

ON THE L_q NORM OF CYCLOTOMIC LITTLEWOOD POLYNOMIALS ON THE UNIT CIRCLE

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Dedicated to the memory of Professor Paul Turán on the occasion of his 100th birthday.

ABSTRACT. Let \mathcal{L}_n be the collection of all (Littlewood) polynomials of degree n with coefficients in $\{-1, 1\}$. We call an algebraic polynomial cyclotomic if each of its coefficients lies on the unit circle of the complex plane. In this paper we prove that if $(P_{2\nu})$ is a sequence of cyclotomic Littlewood polynomials $P_{2\nu} \in \mathcal{L}_{2\nu}$, then

$$M_q(P_{2\nu}) > (2\nu + 1)^a$$

for every $q > 2$ with some $a = a(q) > 1/2$ depending only on q , where

$$M_q(P) := \left(\frac{1}{2\pi} \int_0^{2\pi} |P(e^{it})|^q dt \right)^{1/q}, \quad q > 0.$$

The case $q = 4$ of the above result is due to P. Borwein, Choi, and Ferguson. We also prove that if $(P_{2\nu})$ is a sequence of cyclotomic Littlewood polynomials $P_{2\nu} \in \mathcal{L}_{2\nu}$, then

$$M_q(P_{2\nu}) < (2\nu + 1)^b$$

for every $0 < q < 2$ with some $0 < b = b(q) < 1/2$ depending only on q . Similar results are conjectured for Littlewood polynomials of odd degree. Our main tool here is the Borwein-Choi Factorization Theorem.

1. INTRODUCTION

Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D . Let \mathcal{K}_n be the set of all polynomials of degree n with complex coefficients of modulus 1. Elements of \mathcal{K}_n are often called (complex) unimodular polynomials of degree n . Let \mathcal{L}_n be the set of all polynomials of degree n with coefficients in $\{-1, 1\}$. Elements of \mathcal{L}_n are often called real unimodular polynomials or Littlewood polynomials of degree n . The Parseval formula yields

$$\int_0^{2\pi} |P_n(e^{it})|^2 dt = 2\pi(n + 1)$$

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for all $P_n \in \mathcal{K}_n$. Therefore

$$\min_{z \in \partial D} |P_n(z)| \leq \sqrt{n+1} \leq \max_{z \in \partial D} |P_n(z)|.$$

An old problem (or rather an old theme) is the following.

Problem 1.1 (Littlewood’s Flatness Problem). *How close can a $P_n \in \mathcal{K}_n$ or $P_n \in \mathcal{L}_n$ come to satisfying*

$$(1.1) \quad |P_n(z)| = \sqrt{n+1}, \quad z \in \partial D?$$

Obviously (1.1) is impossible if $n \geq 1$. So one must look for less than (1.1), but then there are various ways of seeking such an “approximate situation”. One way is the following. In his paper [Li1] Littlewood had suggested that, conceivably, there might exist a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ (possibly even $P_n \in \mathcal{L}_n$) such that $(n+1)^{-1/2}|P_n(e^{it})|$ converge to 1 uniformly in $t \in \mathbb{R}$. We shall call such sequences of unimodular polynomials “ultraflat”. More precisely, we give the following definition.

Definition 1.2. *Given a positive number ε , we say that a polynomial $P_n \in \mathcal{K}_n$ is ε -flat if*

$$(1 - \varepsilon)\sqrt{n+1} \leq |P_n(z)| \leq (1 + \varepsilon)\sqrt{n+1}, \quad z \in \partial D.$$

Definition 1.3. *Given a sequence (ε_{n_k}) of positive numbers tending to 0, we say that a sequence (P_{n_k}) of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is (ε_{n_k}) -ultraflat if each P_{n_k} is (ε_{n_k}) -flat. We simply say that a sequence (P_{n_k}) of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is ultraflat if it is (ε_{n_k}) -ultraflat with a suitable sequence (ε_{n_k}) of positive numbers tending to 0.*

