MARKOV-BERNSTEIN TYPE INEQUALITIES FOR CONSTRAINED POLYNOMIALS WITH REAL VERSUS COMPLEX COEFFICIENTS

Tamás Erdélyi

Abstract. Let $\mathcal{P}_{n,k}^c$ denote the set of all polynomials of degree at most n with *complex coefficients* and with at most k $(0 \le k \le n)$ zeros in the open unit disk. Let $\mathcal{P}_{n,k}$ denote the set of all polynomials of degree at most n with *real coefficients* and with at most k $(0 \le k \le n)$ zeros in the open unit disk. Associated with $0 \le k \le n$ and $x \in [-1,1]$, let

$$B_{n,k,x}^* := \max\left\{\sqrt{\frac{n(k+1)}{1-x^2}}\,,\ n\log\left(\frac{e}{1-x^2}\right)\right\}, \qquad B_{n,k,x} := \sqrt{\frac{n(k+1)}{1-x^2}}\,,$$

and

$$M_{n,k}^* := \max\{n(k+1),\ n\log n\}\,, \qquad M_{n,k} := n(k+1)\,.$$

It is shown that

$$c_1 \min\{B_{n,k,x}^*, M_{n,k}^*\} \le \sup_{p \in \mathcal{P}_{n,k}^c} \frac{|p'(x)|}{\|p\|_{[-1,1]}} \le c_2 \min\{B_{n,k,x}^*, M_{n,k}^*\}$$

for every $x \in [-1,1]$, where $c_1 > 0$ and $c_2 > 0$ are absolute constants. Here $\|\cdot\|_{[-1,1]}$ denotes the supremum norm on $[-1,1]$. This result should be compared with the inequalities

$$c_3 \min\{B_{n,k,x}, M_{n,k}\} \le \sup_{p \in \mathcal{P}_{n,k}} \frac{|p'(x)|}{\|p\|_{[-1,1]}} \le c_4 \min\{B_{n,k,x}, M_{n,k}\}\,,$$

for every $x \in [-1,1]$, where $c_3 > 0$ and $c_4 > 0$ are absolute constants. The upper bound of this second result is also fairly recent, and it may be surprising that there is a significant difference between the real and complex cases as far as Markov-Bernstein type inequalities are concerned. The lower bound of the second result is proved in this paper. It is the final piece of a long series of papers on this topic by a number of authors starting with Erdős in 1940.

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1. Introduction

We introduce the following classes of polynomials. Let

$$\mathcal{P}_n := \left\{ f : f(x) = \sum_{i=0}^{n} a_i x^i \,, \quad a_i \in \mathbb{R} \right\}$$

denote the set of all algebraic polynomials of degree at most n with real coefficients.

Let

$$\mathcal{P}_n^c := \left\{ f : f(x) = \sum_{i=0}^{n} a_i x^i \,, \quad a_i \in \mathbb{C} \right\}$$

denote the set of all algebraic polynomials of degree at most n with complex coefficients.

Let $\mathcal{P}_{n,k}$ denote the set of all polynomials of degree at most n with *real coefficients* and with at most k $(0 \le k \le n)$ zeros in the open unit disk.

Let $\mathcal{P}_{n,k}^c$ denote the set of all polynomials of degree at most n with *complex coefficients* and with at most k $(0 \le k \le n)$ zeros in the open unit disk.

In this paper we always assume that k and n are integers satisfying $0 \le k \le n$.

Our starting point is the following two inequalities the usefulness of which is well known especially in approximation theory. See, for example, A.A. Markov [22], V.A. Markov [23], Duffin and Schaeffer [31], Bernstein [1], Cheney [8], Lorentz [20], DeVore and Lorentz [9], or Natanson [27] (some of these references discuss only the case when the polynomial has real coefficients).

Markov Inequality. *The inequality*

$$\|p'\|_{[-1,1]} \le n^2 \|p\|_{[-1,1]}$$

holds for every $p \in \mathcal{P}_n^c$.

Bernstein Inequality. *The inequality*

$$|p'(x)| \le \frac{n}{\sqrt{1-x^2}} \, \|p\|_{[-1,1]}$$

holds for every $p \in \mathcal{P}_n^c$ *and* $x \in (-1,1)$.

In the above two theorems and throughout the paper $\|\cdot\|_A$ denotes the supremum norm on $A \subset \mathbb{R}$.

Markov- and Bernstein-type inequalities in L_p norms are discussed, for example, in DeVore and Lorentz [9], Lorentz, Golitschek, and Makovoz [21], Golitschek and Lorentz [18], Nevai [28], Máté and Nevai [25], Rahman and Scmeisser [30], and Milovanović, Mitrinović, and Rassias [26].

