| \le \int_{\partial D} \log|T_n(z)|\,|dz|\,, \qquad T_n \in \mathcal{F}_n^c\,.$$

Theorem 1.2. *With the notation* $\log^+ |a| := \max\{\log |a|, 0\}$ *we have*

$$\int_{\partial D} \log^+ |T_n'(z)/n| \, |dz| \leq \int_{\partial D} \log^+ |T_n(z)| \, |dz| \,, \qquad T_n \in \mathcal{F}_n^c \,.$$

Theorem 1.3. *We have*

$$\|T_n'\|_p \leq n \|T_n\|_p \,, \qquad T_n \in \mathcal{F}_n^c \,,$$

for every $p > 0$.

2. Lemmas

To prove Theorem 1.1 we need two lemmas.

Lemma 2.1. *Associated with* $S_n \in \mathcal{F}_n^c$, *let* $P_{2n} \in \mathcal{P}_{2n}^c$ *be defined by* $P_{2n}(z) = z^n S_n(z)$. *If* P_{2n} *has each of its* $2n$ *zeros in* D, *then* S_n' *has each of its zeros in* D *as well. The same is true if* D *is replaced by the closed unit disk* $\overline{D}$.

Proof. We prove the lemma for D, the case of the closed unit disk $\overline{D}$ follows from this by a straightforward limiting argument. Suppose $a \notin D$, that is, $a \in \mathbb{C}$ and $|a| \geq 1$. Suppose also that

$$P_{2n}(z) = c \prod_{j=1}^{2n} (z - z_j) \,, \qquad 0 \neq c \in \mathbb{C} \,, \quad z_j \in D \,.$$

We have

$$\frac{aS_n'(a)}{S_n(a)} = \sum_{j=1}^{2n} \frac{a}{a - z_j} - \frac{an}{a} = \sum_{j=1}^{2n} \frac{1}{1 - z_j/a} - n \,. \tag{2.1}$$

Observe that $|z_j/a| < 1$ for each $j = 1, 2, \ldots, 2n$, and hence

$$\mathrm{Re}\left(\frac{1}{1 - z_j/a}\right) > \frac{1}{2} \,, \qquad j = 1, 2, \ldots, 2n \,. \tag{2.2}$$

Combining (2.1) and (2.2) we obtain

$$\mathrm{Re}\left(\frac{aS_n'(a)}{S_n(a)}\right) > \frac{2n}{2} - n = 0 \,.$$

We conclude that $S_n'(a) \neq 0$. □

We remark that, using the notation in Lemma 2.1, we have $S_n(z) = P_{2n}(z) z^{-n}$, hence

$$S_n'(z) = P_{2n}'(z) z^{-n} - n P_{2n}(z) z^{-n-1} = z^{-n-1}(z P_{2n}'(z) - n P_{2n}(z)) \,.$$

Thus the statement of the lemma is equivalent to the fact that the polynomial $zP_{2n}'(z) - nP_{2n}(z)$ has all its zeros in D ($\overline{D}$). However, this statement is a special case of a problem going back to Laguerre [PSz-98, Part Five, Chapter 2, Section 2, Problem 117.]

Lemma 2.2. *Associated with $V_n \in \mathcal{F}_n^c$, let $R_{2n} \in \mathcal{P}_{2n}^c$ be defined by $R_{2n}(z) = z^n V_n(z)$ and suppose that R_{2n} has each of its $2n$ zeros in the closed unit disk $\overline{D}$. If $T_n \in \mathcal{F}_n^c$ and*

$$|T_n(z)| \le |V_n(z)|\,, \qquad z \in \partial D\,, \tag{2.3}$$

then

$$|T_n'(z)| \le |V_n'(z)|, \qquad z \in \partial D\,.$$

Proof. Without loss of generality we may assume that R_{2n} has each of its $2n$ zeros in D, the case when R_{2n} has each of its $2n$ zeros in the closed unit disk $\overline{D}$ follows from this by a straightforward limiting argument. Let $Q_{2n} \in \mathcal{P}_{2n}^c$ be defined by $Q_{2n}(z) := z^n T_n(z)$. Let $\alpha \in \mathbb{C}$, $|\alpha| < 1$, and

$$S_n(z) := (V_n - \alpha T_n)(z) = z^{-n}(R_{2n} - \alpha Q_{2n})(z)\,. \tag{2.4}$$