The existence of an ultraflat sequence of unimodular polynomials seemed very unlikely, in view of a 1957 conjecture of P. Erdős (Problem 22 in [Er]) asserting that, for all $P_n \in \mathcal{K}_n$ with $n \geq 1$,

$$(1.2) \quad \max_{z \in \partial D} |P_n(z)| \geq (1 + \varepsilon)\sqrt{n+1},$$

where $\varepsilon > 0$ is an absolute constant (independent of n). Yet, refining a method of Körner [Kö], Kahane [Ka] proved that there exists a sequence (P_n) with $P_n \in \mathcal{K}_n$ which is (ε_n) -ultraflat, where

$$\varepsilon_n = O\left(n^{-1/17} \sqrt{\log n}\right).$$

See also [QS]. A recent paper of Bombieri and Bourgain [BB] is devoted to the construction of ultraflat sequences of unimodular polynomials. In particular, one obtains a much improved estimate for the error term. A major part of this paper deals also with the long-standing problem of the effective construction of ultraflat sequences of unimodular polynomials.

Thus the Erdős conjecture (1.2) was disproved for the classes \mathcal{K}_n . For the more restricted class \mathcal{L}_n the analogous Erdős conjecture is unsettled to this date. It is a common belief

that the analogous Erdős conjecture for \mathcal{L}_n is true, and consequently there is no ultraflat sequence of polynomials $P_n \in \mathcal{L}_n$. An interesting result related to Kahane's breakthrough is given in [Be]. For an account of some of the work done till the mid 1960's, see Littlewood's book [Li2] and [QS]. The structure of ultraflat sequences of unimodular polynomials is studied in [Er1], [Er2], [Er3], and [Er4], where several conjectures of Saffari are proved.

The Rudin-Shapiro polynomials appear in Harold Shapiro's 1951 thesis at MIT and are sometimes called just Shapiro polynomials. See Chapter 4 of [Bo] for the construction(s). Cyclotomic properties of the Rudin-Shapiro polynomials are discussed in [BLM]. For $n = 2^k - 1$ a sequence (P_n) of Rudin-Shapiro polynomials satisfies $P_n \in \mathcal{L}_n$ and

$$|P_n(z)| \leq C\sqrt{n+1}, \quad z \in \partial D,$$

with an absolute constant C .

An algebraic polynomial is called cyclotomic if each of its coefficients lies on the unit circle ∂D . We prove that a sequence of cyclotomic Littlewood polynomials of even degree is far from having the above "flatness" property of the sequence of Rudin-Shapiro polynomials. Note that (see page 271 of [BE], for instance) a Littlewood polynomial has Mahler measure one if and only if it is cyclotomic, that is, it has all its zeros on the unit circle D . For a polynomial P let

$$M_q(P) := \left(\frac{1}{2\pi} \int_0^{2\pi} |P(e^{it})|^q dt \right)^{1/q}, \quad q > 0,$$

and

$$M_\infty(P) := \max_{t \in [0, 2\pi]} |P(e^{it})|.$$

2. PRELIMINARY RESULTS

An unpublished observation of the author is the following.

Theorem 2.1. *If $(P_{2\nu})$ is a sequence of cyclotomic polynomials $P_{2\nu} \in \mathcal{L}_{2\nu}$, then*

$$M_\infty(P_{2\nu}) > (2\nu + 1)^a,$$

where $a := 1 - \log_3 \frac{\pi}{2} = 0.5889\dots > \frac{1}{2}$.

The stronger result below is due to P. Borwein, Choi, and Ferguson [BCF].

Theorem 2.2. *If $(P_{2\nu})$ is a sequence of cyclotomic polynomials $P_{2\nu} \in \mathcal{L}_{2\nu}$, then*

$$M_4(P_{2\nu}) > (2\nu + 1)^a$$

with $a = \frac{1}{4} \log_2(1 + \sqrt{17}) = 0.5892\dots > 1/2$.

In the proof of both theorems above the result of Borwein and Choi [BC] stated below has been a key. Our new results follow from it too by a standard technology. Similar arguments were used, for example, by Eminyan [Em] and by Green and Ruzsa [GR].

Theorem 2.3. *Every cyclotomic polynomial $P \in \mathcal{L}_n$ of even degree can be factorized as*

$$P(z) = \pm \Phi_{p_1}(\pm z) \Phi_{p_2}(\pm z^{p_1}) \cdots \Phi_{p_r}(\pm z^{p_1 p_2 \cdots p_{r-1}}),$$

where $n + 1 = p_1 p_2 \cdots p_r$, the numbers p_j are primes, not necessarily distinct, and

$$\Phi_p(z) = \sum_{j=0}^{p-1} z^j = \frac{z^p - 1}{z - 1}$$

is the p -th cyclotomic polynomial.