After a number of less general and weaker results of Erdős [17], Lorentz [19], Scheick [32], Szabados and Varma [34], Szabados [33], and Máté [24], the essentially sharp Markov-type estimate

$$c_5 n(k+1) \le \sup_{p \in \mathcal{P}_{n,k}} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \le c_6 n(k+1) \tag{1.1}$$

was proved by Borwein [3] (in a slightly less general formulation) and by Erdélyi [11] (in the above form). Here $c_5 > 0$ and $c_6 > 0$ are absolute constants. A simpler proof is given by Erdélyi [14] that relates the upper bound in (1.1) to a beautiful Markov-type inequality of Newman [29] for Müntz polynomials. See also Borwein and Erdélyi [6] and Lorentz, Golitschek, and Makovoz [21]. A sharp extension of (1.1) to L_p norms is also proved by Borwein and Erdélyi [7]. The lower bound in (1.1) was proved and the upper bound was conjectured by Szabados [33] earlier. Another example that shows the lower bound in (1.1) is given by Erdélyi [12].

Erdős [17] proved a Bernstein-type inequality on $[-1,1]$ for polynomials from $\mathcal{P}_{n,0}$ having only real zeros. Lorentz [19] improved this by establishing the "right" Bernstein-type inequality on $[-1,1]$ for all polynomials from $\mathcal{P}_{n,0}$. Improving weaker results of Erdélyi [12] and Szabados and Erdélyi [16], Borwein and Erdélyi [5] obtained a Bernstein-type analogue of the upper bound in (1.1) which was believed to be essentially sharp. Namely we proved the upper bound in

$$c_3 \min\{B_{n,k,x}, M_{n,k}\} \le \sup_{p \in \mathcal{P}_{n,k}} \frac{|p'(x)|}{\|p\|_{[-1,1]}} \le c_4 \min\{B_{n,k,x}, M_{n,k}\} \tag{1.2}$$

for every $x \in [-1,1]$, where

$$B_{n,k,x} := \sqrt{\frac{n(k+1)}{1-x^2}}\,, \qquad \text{and} \qquad M_{n,k} := n(k+1)\,,$$

and where $c_3 > 0$ and $c_4 > 0$ are absolute constants. Although it was expected that this is the "right" Bernstein-type inequality for the classes $\mathcal{P}_{n,k}$, its sharpness was proved only in the special cases when $x = 0$ or $x = \pm 1$; when $k = 0$; and when $k = n$. The lower bound of (1.2) in full generality is proved in this paper. See Theorem 2.2. It shows that the result in Borwein and Erdélyi [5] is the "right" Markov-Bernstein type inequality for $\mathcal{P}_{n,k}$ on $[-1,1]$.

We also establish the "right" analogue of (1.2) for the classes $\mathcal{P}_{n,k}^c$. See Theorem 2.1.

2. New Results

Our first result is the "right" Markov-Bernstein type inequality on $[-1,1]$ for the classes $\mathcal{P}_{n,k}^c$.

Theorem 2.1. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 \min\{B^*_{n,k,x}, M^*_{n,k}\} \leq \sup_{p \in \mathcal{P}^c_{n,k}} \frac{|p'(x)|}{\|p\|_{[-1,1]}} \leq c_2 \min\{B^*_{n,k,x}, M^*_{n,k}\}$$

for every $x \in [-1,1]$, where

$$B^*_{n,k,x} := \max\left\{\sqrt{\frac{n(k+1)}{1-x^2}}\,,\ \ n\log\left(\frac{e}{1-x^2}\right)\right\}$$

and

$$M^*_{n,k} := \max\{n(k+1),\ \ n\log n\}\,.$$

Theorem 2.1 should be compared with the result below that may be interpreted as the "right" Markov-Bernstein type inequality on $[-1,1]$ for the classes $\mathcal{P}_{n,k}$. It may be surprising that there is a significant difference between the real and complex cases as far as Markov- and Bernstein-type inequalities for polynomials with restricted zeros are concerned.

Theorem 2.2. *There are absolute constants $c_3 > 0$ and $c_4 > 0$ such that*

$$c_3 \min\{B_{n,k,x}, M_{n,k}\} \leq \sup_{p \in \mathcal{P}_{n,k}} \frac{|p'(x)|}{\|p\|_{[-1,1]}} \leq c_4 \min\{B_{n,k,x}, M_{n,k}\}$$

for every $x \in [-1,1]$, where

$$B_{n,k,x} := \sqrt{\frac{n(k+1)}{1-x^2}} \qquad \text{and} \qquad M_{n,k} := n(k+1)\,.$$

Remark 2.3. The "standard" argument to derive Markov's inequality for $\mathcal{P}^c_n$ from Markov's inequality for $\mathcal{P}_n$ goes as follows. Suppose

$$\|q'\|_{[-1,1]} \leq n^2 \|q\|_{[-1,1]}$$

for every $q \in \mathcal{P}_n$. Now let $p \in \mathcal{P}^c_n$ be arbitrary. Fix an arbitrary point $a \in [-1,1]$, and choose a constant $c \in \mathbb{C}$ with $|c| = 1$ so that $cp'(a)$ is real. We introduce $q \in \mathcal{P}_n$ defined by

$$q(x) := \mathrm{Re}(cp(x))\,, \qquad x \in \mathbb{R}\,.$$

Then

$$|p'(a)| = |cp'(a)| = |q'(a)| \leq n^2 \|q\|_{[-1,1]} \leq n^2 \|p\|_{[-1,1]}\,.$$

Since this holds for every $p \in \mathcal{P}^c_n$ and $a \in [-1,1]$, we have

$$\|p'\|_{[-1,1]} \leq n^2 \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}^c_n$.