It follows from (2.3) that

$$|Q_{2n}(z)| \le |R_{2n}(z)|\,, \qquad z \in \partial D\,,$$

and hence $|\alpha| < 1$ and the fact that $R_{2n} \in \mathcal{P}_{2n}^c$ does not vanish on ∂D imply that

$$|\alpha Q_{2n}(z)| < |R_{2n}(z)|\,, \qquad z \in \partial D\,.$$

Therefore Rouché's Theorem implies that the polynomial $P_{2n} := R_{2n} - \alpha Q_{2n} \in \mathcal{P}_{2n}^c$ and R_{2n} has the same number of zeros in D, that is, $R_{2n} - \alpha Q_{2n}$ has each of its $2n$ zeros in D. By Lemma 2.1 and (2.4) we can deduce that $S_n' = V_n' - \alpha T_n'$ has each of its zeros in D. In particular,

$$S_n'(z) = V_n'(z) - \alpha T_n'(z) \ne 0, \qquad z \in \partial D\,,$$

for all $\alpha \in \mathbb{C}$, $|\alpha| < 1$. We conclude that

$$|T_n'(z)| \le |V_n'(z)|, \qquad z \in \partial D\,.$$

□

3. Proof of Theorems 1.1, 1.2, and 1.3

Proof of Theorem 1.1. Associated with $T_n \in \mathcal{F}_n^c$, let $P_{2n} \in \mathcal{P}_{2n}^c$ be defined by $P_{2n}(z) = z^n T_n(z)$. Without loss of generality we may assume that P_{2n} has exactly $2n$ complex zeros, the case when P_{2n} has less than $2n$ complex zeros follows from this by a straightforward limiting argument.

Case 1. Suppose P_{2n} has all its $2n$ zeros in the closed unit disk $\overline{D}$. It follows from Lemma 2.1 that T_n' has all its zeros in the closed unit disk $\overline{D}$. Let

$$T_n(z) = \sum_{j=-n}^{n} a_j z^j\,, \qquad a_j \in \mathbb{C}\,, \quad z \in \mathbb{C} \setminus \{0\}\,.$$

Using Jensen's formula and the multiplicative property of the Mahler measure we can easily deduce that

$$\|T_n'\|_0 = n\|T_n\|_0 = n|a_n|\,.$$

Case 2. Suppose that some of the zeros of P_{2n} are outside the closed unit disk $\overline{D}$. Let $z_1, z_2, \dots, z_m$ be the zeros of T_n outside the closed unit disk $\overline{D}$ and let $z_{m+1}, z_{m+2}, \dots, z_{2n}$ be the zeros of T_n in the closed unit disk $\overline{D}$. We have

$$T_n(z) = a_n z^{-n} \prod_{j=1}^{m} (z - z_j) \prod_{j=m+1}^{2n} (z - z_j)\,.$$

We define

$$V_n(z) := a_n z^{-n} \prod_{j=1}^{m} (1 - \overline{z_j} z) \prod_{j=m+1}^{2n} (z - z_j)\,.$$

Observe that (2.3) holds, $V_n \in \mathcal{F}_n^c$, and $R_{2n} \in \mathcal{P}_{2n}^c$ defined by $R_{2n}(z) := z^n V_n(z)$ has each of its $2n$ zeros in the closed unit disk $\overline{D}$. Using Lemma 2.2, the (in)equality of the theorem in Case 1, and Jensen's formula, we obtain

$$\|T_n'\|_0 \le \|V_n'\|_0 = n\|V_n\|_0 = n|a_n| \prod_{j=1}^{m} |\overline{z_j}| = n|a_n| \prod_{j=1}^{m} |z_j| = n\|T_n\|_0\,.$$

□

Proof of Theorem 1.2. We follow the argument given first in [A-81] and used recently in [QZ-19] as well to base our proof on Theorem 1.1. It is well-known, and by applying Jensen's formula it is easy to see, that

$$\log^+ |v| = \frac{1}{2\pi} \int_{\partial D} \log |v + w|\, |dw|\,, \qquad v \in \mathbb{C}\,,$$

and hence

$$\log^+ |v| = \frac{1}{2\pi} \int_{\partial D} \log |v + wu|\, |dw|\,, \qquad u \in \partial D\,, \quad v \in \mathbb{C}\,. \tag{3.1}$$