It is conjectured that this characterization also holds for polynomials $P \in \mathcal{L}_n$ of odd degree. This conjecture is based on substantial computation together with a number of special cases.

3. NEW RESULTS

Theorem 3.1. *If $(P_{2\nu})$ is a sequence of cyclotomic polynomials $P_{2\nu} \in \mathcal{L}_{2\nu}$, then*

$$M_q(P_{2\nu}) > (2\nu + 1)^a$$

for every $q > 2$ with some $a = a(q) > 1/2$ depending only on q .

Theorem 3.1 follows from our next result.

Theorem 3.2. *If $(P_{2\nu})$ is a sequence of cyclotomic polynomials $P_{2\nu} \in \mathcal{L}_{2\nu}$, then*

$$M_1(P_{2\nu}) < (2\nu + 1)^b$$

with some absolute constant $0 < b < 1/2$.

In fact, Theorem 3.2 is a special case of our result below.

Theorem 3.3. *If $(P_{2\nu})$ is a sequence of cyclotomic polynomials $P_{2\nu} \in \mathcal{L}_{2\nu}$, then*

$$M_q(P_{2\nu}) < (2\nu + 1)^b$$

for every $0 < q < 2$ with some $0 < b = b(q) < 1/2$ depending only on q .

It is conjectured that similar results hold for cyclotomic Littlewood polynomials of odd degree.

4. PROOFS

Although Theorem 2.2 beats Theorem 2.1, we present the short proof of Theorem 2.1 that is simpler than and quite different from that of Theorem 2.2.

Proof of Theorem 2.1. We use the factorization theorem of Borwein and Choi. We prove the theorem by induction on the number of factors. The theorem is obviously true when $P_{2\nu}$ has only one factor. The proof of the inductive step goes as follows.

Suppose the theorem is true for f , where f has $k - 1$ factors. We have to prove that the theorem is true for

$$g(z) := \Phi_p(\pm z)f(z^p).$$

Let $M(f)$ be the maximum modulus of f on the unit circle ∂D . The key observation is that $M(f)$ is achieved by $|f(z^p)|$ at a system of p equidistant points on the unit circle ∂D . Denote these by z_1, z_2, \dots, z_p . Then there is at least one z_j such that the angular distance between 1 and z_j is at most $2\pi/(2p)$. Similarly there is at least one z_j such that the angular distance between -1 and z_j is at most $2\pi/(2p)$. Now the proof can be finished by Lemma 4.1 below the proof of which is a straightforward geometric argument. Using Lemma 4.1 the proof of the inductive step is obvious, since $a := 1 - \log_3 \frac{\pi}{2}$ ensures $(2/\pi)p \geq p^a$ for every $p \geq 3$. In fact, using the prime factorization of $2\nu + 1$, where 2ν is the degree of $P_{2\nu}$, one can get a larger value of the exponent a in the theorem if the primes in the factorization of $2\nu + 1$ are large. \square

Lemma 4.1. *If z is a point on the unit circle ∂D such that the angular distance of z from 1 is at most $2\pi/(2p)$, then $|\Phi_p(z)| \geq (2/\pi)p$. If z is a point on the unit circle ∂D such that the angular distance of z from -1 is at most $2\pi/(2p)$. Then $|\Phi_p(-z)| \geq (2/\pi)p$.*

Proof of Lemma 4.1. Recall that

$$\Phi_p(z) = \frac{z^p - 1}{z - 1}$$

and $|\sin t| \leq |t|$ for every $t \in \mathbb{R}$. \square

To prove Theorem 3.2 we proceed as follows. First we introduce some notation. Associated with a positive integer p and a function f defined on the unit circle ∂D let

$$g(t) := f(e^{it}), \quad t \in \mathbb{R},$$

$$f_p(z) := f(z^p)\Phi_p(z), \quad \Phi_p(z) := \sum_{j=0}^{p-1} z^j, \quad g_p(t) := f_p(e^{it}),$$

that is,

$$g_p(t) := f(e^{ipt})h_p(t), \quad h_p(t) := \Phi_p(e^{it}).$$

Let

$$I(f) := \int_{\partial D} |f(z)| |dz| = \int_0^{2\pi} |f(e^{it})| dt, \quad I(g) := \int_0^{2\pi} |g(t)| dt,$$

so $I(g) = I(f)$. The key to the proof of Theorem 3.2 is the following lemma that allows an induction on the number of factors in the decomposition of the cyclotomic polynomial $P_{2\nu}$ given by Theorem 2.3.