Observe that, while $p \in \mathcal{P}_n^c$ implies $q := \mathrm{Re}(cp) \in \mathcal{P}_n$ (restricted to the real line), $p \in \mathcal{P}_{n,k}^c$ does not imply $q := \mathrm{Re}(cp) \in \mathcal{P}_{n,k}$. This suggests that in order to establish the "right" Markov-type inequalities for $\mathcal{P}_{n,k}^c$, the arguments need to be more clever than the above standard extension.

Remark 2.4. The Markov-type upper bound

$$\sup_{p \in \mathcal{P}_{n,k}^c} \frac{|p'(x)|}{\|p\|_{[-1,1]}} \le c_2 M_{n,k}^*$$

of Theorem 2.1 is proved by Erdélyi [15].

Remark 2.5. The Markov-type upper bound

$$\sup_{p \in \mathcal{P}_{n,k}} \frac{|p'(x)|}{\|p\|_{[-1,1]}} \le c_4 M_{n,k}$$

in Theorem 2.2 is a combination of the results in Borwein [3] and Erdélyi [12]. See also Erdélyi [14], Borwein and Erdélyi [6], and Lorentz, Golitschek, and Makovoz [21].

Remark 2.6. The Bernstein-type upper bound

$$\sup_{p \in \mathcal{P}_{n,k}} \frac{|p'(x)|}{\|p\|_{[-1,1]}} \le c_2 B_{n,k,x}$$

in Theorem 2.2 is proved in Borwein and Erdélyi [5].

3. Lemmas for Theorem 2.1

Our first lemma is proved in Erdélyi [15, Lemma 3.1]. It can also be derived easily from a quite similar result proved in [4, Theorem 3.2].

Lemma 3.1. *Let $0 \le k \le n$ be integers and let $s \in [0,1]$. We have*

$$\|p\|_{[-1-s,1+s]} \le \exp\left(18\left(\sqrt{nks} + ns\right)\right)\|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_{n,k}^c$.

Our next lemma shows that how fast a polynomial $p \in \mathcal{P}_{n,k}^c$ can grow on the vertical lines $\mathrm{Re}(z) = x,\ \ x \in (-1,1)$, subject to $\|p\|_{[-1,1]} = 1$.

Lemma 3.2. *Let $0 \le k \le n$ be integers. Let $x \in (-1,1)$. There is an absolute constant c_7 such that*

$$|p(z)| \le c_7 \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_{n,k}^c$ and for every $z \in \mathbb{C}$ satisfying

$$\mathrm{Re}(z) = x \qquad \text{and} \qquad |\mathrm{Im}(z)| \le \frac{1}{B_{n,k,x}^*}\,.$$

Lemma 3.2 will follow by a combination of Lemma 3.1 and our next lemma. The proof of Lemma 3.3 below (in fact a more general result) may be found in Boas [2].

Lemma 3.3 (Nevanlinna's Inequality). *The inequality*

$$\log|p(x+iy)| \le \frac{|y|}{\pi}\int_{-\infty}^{\infty} \frac{\log|p(t)|}{(t-x)^2+y^2}\,dt$$

holds for every polynomial p with complex coefficients.

The Bernstein-type upper bound of Theorem 2.1 will be obtained by a combination of the Cauchy integral formula and Lemma 3.2.

4. Proof of Theorem 2.1

Proof of Lemma 3.2. Let $p \in \mathcal{P}_{n,k}^c$. Without loss of generality we may assume that $n \ge 1$ and $x \in [0,1)$. We normalize so that

$$\|p\|_{[-1,1]} = 1\,, \tag{4.1}$$

that is,

$$\log|p(t)| \le 0\,, \qquad -1 \le t \le 1\,. \tag{4.2}$$

In the rest of the proof, associated with a fixed $x \in [0,1)$, let

$$z = x+iy\,, \qquad y \in \mathbb{R}\,, \quad |y| \le \frac{1}{B_{n,k,x}^*}\,. \tag{4.3}$$