Let $T_n \in \mathcal{F}_n^c$, $w \in \partial D$, and $E_n(z) := z^n$. Applying Theorem 1.1 with T_n replaced by $T_n + wE_n$ we obtain

$$\int_{\partial D} \log |T_n'(z)/n + wE_{n-1}(z)|\, |dz| \le \int_{\partial D} \log |T_n(z) + wE_n(z)|\, |dz|\,.$$

Integrating both sides on ∂D with respect to $|dw|$, then using Fubini's theorem and (3.1), we get the theorem. □

Proof of Theorem 1.3. We follow the argument given in [QZ-19] to base our proof on Theorem 1.2. Observe that

$$u^p = \int_0^\infty \log^+(u/a) p^2 a^{p-1}\, da \qquad p > 0\,, \quad u \geq 0\,. \tag{3.2}$$

Indeed, the integration by parts formula gives

$$\int_0^\infty \log^+(u/a) p^2 a^{p-1}\, da = \int_0^u \log^+(u/a) p^2 a^{p-1}\, da = \int_0^u p a^{p-1}\, da = u^p\,.$$

For the sake of brevity we will use the notation $d\mu(a) = p^2 a^{p-1}\, da$. Using (3.2), Fubini's theorem, Theorem 1.2, and Fubini's theorem again, we obtain

$$\begin{aligned}
\int_{\partial D} |T_n'(z)/n|^p \; |dz| &= \int_{\partial D} \left(\int_0^\infty \log^+ |T_n'(z)/(na)| d\mu(a) \right) |dz| \\
&= \int_0^\infty \left(\int_{\partial D} \log^+ |T_n'(z)/(na)||dz| \right) d\mu(a) \\
&\leq \int_0^\infty \left(\int_{\partial D} \log^+ |T_n(z)/a||dz| \right) d\mu(a) \\
&= \int_{\partial D} \left(\int_0^\infty \log^+ |T_n(z)/a| d\mu(a) \right) |dz| \\
&= \int_{\partial D} |T_n(z)|^p \; |dz|\,.
\end{aligned}$$

□

4. Additional Remarks

Let $\mathcal{T}_n^c$ be the set of trigonometric polynomials f_n of the form

$$f_n(t) = \sum_{k=-n}^{n} a_k e^{ikt}\,, \qquad a_k \in \mathbb{C}\,,\;\; t \in \mathbb{R}\,,$$

that is, $f_n(t) = T_n(e^{it})$ with some $T_n \in \mathcal{F}_n^c$. We define

$$\|f_n\|_p := \|T_n\|_p\,, \qquad 0 \leq p \leq \infty\,.$$

The sharp inequality

$$\|f_n'\|_p \leq n \|f_n\|_p\,, \qquad f_n \in \mathcal{T}_n^c\,, \tag{4.1}$$

is associated with the name of S. N. Bernstein, who proved (4.1) for $p = \infty$ with the constant $2n$ and applied it to obtain inverse theorems of approximation theory. In 1914 M. Riesz proved the interpolation formula

$$f_n'(t) = \sum_{k=1}^{2n} (-1)^{k+1} \alpha_k f_n(t + t_k)\,, \qquad f_n \in \mathcal{T}_n^c\,,$$

where

$$t_k = \frac{2k-1}{2n}\,\pi\,, \qquad \alpha_k = \frac{1}{4n\sin^2(t_k/2)}\,, \qquad 1 \le k \le 2n\,. \tag{4.2}$$

The coefficients α_k in (4.2) are nonnegative and $\sum_{k=1}^{2n} \alpha_k = n$. The formula (4.2) gives inequality (4.1) for $1 \le p \le \infty$. Moreover, (4.2) gives the sharp inequality

$$\int_0^{2\pi} \varphi(|f_n'(t)|)\,dt \le \int_0^{2\pi} \varphi(n|f_n(t)|)\,dt\,, \qquad f_n \in \mathcal{T}_n^c\,, \tag{4.3}$$

for an arbitrary nondecreasing function φ convex downwards on the half-line $[0,\infty)$. In particular, the function $\varphi(u) = u^p$ for $1 \le p < \infty$ has such properties, and in this case (4.3) turns into (4.1). The details may be found in [Z-59] or [BE-95], for example.