Lemma 4.2. *If f is a continuous function on ∂D such that $|f(z)| = |f(\bar{z})|$ for every $z \in \partial D$, then*

$$I(f_p) \leq p^\alpha I(f)$$

for every odd prime p with an absolute constant $0 < \alpha < 1/2$.

Proof Lemma 4.2. Let $k > 0$ be an integer and we define

$$L_{j,k} := \int_{(j-1)\pi/k}^{j\pi/k} |g(t)| dt, \quad j = 1, 2, \dots$$

Then

$$L_{j,k} = L_{j+2k,k}, \quad j = 1, 2, \dots,$$

and

$$\begin{aligned} I(g_p) &= \int_0^{2\pi} |g_p(t)| dt = 2 \int_0^\pi |g_p(t)| dt = 2 \sum_{j=1}^{kp} \int_{(j-1)\pi/(kp)}^{j\pi/(kp)} |g_p(t)| dt \\ &= 2 \sum_{j=1}^{kp} \int_{(j-1)\pi/(kp)}^{j\pi/(kp)} |f(e^{ipt})| |h_p(t)| dt \leq 2 \sum_{j=1}^{kp} \left(\int_{(j-1)\pi/(kp)}^{j\pi/(kp)} |f(e^{ipt})| dt \right) M_{j,k,p} \\ &= 2 \sum_{j=1}^{kp} \left(\int_{(j-1)\pi/k}^{j\pi/k} |f(e^{iu})| du \right) \frac{1}{p} M_{j,k,p} \\ &= 2 \sum_{\mu=1}^k L_{\mu,k} \frac{1}{p} \left(\sum_{\nu=0}^{(p-1)/2} M_{\mu+2\nu k,k,p} + \sum_{\nu=0}^{(p-3)/2} M_{2k+1-\mu+2\nu k,k,p} \right) \\ &= 2 \sum_{\mu=1}^k A_{\mu,k,p} L_{\mu,k}, \end{aligned}$$

where

$$M_{j,k,p} := \max_{I_{j,k,p}} |h_p(t)| = \max_{I_{j,k,p}} |\Phi_p(e^{it})|$$

with

$$I_{j,k,p} := \left[\frac{(j-1)\pi}{kp}, \frac{j\pi}{kp} \right], \quad j = 1, 2, \dots, kp,$$

and

$$(4.1) \quad A_{\mu,k,p} := \frac{1}{p} \left(\sum_{\nu=0}^{(p-1)/2} M_{\mu+2\nu k,k,p} + \sum_{\nu=0}^{(p-3)/2} M_{2k+1-\mu+2\nu k,k,p} \right), \quad \mu = 1, 2, \dots, k.$$

Here we used that the assumptions on f imply that the value of

$$\int_{(\mu-1)\pi/k}^{\mu\pi/k} |f(e^{iu})| du$$

remains the same when μ is replaced with $\mu + 2\nu k$ or $2k + 1 - \mu + 2\nu k$. The proof of Lemma 4.2 now follows from Lemma 4.3. \square

Lemma 4.3. *Let the numbers $A_{\mu,k,p}$, $\mu = 1, 2, \dots, k$, be defined by (4.1). There is an absolute constant $0 < b < 1/2$ such that for every odd prime p there is a positive integer k such that*

$$(4.2) \quad A_{\mu,k,p} \leq p^b, \quad \mu = 1, 2, \dots, k.$$

Proof of Lemma 4.3. It turns out that for large primes even $A_{\mu,1,p} \leq c \log p$ is true, while for smaller primes we choose larger values of k to establish (4.2). Observe that

$$|\Phi_p(e^{it})| = \left| \frac{\sin(pt/2)}{\sin(t/2)} \right|,$$

hence

$$|\Phi_p(e^{it})| \leq \frac{1}{\sin(t/2)} \leq \frac{1}{\frac{2}{\pi} \frac{t}{2}} = \frac{\pi}{t}, \quad t \in (0, \pi].$$

This implies

$$(4.3) \quad |\Phi_p(e^{it})| \leq \min \left\{ p, \frac{\pi}{t} \right\}, \quad t \in (0, \pi].$$