We have

$$\begin{aligned} \frac{|y|}{\pi}\int_{-\infty}^{-1} \frac{\log|p(t)|}{(t-x)^2+y^2}\,dt &\le \frac{|y|}{\pi}\int_{-\infty}^{-1} \frac{n\log(2|t|)}{(t-x)^2+y^2}\,dt \\ &\le \frac{n}{\pi B_{n,k,x}^*}\int_{-\infty}^{-1} \frac{\log(2|t|)}{|t|^2}\,dt \le c \end{aligned} \tag{4.4}$$

with an absolute constant c. Here we used the well-known inequality $|p(t)| \le |2t|^n$ valid for all $p \in \mathcal{P}_n^c$ with $\|p\|_{[-1,1]} \le 1$ and for all $t \in \mathbb{R} \setminus (-1,1)$. Obviously

$$\frac{|y|}{\pi}\int_{-1}^{1} \frac{\log|p(t)|}{(t-x)^2+y^2}\,dt \le 0\,. \tag{4.5}$$

Now we use Lemma 3.1 and (4.3) to obtain

$$\begin{aligned} &\frac{|y|}{\pi}\int_{1}^{2} \frac{\log|p(t)|}{(t-x)^2+y^2}\,dt \\ &\quad\le \frac{|y|}{\pi}\int_{1}^{2} \frac{18\left(\sqrt{nk(t-1)}+n(t-1)\right)}{(t-x)^2+y^2}\,dt \\ &\quad\le \frac{|y|}{\pi}\int_{1}^{2} \frac{18\left(\sqrt{nk(t-x)}+n(t-x)\right)}{(t-x)^2}\,dt \\ &\quad\le \frac{18\sqrt{nk}}{\pi B_{n,k,x}^*}\int_{1}^{2} (t-x)^{-3/2}\,dt + \frac{18n}{\pi B_{n,k,x}^*}\int_{1}^{2} (t-x)^{-1}\,dt \\ &\quad\le \frac{36\sqrt{nk}\,(1-x)^{-1/2}}{\pi B_{n,k,x}^*} + \frac{18n\log\left(\frac{2-x}{1-x}\right)}{\pi B_{n,k,x}^*} \le c \end{aligned} \tag{4.6}$$

with an absolute constant c. Finally, similarly to (4.4), we have

$$\begin{aligned}\frac{|y|}{\pi}\int_2^\infty \frac{\log|p(t)|}{(t-x)^2+y^2}\,dt &\leq \frac{|y|}{\pi}\int_2^\infty \frac{n\log(2|t|)}{(t-x)^2+y^2}\,dt \\ &\leq \frac{n}{\pi B^*_{n,k,x}}\int_2^\infty \frac{\log(2|t|)}{|t-1|^2}\,dt \leq c\end{aligned} \tag{4.7}$$

with an absolute constant c. Now (4.1) – (4.7) and Lemma 3.3 (Nevanlinna's inequality) yield that

$$|p(z)| \leq \exp\left(\frac{|y|}{\pi}\int_{-\infty}^\infty \frac{\log|p(t)|}{(t-x)^2+y^2}\,dt\right) \leq c = c\,\|p\|_{[-1,1]}$$

with an absolute constant c. $\square$

Proof of Theorem 2.1. It follows from Lemma 3.2 and Cauchy's integral formula in a standard fashion that there is an absolute constant $c_8 > 0$ such that

$$|p'(x)| \leq c_8 B^*_{n,k,x}\|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}^c_{n,k}$ and for every $x \in (-1,1)$ such that

$$\frac{1-|x|}{2} \geq \frac{1}{B^*_{n,k,x}}\,. \tag{4.8}$$

Now assume that $x \in [-1,1]$ and (4.8) does not hold. Note that in this case $B^*_{n,k,x} \leq c_9 M^*_{n,k}$ with an absolute constant $c_9 > 0$, and the upper bound of the theorem follows from Remark 2.4.

Now we prove the lower bound of the theorem. Because of the the lower bound in Theorem 2.2 (which we prove in the next section), it is sufficient to prove that there is an absolute constant $c_{10} > 0$ such that

$$c_{10}\min\left\{n\log\left(\frac{e}{1-x^2}\right)\,,\ n\log n\right\} \leq \sup_{p\in\mathcal{P}^c_{n,0}} \frac{|p'(x)|}{\|p\|_{[-1,1]}} \leq \sup_{p\in\mathcal{P}^c_{n,k}} \frac{|p'(x)|}{\|p\|_{[-1,1]}} \tag{4.9}$$

for every $x \in [0,1]$. To see (4.9) let

$$z_m := \exp\left(\frac{(2m-1)\pi i}{2n}\right), \qquad m = 1,2,\dots,n$$

be the zeros of $z^{2n}+1$ in the open upper half-plane. Let

$$p_n(z) := \prod_{m=1}^n (z-z_m)^2\,.$$

Then $p_n \in \mathcal{P}^c_{2n,0}$ and $|p_n(t)| = t^{2n}+1$ for every $t \in \mathbb{R}$. Note that this implies

$$1 \leq |p_n(t)|\,, \qquad t \in [-1,1]\,,$$

and

$$|p_n(1)| = \|p_n\|_{[-1,1]} = 2\,.$$

Let $x \in [0,1]$ be fixed. Then

$$\operatorname{Im}\left(\frac{1}{x - z_m}\right) > 0\,, \qquad m = 1, 2, \dots, n\,,$$

and there is an absolute constant $c_{11} > 0$ such that

$$\operatorname{Im}\left(\frac{1}{x - z_m}\right) > \frac{c_{11} n}{m}\,, \qquad \frac{n(1-x)}{4} \le m \le \frac{n+1}{2}\,.$$

