ON THE DERIVATIVES OF UNIMODULAR POLYNOMIALS

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Abstract. Let D be the open unit disk of the complex plane; its boundary, the unit circle of the complex plane, is denoted by ∂D. Let $\mathcal{P}_n^c$ denote the set of all algebraic polynomials of degree at most n with complex coefficients. For $\lambda \geq 0$, let

$$\mathcal{K}_n^\lambda \stackrel{\text{def}}{=} \left\{ P_n : P_n(z) = \sum_{k=0}^n a_k k^\lambda z^k \,, \;\; a_k \in \mathbb{C}\,, \;\; |a_k| = 1 \right\} \subset \mathcal{P}_n^c \,.$$

The class $\mathcal{K}_n^0$ is often called the collection of all (complex) unimodular polynomials of degree n. Given a sequence (ε_n) of positive numbers tending to 0, we say that a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n^\lambda$ is $\{\lambda, (\varepsilon_n)\}$-ultraflat if

$$(1-\varepsilon_n)\frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}} \leq |P_n(z)| \leq (1+\varepsilon_n)\frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}} \,, \qquad z \in \partial D \,, \quad n \in \mathbb{N}_0 \,.$$

Although we do not know, in general, whether or not $\{\lambda, (\varepsilon_n)\}$-ultraflat sequences of polynomials $P_n \in \mathcal{K}_n^\lambda$ exist for each fixed $\lambda > 0$, we make an effort to prove various interesting properties of them. These allow us to conclude that there are no sequences (P_n) of either conjugate, or plain, or skew reciprocal unimodular polynomials $P_n \in \mathcal{K}_n^0$ such that (Q_n) with $Q_n(z) \stackrel{\text{def}}{=} zP_n'(z) + 1$ is a $\{1, (\varepsilon_n)\}$-ultraflat sequence of polynomials.

1. Introduction

Let $\mathbb{N}_0$ denote the nonnegative integers. Let D be the open unit disk of the complex plane; its boundary, the unit circle of the complex plane, is denoted by ∂D. For $n \in \mathbb{N}_0$, let $\mathcal{P}_n^c$ denote the set of all algebraic polynomials of degree at most n with complex coefficients. For $\lambda \geq 0$, let

$$\mathcal{K}_n^\lambda \stackrel{\text{def}}{=} \left\{ P_n : P_n(z) = \sum_{k=0}^n a_k k^\lambda z^k \,, \;\; a_k \in \mathbb{C}\,, \;\; |a_k| = 1 \right\} \subset \mathcal{P}_n^c \,.$$

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The class $\mathcal{K}_n^0$ is often called the collection of all (complex) unimodular polynomials of degree n.

For $\lambda \geq 0$, let

$$\mathcal{L}_n^\lambda \stackrel{\text{def}}{=} \left\{ P_n : P_n(z) = \sum_{k=0}^{n} a_k k^\lambda z^k, \;\; a_k \in \{-1, 1\} \right\} \subset \mathcal{P}_n^c \,.$$

Elements of $\mathcal{L}_n^0$ are often called real unimodular polynomials or Littlewood polynomials of degree n.

Parseval's formula yields

$$\left(\int_0^{2\pi} |P_n(e^{it})|^2 \, dt \right)^{1/2} = (2\pi + \delta_n)^{1/2} \frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}}$$

for all $P_n \in \mathcal{K}_n^\lambda$, where the sequence (δ_n) converges to 0 as $n \to \infty$, in fact, $\delta_n = \mathcal{O}(1/n)$. Therefore

$$\min_{z \in \partial D} |P_n(z)| \leq \left(1 + \frac{\delta_n}{2\pi} \right)^{1/2} \frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}} \leq \max_{z \in \partial D} |P_n(z)| \,.$$

Given $\lambda \geq 0$ and $n \in \mathbb{N}_0$, we say that $P \in \mathcal{P}_n^c$ is $\{\lambda, n, \varepsilon\}$-flat if $P \in \mathcal{K}_n^\lambda$ and

$$(1-\varepsilon) \frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}} \leq |P(z)| \leq (1+\varepsilon) \frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}} \,, \qquad z \in \partial D \,.$$

Generalizing flatness, we say that a sequence (P_n) of polynomials is $\{\lambda, (\varepsilon_n)\}$-ultraflat if, for each $n \in \mathbb{N}_0$, we have $P_n \in \mathcal{K}_n^\lambda$ and

$$(1-\varepsilon_n) \frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}} \leq |P_n(z)| \leq (1+\varepsilon_n) \frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}} \,, \qquad z \in \partial D \,, \quad n \in \mathbb{N}_0 \,.$$

Here and throughout this paper, when we talk about a sequence of $\{\lambda, (\varepsilon_n)\}$-ultraflat polynomials, we always assume that $\lambda \geq 0$ and that the sequence (ε_n) of positive numbers converges to 0 as $n \to \infty$.