Observe also that

$$|\Phi_p(e^{it})| \leq \frac{1}{\sin(t/2)} \leq \frac{1}{\frac{\sin(\pi/4)}{\pi/4} \frac{t}{2}} \leq \frac{\pi}{\sqrt{2}t}, \quad t \in (0, \pi/2],$$

hence

$$(4.4) \quad |\Phi_p(e^{it})| \leq \min \left\{ p, \frac{\pi}{\sqrt{2}t} \right\}, \quad t \in (0, \pi/2].$$

If $\mu = k = 1$, then using (4.3) and (4.4) we easily obtain

$$\begin{aligned} A_{1,1,p} &:= \frac{1}{p} \left(\sum_{\nu=0}^{(p-1)/2} M_{1+2\nu,1,p} + \sum_{\nu=0}^{(p-1)/2} M_{2+2\nu,1,p} \right) = \frac{1}{p} \sum_{j=1}^p M_{j,1,p} \\ &\leq \frac{1}{p} \left(p + \sum_{j=1}^{(p-1)/2-1} \frac{p}{\sqrt{2}j} + \sum_{j=(p-1)/2}^{p-1} \frac{p}{j} \right) \\ &\leq 1 + \sum_{j=1}^{(p-1)/2-1} \frac{1}{\sqrt{2}j} + \sum_{j=(p-1)/2}^{p-1} \frac{1}{j} \\ &\leq 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \int_1^{(p-1)/2} \frac{dx}{x} + \frac{2}{p-1} + \int_{(p-1)/2}^{p-1} \frac{dx}{x} \\ &= 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} (\ln(p-1) - \ln 2) + \frac{2}{p-1} + \ln 2 \\ &\leq 1 + \frac{1}{\sqrt{2}} + \left(1 - \frac{1}{\sqrt{2}}\right) \ln 2 + \frac{2}{p-1} + \frac{1}{\sqrt{2}} \ln p \\ &\leq p^{0.48} \end{aligned}$$

for every prime $p \geq 23$. Here we used the fact that $\ln p/p^{0.48}$ is decreasing for $p \geq 23$. Hence the lemma holds for all primes $p \geq 23$ (we choose $k = 1$). Further, the estimates

$$A_{1,1,19} \leq 1 + \frac{1}{\sqrt{2}} \sum_{j=1}^8 \frac{1}{j} + \sum_{j=9}^{18} \frac{1}{j} \leq 3.7 < \sqrt{19},$$

$$A_{1,1,17} \leq 1 + \frac{1}{\sqrt{2}} \sum_{j=1}^7 \frac{1}{j} + \sum_{j=8}^{16} \frac{1}{j} \leq 3.63 < \sqrt{17},$$

$$A_{1,1,13} \leq 1 + \frac{1}{\sqrt{2}} \sum_{j=1}^5 \frac{1}{j} + \sum_{j=6}^{12} \frac{1}{j} \leq 3.42 < \sqrt{13},$$

and

$$A_{1,1,11} \leq 1 + \frac{1}{11 \sin(\pi/22)} + \frac{1}{\sqrt{2}} \sum_{j=2}^4 \frac{1}{j} + \sum_{j=5}^{10} \frac{1}{j} \leq 3.25 < \sqrt{11}$$

show that the lemma holds for all primes $11 \leq p \leq 19$ (we choose $k = 1$).

Now we study the case $p = 7$. We have

$$A_{1,4,7} \leq \frac{1}{7}(7 + 1 + 2.31 + 1.35 + 1.38 + 1.05 + 1.03) \leq 2.16 < \sqrt{7},$$

$$A_{2,4,7} \leq \frac{1}{7}(6.83 + 2.15 + 2.07 + 1.42 + 1.23 + 1.06 + 1.02) \leq 2.26 < \sqrt{7},$$

$$A_{3,4,7} \leq \frac{1}{7}(6.32 + 3.34 + 1.88 + 1.51 + 1.19 + 1.09 + 1.01) \leq 2.34 < \sqrt{7},$$

and

$$A_{4,4,7} \leq \frac{1}{7}(5.52 + 4.50 + 1.73 + 1.61 + 1.15 + 1.11 + 1.01) \leq 2.38 < \sqrt{7}.$$