Hence there is an absolute constant $c_{11} > 0$ such that

$$\begin{aligned}\frac{|p_n'(x)|}{\|p_n\|_{[-1,1]}} &\ge \left|\frac{p_n'(x)}{2p_n(x)}\right| \ge \sum_{\frac{n(1-x)}{4} \le m \le \frac{n+1}{2}} \frac{c_{11} n}{m} \\ &\ge c_{12} \min\left\{ n \log\left(\frac{e}{1-x^2}\right),\ n \log n \right\}.\end{aligned}$$

This completes the proof of the lower bound of the theorem. $\square$

5. Proof of Theorem 2.2

For the upper bound in Theorem 2.2, see Remarks 2.5 and 2.6. What remains to prove is the lower bound in Theorem 2.2.

The following lemma is trivial. However, it plays a crucial role in proving the lower bound in Theorem 2.2.

Lemma 5.1. *Let $a \in [0,1)$. Suppose*

$$\begin{aligned} x &\in \left[a - \tfrac{1}{2}(1-a),\, a\right] && \textit{if } a \in \left[0, \tfrac{3}{4}\right), \\ x &\in [a - 3(1-a),\, a] && \textit{if } a \in \left[\tfrac{3}{4}, 1\right). \end{aligned}$$

If $R_n := R_{n,k,a} \in \mathcal{P}_{n,k}$ satisfies

$$\frac{|R_n'(a)|}{|R_n\|_{[-1,1]}} \ge c \left(\frac{n(k+1)}{1-a^2}\right)^{1/2},$$

then $\widetilde{R}_n := \widetilde{R}_{n,k,x} \in \mathcal{P}_{n,k}$ defined by

$$\widetilde{R}_n(z) = R_n\left(1 - \frac{1-a}{1-x}(1-z)\right)$$

satisfies

$$\frac{|\widetilde{R}_n'(x)|}{\|\widetilde{R}_n\|_{[-1,1]}} \ge \frac{c}{4}\left(\frac{n(k+1)}{1-x^2}\right)^{1/2}.$$

Proof of the lower bound of the Theorem 2.2.

Case 1. $0 \le k \le 3$.

In the case $k = 0$, and hence in the case $0 \le k \le 3$, the lower bound of the theorem is proved in Borwein and Erdélyi [6, pages 433–434].

Case 2. $0 \le x \le 1 - \frac{k}{n-k}$, $0 < 4k \le n$, k is odd, and $x = 1 - \frac{2j}{n-k}$ for a positive integer j.

Let $m_1 := (n-k)(1-x)$ and $m_2 := (n-k)(1+x)$. Let

$$P_n(z) := P_{n,k,x}(z) := (z-1)^{m_1}(z+1)^{m_2}(z-x)^k\,.$$

Observe that if α is an extreme point of P_n in $(-1,1)$, then

$$\frac{P_n'(\alpha)}{P_n(\alpha)} = \frac{m_1}{\alpha - 1} + \frac{m_2}{\alpha + 1} + \frac{k}{\alpha - x} = 0\,,$$

that is,

$$(\alpha - x)^2 = \frac{k(1-\alpha^2)}{2(n-k)}\,.$$

Solving the quadratic equation, we obtain that

$$\alpha_1, \alpha_2 = \frac{2x \pm \sqrt{\frac{2k}{n-k}(1-x^2) + \frac{k^2}{(n-k)^2}}}{2 + \frac{k}{n-k}}$$

with

$$\alpha_2 - \alpha_1 \le \sqrt{\frac{2k}{n-k}(1-x^2) + \frac{k^2}{(n-k)^2}}$$

and $x \in (\alpha_1, \alpha_2)$. Using the assumptions $0 \le x \le 1 - \frac{k}{n-k}$ and $0 < 4k \le n$, we have

$$x - \left(\frac{3k(1-x^2)}{n-k}\right)^{1/2} \le \alpha_1,\ \alpha_2 \le x + \left(\frac{3k(1-x^2)}{n-k}\right)^{1/2}. \tag{5.1}$$

Let $Q_k := Q_{n,k,x}$ be the kth Chebyshev polynomial T_k transformed linearly from $[-1,1]$ to

$$I := I_{n,k,x} := \left[x - \left(\frac{12k(1-x^2)}{(n-k)}\right)^{1/2},\ x + \left(\frac{12k(1-x^2)}{(n-k)}\right)^{1/2}\right].$$

That is,

$$\begin{aligned} Q_k(z) &:= T_k\left(\left(\frac{n-k}{12k(1-x^2)}\right)^{1/2}(z-x)\right) \\ &:= \cos\left(k \arccos\left(\left(\frac{n-k}{12k(1-x^2)}\right)^{1/2}(z-x)\right)\right), \qquad z \in I\,. \end{aligned}$$