V.V. Arestov [A-81] proposed another method for studying inequalities similar to (4.1) for convolution operators, including the differentiation operator. More exactly, he considered inequalities of type (4.1) for the operators of Szegő composition on the set of algebraic polynomials on the unit circle of the complex plane. For polynomials $P_n \in \mathcal{P}_n^c$ and $\Lambda_n \in \mathcal{P}_n^c$ written in the form

$$P_n(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k\,, \qquad \Lambda_n(z) = \sum_{k=0}^{n} \binom{n}{k} \lambda_k z^k\,,$$

the polynomial

$$\Lambda_n P_n(z) := \sum_{k=0}^{n} \binom{n}{k} \lambda_k a_k z^k \tag{4.4}$$

is called the Szegő composition of the polynomials Λ_n and P_n. For a fixed $\Lambda_n \in \mathcal{P}_n^c$ the Szegő composition (4.4) is a linear operator on $\mathcal{P}_n^c$. In particular, writing $f_{n/2}(t) = P_n(e^{it})e^{-int/2}$ for the differentiation operator $f_{n/2} \to f_{n/2}'$ on $\mathcal{T}_{n/2}^c$ it corresponds to the composition operator

$$D_n P_n(z) = zP_n'(z) - \frac{n}{2} P_n(z)$$

with the polynomial

$$D_n(z) = \frac{n}{2}(z+1)^{n-1}(z-1) \tag{4.5}$$

on $\mathcal{P}_n^c$. Let Φ^+ be the set of functions φ nondecreasing and locally absolutely continuous on $(0,\infty)$, for which the function $u\varphi'(u)$ is also nondecreasing on $(0,\infty)$. In particular $\varphi(u) = u^p$ for every $p > 0$ belong to the set Φ^+.

In [A-81, Theorem 3] it is proved that the following sharp inequality holds on $\mathcal{P}_n^c$ for all functions $\varphi \in \Phi^+$ and an arbitrary operator (4.4):

$$\int_0^{2\pi} \varphi\left(\left|\frac{\Lambda_n P_n(e^{it})}{\Lambda_n Q_n(e^{it})}\right|\right) dt \le \int_0^{2\pi} \varphi\left(\left|\frac{P_n(e^{it})}{Q_n(e^{it})}\right|\right) dt\,, \qquad P_n \in \mathcal{P}_n^c\,, \tag{4.6}$$

where $Q_n \in \mathcal{P}_n^c$ is the extremal polynomial for the inequality

$$(4.7) \qquad \int_0^{2\pi} \log \left|\Lambda_n P_n(e^{it})\right| \, dt \le \int_0^{2\pi} \log \left(\|\Lambda_n\|_0 \left|P_n(e^{it})\right|\right) \, dt \,, \qquad P_n \in \mathcal{P}_n^c \,.$$

In [A-81, Theorem 4] for a special class of polynomials $\Lambda_n \in \mathcal{P}_n^c$ (including, in particular, the polynomial D_n defined by (4.5)) and in [A-90, Theorem 1] for an arbitrary polynomial $\Lambda_n \in \mathcal{P}_n^c$ it is proved that that the following inequality holds for all functions $\varphi \in \Phi^+$:

$$(4.8) \qquad \int_0^{2\pi} \varphi \left(\left|\Lambda_n P_n(e^{it})\right|\right) \, dt \le \int_0^{2\pi} \varphi \left(\|\Lambda_n\|_0 \left|P_n(e^{it})\right|\right) \, dt \,, \qquad P_n \in \mathcal{P}_n^c \,.$$

As a special case of (4.8), we have

$$(4.9) \qquad \|\Lambda_n P_n\|_p \le \|\Lambda_n\|_0 \|P_n\|_p \,, \qquad P_n \in \mathcal{P}_n^c \,.$$

for all $0 < p < \infty$.

The following method, called Λ-method, used in the proof of inequalities (4.6) and (4.8) was proposed in [A-81]. The method consists of three steps.