In the above four estimates we used that

$$M_{1,4,7} = 7, \quad M_{2,4,7} = \frac{\sin(\pi/8)}{\sin(\pi/56)} \leq 6.83,$$

$$M_{3,4,7} = \frac{\sin(2\pi/8)}{\sin(2\pi/56)} \leq 6.32, \quad M_{4,4,7} = \frac{\sin(3\pi/8)}{\sin(3\pi/56)} \leq 5.52,$$

$$M_{5,4,7} = \frac{\sin(4\pi/8)}{\sin(4\pi/56)} \leq 4.50, \quad M_{6,4,7} = \frac{\sin(5\pi/8)}{\sin(5\pi/56)} \leq 3.34,$$

$$M_{7,4,7} = \frac{\sin(6\pi/8)}{\sin(6\pi/56)} \leq 2.15, \quad M_{8,4,7} = \frac{\sin(7\pi/8)}{\sin(7\pi/56)} \leq 1,$$

and

$$M_{j,4,7} = \frac{1}{\sin((j-1)\pi/56)}, \quad j = 9, 10, \dots, 28.$$

Hence the lemma is proved for $p = 7$ (we choose $k = 4$).

Now we study the case $p = 5$. We have

$$A_{1,4,5} \leq \frac{1}{5}(5 + 0.74 + 0.59 + 0.42 + 0.40) \leq 1.43 < \sqrt{5},$$

$$A_{2,4,5} \leq \frac{1}{5}(4.88 + 1.56 + 1 + 0.80 + 0.72) \leq 1.80 < \sqrt{5},$$

$$A_{3,4,5} \leq \frac{1}{5}(4.53 + 2.42 + 1.25 + 1.09 + 0.93) \leq 2.05 < \sqrt{5},$$

and

$$A_{4,4,5} \leq \frac{1}{7}(3.96 + 3.24 + 1.25 + 1.25 + 1) \leq 2.15 < \sqrt{5},$$

In the above four estimates we used that

$$M_{1,4,5} = 5, \quad M_{2,4,5} = \frac{\sin(\pi/8)}{\sin(\pi/40)} \leq 4.88,$$

$$M_{3,4,5} = \frac{\sin(2\pi/8)}{\sin(2\pi/40)} \leq 4.53, \quad M_{4,4,5} = \frac{\sin(3\pi/8)}{\sin(3\pi/40)} \leq 3.96,$$

$$M_{5,4,5} = \frac{\sin(4\pi/8)}{\sin(4\pi/40)} \leq 3.24, \quad M_{6,4,5} = \frac{\sin(5\pi/8)}{\sin(5\pi/40)} \leq 2.42,$$

$$M_{7,4,5} = \frac{\sin(6\pi/8)}{\sin(6\pi/40)} \leq 1.56, \quad M_{8,4,5} = \frac{\sin(7\pi/8)}{\sin(7\pi/40)} \leq 0.74,$$

$$M_{9,4,5} = -\frac{\sin(9\pi/8)}{\sin(9\pi/40)} \leq 0.59, \quad M_{10,4,5} = \frac{\sin(10\pi/8)}{\sin(10\pi/40)} = 1,$$

$$M_{11,4,5} \leq 1.25, \quad M_{12,4,5} \leq 1.25,$$

$$M_{13,4,5} \leq 1.25, \quad M_{14,4,5} = -\frac{\sin(13\pi/8)}{\sin(13\pi/40)} \leq 1.09,$$

$$M_{15,4,5} = -\frac{\sin(14\pi/8)}{\sin(14\pi/40)} \leq 0.80, \quad M_{16,4,5} = \frac{\sin(15\pi/8)}{\sin(15\pi/40)} \leq 0.42,$$

$$M_{17,4,5} = \frac{\sin(17\pi/8)}{\sin(17\pi/40)} \leq 0.40, \quad M_{18,4,5} = \frac{\sin(18\pi/8)}{\sin(18\pi/40)} \leq 0.72,$$

and

$$M_{19,4,5} = \frac{\sin(19\pi/8)}{\sin(19\pi/40)} \leq 0.93, \quad M_{20,4,5} = \frac{\sin(20\pi/8)}{\sin(20\pi/40)} \leq 1.$$

Hence the lemma holds for $p = 5$ (we choose $k = 4$).