Note that

$$Q_k(z) = 2^{k-1}\left(\frac{n-k}{12k(1-x^2)}\right)^{k/2}\prod_{j=1}^{k}(z-x_j)\,, \qquad x_j \in I\,,$$

where the set $\{x_1, x_2, \dots, x_k\}$ is symmetric with respect to x. Hence

$$|Q_k(z)| \le \left(\frac{n-k}{3k(1-x^2)}\right)^{k/2}|z-x|^k\,, \qquad z \in \mathbb{R}\setminus I\,. \tag{5.2}$$

Let

$$R_n(z) := R_{n,k,x}(z) := (z-1)^{m_1}(z+1)^{m_2}Q_k(z) \in \mathcal{P}_{n,k}\,.$$

Let $S_{n-k}(z) := (z-1)^{m_1}(z+1)^{m_2}$, and let

$$L := L_{n,k,x} := \max_{-1\le z\le 1}|S_{n-k}(z)| = S_{n-k}(x)\,.$$

Then

$$\max_{z\in I}|R_n(z)| \le L\,, \tag{5.3}$$

and by (5.2) and (5.1)

$$\begin{aligned}
\max_{z\in[-1,1]\setminus I}|R_n(z)| &= \max_{z\in[-1,1]\setminus I}|(z-1)^{m_1}(z+1)^{m_2}Q_k(z)| \\
&\le \max_{z\in[-1,1]\setminus I}\left|(z-1)^{m_1}(z+1)^{m_2}\left(\frac{n-k}{3k(1-x^2)}\right)^{k/2}(z-x)^k\right| \\
&\le \left(\frac{n-k}{3k(1-x^2)}\right)^{k/2}\left(\frac{3k(1-x^2)}{n-k}\right)^{k/2}L = L\,.
\end{aligned} \tag{5.4}$$

Combining (5.3) and (5.4), we obtain

$$\max_{z\in[-1,1]}|R_n(z)| \le L\,. \tag{5.5}$$

Also,

$$\begin{aligned}
|R_n'(x)| &= |Q_k'(x)S_{n-k}(x) + Q_k(x)S_{n-k}'(x)| \\
&= |Q_k'(x)|\,L = \left(\frac{n-k}{12k(1-x^2)}\right)^{1/2}|T_k'(0)|\,L \\
&= \left(\frac{n-k}{12k(1-x^2)}\right)^{1/2}kL = \left(\frac{(n-k)k}{12(1-x^2)}\right)^{1/2}L \\
&\ge \frac{1}{4}\left(\frac{nk}{1-x^2}\right)^{1/2}L\,.
\end{aligned} \tag{5.6}$$

Hence, by (5.5) and (5.6),

$$\frac{|R_n'(x)|}{\|R_n\|_{[-1,1]}} \ge \frac{1}{4}\left(\frac{nk}{1-x^2}\right)^{1/2}\,.$$

Case 3. $0 \le x \le 1 - \frac{k}{n-k}$, $0 < 4k \le n$, $k \ge 5$ is odd.

Then choose a positive integer j so that $1-\frac{2(j+1)}{n-k} \le x \le 1-\frac{2j}{n-k}$. The lower bound of the theorem follows from Case 2 and Lemma 5.1.

Before examining the next cases we introduce some notation that should be kept in mind throughout Cases 4, 5, and 6. Let $Q_k := Q_{n,k,x}$ be the kth Chebyshev polynomial T_k transformed linearly from $[-1,1]$ to

$$I := I_{n,k} := \left[1 - \frac{16k}{n-k}\,, 1\right].$$

That is, with $b := 1 - \frac{8k}{n-k}$,

$$\begin{aligned} Q_k(z) &:= T_k\left(\frac{n-k}{8k}(z-b)\right) \\ &:= \cos\left(k \arccos\left(\frac{n-k}{8k}\,(z-b)\right)\right), \qquad z \in I\,. \end{aligned} \tag{5.7}$$

Let $\eta_1 > \eta_2 > \ldots > \eta_u$ be the zeros of Q_k in $\left[1 - \frac{k}{n-k}\,, 1\right]$. That is,

$$\eta_j := \frac{8k}{n-k}\cos\frac{(2j-1)\pi}{2k} + \left(1 - \frac{8k}{n-k}\right)$$

and u is the largest positive integer j for which $\eta_j \ge 1 - \frac{k}{n-k}$.

Case 4. $x \in \{\eta_1, \eta_2, \ldots, \eta_u\}$ and $0 < 9k \le n$.