Step 1. Let $\varphi(u) = \log|u|$. In this case, inequalities (4.6) and (4.8) coinside with (4.7), which can also be written in the form

$$(4.10) \qquad \|\Lambda_n P_n\|_0 \le \|\Lambda_n\|_0 \|P_n\|_0 \,, \qquad P_n \in \mathcal{P}_n^c \,.$$

Inequality (4.10) was known; the general case was proved by N.G. de Bruijn and T.A. Springer in [DS-47] (see the theorem and its proof, in particular, (8) and the last formula in the proof); K. Mahler [M-61] proved (4.7) in a special case.

Step 2. Let $\varphi(u) = \log^+|u|$. In this case the formula

$$(4.11) \qquad \log^+|z| = \frac{1}{2\pi} \int_0^{2\pi} \log|z + e^{i\theta}| \, d\theta \,, \qquad z \in \mathbb{C} \,,$$

and the result of Step 1 were used.

Step 3. Let $\varphi \in \Phi^+$ be arbitrary. In [A-81] a representation of the function φ in terms of the function $\log^+$ was presented and used as well as the result in Step 2. If the function $\varphi \in \Phi^+$ is defined and continuous on the half-line $[0, \infty)$, then this representation has the form [A-81, (3.5)]

$$(4.12) \qquad \varphi(u) = \varphi(0) + \int_0^{\infty} \log^+ \frac{u}{a} \, d\chi(a) \,, \qquad u \ge 0 \,,$$

where $\chi(a) = a\varphi'(a)$. In particular,

$$u^p = \int_0^{\infty} \log^+(u/a) p^2 a^{p-1} \, da \,.$$

In [GL-89] inequality (4.1) was discussed for the operator

$$S_n f_n = A f_n + B \frac{f_n'}{n}\,, \qquad f_n \in \mathcal{F}_n^c\,,$$

which is more general in comparison with the differentiation operator $f_n \to f_n'$, and the Λ-method was used. In [GL-89, Lemma 1], for real A and B, a classical property of the zeros of the function $S_n f_n$ was observed depending on the zeros of the function $f_n \in \mathcal{T}_n^c$. In short this property means that an analogue of the Lagrange theorem on the zeros of the derivative holds for the operator S_n. In this way, in Theorem 1 (by means of the considerations applied earlier in [A-81] and [M-61]) an analogue of the inequality (4.1) with $p = 0$ was proved, thus Step 1 was realized. After that Steps 2 and 3 from [A-81] were applied. Note that the operator S_n corresponds to the operator of Szegő composition on $\mathcal{P}_{2n}^c$ satisfying the assumptions of [A-81, Theorem 4]. Thus the final result of [GL-89] was not new.

In [QZ-19, Sections 5 and 6] inequality (4.1) was proved by the Λ-method. In [QZ-19, Sections 5] inequality 4.1 for $p = 0$ called the inequality of K. Mahler, was discussed, though, in fact, this inequality appeared in the essentially earlier paper [DS-47]. In [QZ-19, Sections 6] the passage to the function $\log^+$ by formula (4.11), and then the passage to the function u^p, $p > 0$, by formula (4.12) have been made.

The main result of the present paper is the proof of Theorem 1.1 contained in Lemmas 2.1 and 2.2. In these two lemmas we show that analogues of these lemmas, known earlier to be valid for algebraic polynomials $P_n \in \mathcal{P}_n^c$ also hold for trigonometric polynomials $T_n \in \mathcal{F}_n^c$. As a result we offer quite a simple proof of Bernstein's inequality (4.1) in L_p for all $p \geq 0$.

In particular, note that Lemma 2.2 in terms of polynomials $Q_{2n}, R_{2n} \in \mathcal{P}_{2n}$ means that if R_{2n} has each of its $2n$ zeros in the closed unit disk $\overline{D}$ and

$$|Q_{2n}(z)| \leq |R_{2n}(z)|\,, \qquad z \in \partial D\,,$$

then

$$|zQ_{2n}'(z) - nQ_{2n}(z)| \leq |zR_{2n}'(z) - nR_{2n}(z)|\,, \qquad z \in \partial D\,.$$

This lemma can be considered as an extension of the Bernstein-De Bruijn theorem [M-96, Chapter 2, Theorem 6.4].

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Department of Mathematics, Texas A&M University, College Station, Texas 77843, College Station, Texas 77843

E-mail address: `terdelyi@math.tamu.edu`