Now we study the case $p = 3$. We have

$$M_{1,k,3} = 3,$$

$$M_{j,k,3} = 1 + 2 \cos \frac{(j-1)\pi}{3k}, \quad j = 2, 3, \dots, 2k,$$

and

$$M_{j,k,3} = - \left(1 + 2 \cos \frac{j\pi}{3k} \right), \quad j = 2k+1, 2k+2, \dots, 3k,$$

hence with

$$y := \frac{(\mu-1)\pi}{3k} \in [0, \pi/3], \quad \mu = 1, 2, \dots, k,$$

we have

$$\begin{aligned} 3A_{\mu,k,3} &= M_{\mu,k,3} + M_{2k+1-\mu,k,3} + M_{2k+\mu,k,3} \\ &= \left(1 + 2 \cos \frac{(\mu-1)\pi}{3k} \right) + \left(1 + 2 \cos \frac{(2k-\mu)\pi}{3k} \right) - \left(1 + 2 \cos \frac{(2k+\mu)\pi}{3k} \right) \\ &= 1 + 2 \cos y + 2 \cos \left(\frac{2\pi}{3} - \frac{2\pi}{3k} - y \right) - 2 \cos \left(\frac{2\pi}{3} + \frac{2\pi}{3k} + y \right) \\ &= 1 + 2(\cos y + \sqrt{3} \sin y) + c(k) = 1 + 4 \sin \left(y + \frac{\pi}{6} \right) + c(k) \leq 5 + c(k) \end{aligned}$$

with $c(k) \rightarrow 0$ as $k \rightarrow \infty$. Therefore

$$A_{\mu,k,3} \leq \frac{5}{3} + \frac{c(k)}{3} < \sqrt{3}, \quad k = 1, 2, \dots, \mu,$$

for all sufficiently large k . Thus the lemma is proved for $p = 3$ (we choose a sufficiently large k). \square

Proof of Theorem 3.2. The polynomials $P_{2\nu}$ can be factorized as it is given in Theorem 2.3. The theorem follows by induction on the number of factors in $P_{2\nu}$. We use Lemma 4.2 in the inductive step. \square

Proof of Theorem 3.1. Let f be a continuous function on ∂D and let

$$I_q(f) := M_q(f)^q = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^q dt.$$

Then $h(q) := \log(I_q(f)) = q \log(M_q(f))$ is a convex function of q on $(0, \infty)$. This is a simple consequence of Hölder's inequality. For the sake of completeness we present the short proof of it. We need to see that if $q < r < p$, then

$$I_r(f) \leq I_p(f)^{\frac{r-q}{p-q}} I_q(f)^{\frac{p-r}{p-q}},$$

that is,

$$(4.5) \quad \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^r dt \right)^{p-q} \leq \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^p dt \right)^{r-q} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^q dt \right)^{q-r}.$$

To see this let

$$\alpha := \frac{p-q}{r-q}, \quad \beta := \frac{p-q}{p-r}, \quad \gamma := \frac{p}{\alpha}, \quad \delta := \frac{q}{\beta},$$

hence $1/\alpha + 1/\beta = 1$ and $\gamma + \delta = r$. Let

$$F(t) := |f(e^{it})|^\gamma = |f(e^{it})|^{\frac{p(r-q)}{p-q}},$$

and

$$G(t) := |f(e^{it})|^\delta = |f(e^{it})|^{\frac{q(p-r)}{p-q}}.$$

Then by Hölder's inequality we conclude

$$\int_0^{2\pi} F(t)G(t) dt \leq \left(\int_0^{2\pi} F(t)^\alpha dt \right)^{1/\alpha} \left(\int_0^{2\pi} G(t)^\beta dt \right)^{1/\beta},$$

and (4.5) follows. Using the convexity of $\log(I_q(P_{2\nu}))$ on $(0, \infty)$, for $q > 2$ we have

$$I_2(P_{2\nu}) \leq (I_1(P_{2\nu}))^{\frac{q-2}{q-1}} (I_q(P_{2\nu}))^{\frac{1}{q-1}},$$

and from Theorem 3.2 we obtain

$$2\nu + 1 \leq ((2\nu + 1)^b)^{\frac{q-2}{q-1}} (I_q(P_{2\nu}))^{\frac{1}{q-1}},$$

that is,

$$(2\nu + 1)^{(q-1)-(q-2)b} \leq I_q(P_{2\nu}) = (M_q(P_{2\nu}))^q$$

with an absolute constant $0 < b < 1/2$. Hence with $a = a(q) := (q-1-(q-2)b)/q$ we have