Let $x_* := 1 - \frac{2v}{n-k}$, where the nonnegative integer v is chosen so that

$$1 - \frac{k+2}{n-k} \le 1 - \frac{2(v+1)}{n-k} < x \le 1 - \frac{2v}{n-k}\,.$$

Let $m_1 := (n-k)(1-x_*)$ and $m_2 := (n-k)(1+x_*)$. Let

$$P_n(z) := P_{n,k,x}(z) := (z-1)^{m_1}(z+1)^{m_2}(z-b)^k\,.$$

Recall that $b := 1 - \frac{8k}{n-k}$. Observe that if α is an extreme point of P_n in $(-1,1)$, then

$$\frac{P_n'(\alpha)}{P_n(\alpha)} = \frac{m_1}{\alpha - 1} + \frac{m_2}{\alpha+1} + \frac{k}{\alpha - b} = 0\,,$$

that is,

$$(\alpha - x_*)(\alpha - b) = \frac{k(1-\alpha^2)}{2(n-k)}\,.$$

Solving the quadratic equation, we obtain that

$$\alpha_1, \alpha_2 = \frac{(x_* + b) \pm \sqrt{(x_* - b)^2 + \frac{2k}{n-k}(1 - x_* b) + \frac{k^2}{(n-k)^2}}}{2 + \frac{k}{n-k}}$$

with
(5.8)

$$\begin{aligned}
1 \geq \alpha_1, \alpha_2 &\geq \frac{(x_* + b) - \sqrt{(x_* - b)^2 + \frac{2k}{n-k}(1 - x_* b) + \frac{k^2}{(n-k)^2}}}{2 + \frac{k}{n-k}} \\
&\geq \frac{(x_* + b) - \sqrt{(x_* - b)^2 + \frac{2k}{n-k}\left(1 - \left(1 - \frac{k}{n-k}\right)\left(1 - \frac{8k}{n-k}\right)\right) + \frac{k^2}{(n-k)^2}}}{2 + \frac{k}{n-k}} \\
&\geq \frac{(x_* + b) - (x_* - b) - \sqrt{19}\,\frac{k}{n-k}}{2 + \frac{k}{n-k}} \geq \frac{2 - \frac{16k}{n-k} - \sqrt{19}\,\frac{k}{n-k}}{2 + \frac{k}{n-k}} \\
&\geq 1 - \frac{12k}{n-k}\,,
\end{aligned}$$

since $0 < 9k \leq n$ implies

$$\begin{aligned}
\left(1 - \frac{12k}{n-k}\right)\left(2 + \frac{k}{n-k}\right) &= 2 - \frac{23k}{n-k} + \frac{12k^2}{(n-k)^2} \\
&\leq 2 - \frac{23k}{n-k} + \frac{12k}{8(n-k)} \\
&\leq 2 - \frac{16k}{n-k} - \sqrt{19}\,\frac{k}{n-k}\,.
\end{aligned}$$

Note that the polynomial Q_k defined by (5.7) is of the form

$$Q_k(z) = 2^{k-1}\left(\frac{n-k}{8k}\right)^k \prod_{j=1}^{k}(z - x_j)\,, \qquad x_j \in I\,,$$

where the set $\{x_1, x_2, \ldots, x_k\}$ is symmetric with respect to b. Hence

$$|Q_k(z)| \leq \left(\frac{n-k}{4k}\right)^k |z - b|^k\,, \qquad z \in \mathbb{R} \setminus I\,. \tag{5.9}$$

Let

$$R_n(z) := R_{n,k,x}(z) := (z-1)^{m_1}(z+1)^{m_2} Q_k(z) \in \mathcal{P}_{n,k}\,.$$

Let $S_{n-k}(z) := (z-1)^{m_1}(z+1)^{m_2}$, and let

$$L := L_{n,k,x} := \max_{-1 \leq z \leq 1} |S_{n-k}(z)|\,.$$

Then

$$\max_{z \in I} |R_n(z)| \leq L\,, \tag{5.10}$$

and by (5.9) and (5.8),
(5.11)

$$\begin{aligned}
\max_{z \in [-1,1] \setminus I} |R_n(z)| &\leq \max_{z \in [-1,1] \setminus I} |(z-1)^{m_1}(z+1)^{m_2} Q_k)(z)| \\
&\leq \max_{z \in [-1,b]} \left|(z-1)^{m_1}(z+1)^{m_2}\left(\frac{n-k}{4k}\right)^k (z-b)^k\right| \\
&\leq \left(\frac{n-k}{4k}\right)^k \left(\frac{(12-8)k}{n-k}\right)^k L \leq L\,.
\end{aligned}$$