$$(2\nu + 1)^a = (2\nu + 1)^{(q-1-(q-2)b)/q} \leq M_q(P_{2\nu}).$$

Here $a = a(q) := (q-1-(q-2)b)/q > 1/2$, since $(1-2b)(q/2-1) > 0$. \square

To prove Theorem 3.3 we need the lemma below. As before, associated with a positive integer p and a function f defined on the unit circle ∂D let

$$g(t) := f(e^{it}), \quad t \in \mathbb{R},$$

$$f_p(z) := f(z^p)\Phi_p(z), \quad \Phi_p(z) := \sum_{j=0}^{p-1} z^j, \quad g_p(t) := f_p(e^{it}),$$

that is,

$$g_p(t) := f(e^{ipt})h_p(t), \quad h_p(t) := \Phi_p(e^{it}).$$

For $q > 0$ we define

$$I_q(f) := (M_q(f))^q := \int_{\partial D} |f(z)|^q |dz| = \int_0^{2\pi} |f(e^{it})|^q dt,$$

and

$$I_q(g) := (M_q(g))^q := \int_0^{2\pi} |g(t)|^q dt,$$

so $M_q(g) = M_q(f)$.

Lemma 4.4. *Let $0 < q < 2$. If f is a continuous function on ∂D , then*

$$M_q(g_p) \leq p^\alpha M_q(g)$$

for every odd prime p with some $0 < \alpha = \alpha(q) < 1/2$ depending only on q .

Note that Lemma 4.2 is the special case $q = 1$ in Lemma 4.4 below. However, in the case $q = 1$ the method of the proof of Lemma 4.2 offers a reasonably good explicit value of the exponent $0 < \alpha < 1/2$ by brute force.

Proof of Lemma 4.4. We have

$$M_q(g_p) \leq A_{p,q} M_q(g),$$

where

$$A_{p,q} := \left(\frac{1}{p} \max_{t \in [0, 2\pi)} \sum_{j=0}^{p-1} |h_p(t + 2\pi j/p)|^q \right)^{1/q}.$$

Applying the inequality between q -th mean and the quadratic mean ($0 < q < 2$), we have

$$\sum_{j=0}^{p-1} |h_p(t + 2\pi j/p)|^q \leq p^{1-q/2} \left(\sum_{j=0}^{p-1} |h_p(t + 2\pi j/p)|^2 \right)^{q/2}.$$

Using orthogonality we obtain

$$\sum_{j=0}^{p-1} |h_p(t + 2\pi j/p)|^2 \leq \sum_{j=0}^{p-1} \left| \sum_{k=0}^{p-1} \exp(ik(t + 2\pi j/p)) \right|^2 = p^2.$$

Combining the previous two inequalities we deduce

$$\sum_{j=0}^{p-1} |h_p(t + 2\pi j/p)|^q \leq p^{1-q/2+2q/2} = p^{1+q/2},$$

and hence $A_{p,q} \leq \sqrt{p}$. Equality could hold if and only if there is a $t \in [0, 2\pi)$ such that

$$|h_p(t + 2\pi j/p)| = \sqrt{p}, \quad j = 0, 1, \dots, p-1.$$

However this is impossible for any prime $p \geq 3$ in the light of Lemma 4.1. So $A_{p,q}$ is strictly less than \sqrt{p} . On the other hand, by using the obvious estimate

$$|\Phi_p(z)| \leq \max \left\{ p, \frac{2}{|z-1|} \right\}, \quad |z| = 1,$$

it is an elementary calculus to show that there are constants $c = c(q) > 0$ and $\alpha = \alpha(q) \in (0, 1/2)$ depending only on $q \in (0, 2)$ such that

$$A_{p,q} \leq c \max\{\log p, p^{1-1/q}\} < p^\alpha$$

for all $0 < q < 2$ and for all sufficiently large primes $p \geq 3$. Combining this with $A_{p,q} < \sqrt{p}$, we get the conclusion of the lemma. \square

Proof of Theorem 3.3. The polynomials $P_{2\nu}$ can be factorized as it is given in Theorem 2.3. The theorem follows by induction on the number of factors in $P_{2\nu}$. We use Lemma 4.4 in the inductive step. \square

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