Now (5.10) and (5.11) yield that

(5.12)
$$\max_{z\in[-1,1]} |R_n(z)| \le L\,.$$

Also, if $x=\eta_j$, then

(5.13)
$$\begin{aligned}|R_n'(x)| &= |Q_k'(x)S_{n-k}(x)+Q_k(x)S_{n-k}'(x)| = |Q_k'(x)S_{n-k}(x)|\\
&\ge |Q_k'(x)|\,\frac{L}{16} = \frac{n-k}{8k}\,\left|T_k'\left(\cos\tfrac{(2j-1)\pi}{2k}\right)\right|\,\frac{L}{16}\\
&\ge \frac{n-k}{8k}\,\frac{k}{\sin\frac{(2j-1)\pi}{2k}}\,\frac{L}{16} \ge \frac{n-k}{8k}\,\frac{k}{2\sin\frac{(2j-1)\pi}{4k}}\,\frac{L}{16}\\
&\ge \frac{n-k}{16k}\,\frac{k}{\sqrt{\frac{n-k}{16k}(1-\eta_j)}}\,\frac{L}{16} = \left(\frac{(n-k)k}{1-\eta_j}\right)^{1/2}\,\frac{L}{64}\\
&\ge \frac{1}{128}\,\left(\frac{nk}{1-x^2}\right)^{1/2} L\,.\end{aligned}$$

Here we used the fact that $0<x_*-\frac{2}{n-k}\le x\le x_*\le 1$ and $0<9k\le n$ imply

$$\begin{aligned}\frac{|S_{n-k}(x)|}{L} &= \frac{|S_{n-k}(x)|}{|S_{n-k}(x_*)|} = \frac{(1-x)^{m_1}(x+1)^{m_2}}{(1-x_*)^{m_1}(x_*+1)^{m_2}}\\
&= \left(\frac{1-x}{1-x_*}\right)^{m_1}\left(\frac{x+1}{x_*+1}\right)^{m_2} \ge \left(\frac{x+1}{x_*+1}\right)^{n-k}\\
&= \left(1-\frac{2}{n-k}\right)^{n-k} \ge \frac{1}{16}\,.\end{aligned}$$

From (5.12) and (5.13) we conclude that

$$\frac{|R_n'(x)|}{\|R_n\|_{[-1,1]}} \ge \frac{1}{128}\,\left(\frac{nk}{1-x^2}\right)^{1/2}\,.$$

Case 5. $1-\frac{k}{n-k}\le x\le \eta_1\,,\;\; 0<9k\le n.$

The lower bound of the theorem follows from Case 4 and Lemma 5.1.

Case 6. $x\in[\eta_1,1]$, $0<9k\le n$, and k is odd.

Let $R_n\in\mathcal{P}_{n,k}$ be as in Case 4. Then

$$\frac{|R_n'(x)|}{\|R_n\|_{[-1,1]}} \ge \frac{|R_n'(\eta_1)|}{\|R_n\|_{[-1,1]}} \ge \frac{1}{128}\,\left(\frac{nk}{1-\eta_1^2}\right)^{1/2} \ge \frac{nk}{128\sqrt{24}}\,,$$

where we used the fact that

$$1-\eta_1 = \frac{8k}{n-k}\left(1-\cos\frac{\pi}{2k}\right) \le \frac{8k}{n-k}\,\frac{1}{2}\,\left(\frac{\pi}{2k}\right)^2 \le \frac{12}{nk}\,.$$

Case 7. $0 < k \le n \le 9k$ and x is a zero of the Chebyshev polynomial T_k defined by
$$T_k(x) := \cos k\theta\,, \qquad x = \cos\theta\,, \quad \theta \in [0, \pi]\,.$$

Let
$$R_n := R_{n,k,x} := T_k \in \mathcal{P}_{n,k}\,.$$

Then
$$\frac{|R_n'(x)|}{\|R_n\|_{[-1,1]}} \ge \frac{k}{\sqrt{1-x^2}} \ge \frac{1}{3}\left(\frac{nk}{1-x^2}\right)^{1/2}\,.$$

Case 8. $0 < k \le n \le 9k$ and $x \in [0, \eta]$, where $\eta := \cos\frac{\pi}{2k}$ is the largest zero of the Chebyshev polynomial T_k.

The lower bound of the theorem follows from Case 7 and Lemma 5.1.

Case 9. $0 < k \le n \le 9k$ and $x \in [\eta, 1]$, where $\eta := \cos\frac{\pi}{2k}$ is the largest zero of the Chebyshev polynomial T_k.

Let $R_n := R_{n,k,x} := T_k \in \mathcal{P}_{n,k}$ be as in Case 7. Then
$$\frac{|R_n'(x)|}{\|R_n\|_{[-1,1]}} \ge \frac{|R_n'(\eta)|}{\|R_n\|_{[-1,1]}} \ge \frac{k}{\sin\frac{\pi}{2k}} \ge \frac{2k^2}{\pi} \ge \frac{kn}{5\pi}\,.$$

Case 10. $x \in [0, 1]$, $0 \le k \le n$.

The lower bound of the theorem follows from Cases 1, 3, 5, 6, 8, and 9.

Case 11. $x \in [-1, 0]$, $0 \le k \le n$.

The lower bound of the theorem follows from Case 10 by noting that $R_n \in \mathcal{P}_{n,k}$ implies $\widetilde{R}_n(z) := R_n(-z) \in \mathcal{P}_{n,k}$.

The proof of the lower bound of the theorem is now finished. $\square$

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Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA

E-mail address: terdelyi@math.tamu.edu