Similarly, given an increasing sequence (j_n) of nonnegative integers, we say that a sequence (P_{j_n}) of polynomials is $\{\lambda, (\varepsilon_{j_n})\}$-ultraflat if, for each j_n, we have $P_{j_n} \in \mathcal{K}_{j_n}^\lambda$ and

$$(1-\varepsilon_{j_n}) \frac{{j_n}^{\lambda+1/2}}{\sqrt{2\lambda+1}} \leq |P_{j_n}(z)| \leq (1+\varepsilon_{j_n}) \frac{{j_n}^{\lambda+1/2}}{\sqrt{2\lambda+1}} \,, \qquad z \in \partial D \,, \quad n \in \mathbb{N}_0 \,.$$

Here and throughout this paper, when we talk about a sequence of $\{\lambda, (\varepsilon_{j_n})\}$-ultraflat polynomials, we always assume that $\lambda \geq 0$, (j_n) is an increasing sequence of nonnegative integers, and that the sequence (ε_{j_n}) of positive numbers converges to 0 as $n \to \infty$.

A motivation to study polynomials $P_n \in \mathcal{K}_n^\lambda$ in general is the fact that $P_n \in \mathcal{K}_n^\lambda$ implies that for Q_n defined by $Q_n(z) = zP_n'(z) \pm 1$ we have $Q_n \in \mathcal{K}_n^{\lambda+1}$. In this paper we do not attempt to study more general classes related to unimodular polynomials although we have no doubt that our methods employed in this paper leave some room for further generalizations.

In 1957, the existence of a $\{0, (\varepsilon_n)\}$-ultraflat sequence (P_n) seemed very unlikely in view of a conjecture of P. Erdős, see Problem 22 in [Er], asserting that, for all $P_n \in \mathcal{K}_n^0$ with $n \in \mathbb{N}$,

$$\max_{z \in \partial D} |P_n(z)| \geq (1+\varepsilon) n^{1/2}\,,$$

where $\varepsilon > 0$ is an absolute constant. Yet, combining some probabilistic lemmas from Körner's paper [Kö] with some constructive methods using Gauss polynomials and other tools, that were completely unrelated to the deterministic part of Körner's paper, Kahane [Ka] proved that there exists a sequence (P_n) that is $\{0, (\varepsilon_n)\}$-ultraflat, where $\varepsilon_n = \mathcal{O}\left(n^{-1/17}\sqrt{\log n}\right)$, see p. 240 in [Ka]. Thus the Erdős conjecture was disproved for the classes $\mathcal{K}_n^0$. For the more restricted class $\mathcal{L}_n^0$ the analogous Erdős conjecture is unsettled to this date (Erdős offered \$50 for a solution). It is a common belief that the analogous Erdős conjecture for $\mathcal{L}_n^0$ is true, and consequently there is no ultraflat sequence of polynomials $P_n \in \mathcal{L}_n^0$. For an account of some of the work done till the mid 1960's, see Littlewood's book [L] and [QS]. For a more recent survey on polynomials with Littlewood-type coefficient constraints, see [E4]. The structure of $\{0, (\varepsilon_n)\}$-ultraflat sequences (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n^0$ produced by Kahane in [Ka] is examined by Queffélec and Saffari [QS], where several interesting properties of Kahane's ultraflat sequences of unimodular polynomials have been observed and proved. The structure of ultraflat sequences of unimodular polynomials in general is studied in [E1], [E2], [E3], and [E5], where several conjectures of Saffari raised in [S] and [QS] are proved for all $\{0, (\varepsilon_n)\}$-ultraflat sequences (P_n) of polynomials in general, not necessarily those produced by Kahane in [Ka]. In this paper, following the techniques used in [E1], [E2], [E3], and [E5] earlier, we prove some extensions of these conjectures.

A recent paper [BB] by Bombieri and Bourgain is devoted to the construction of ultraflat sequences of unimodular polynomials. In particular, one obtains a much improved estimate for the error term. A major part of this paper deals also with the long-standing problem of the effective construction of ultraflat sequences of unimodular polynomials.

Whether or not $\{\lambda, (\varepsilon_{j_n})\}$-ultraflat sequences of (P_{j_n}) with $P_{j_n} \in \mathcal{K}_{j_n}^\lambda$ exist for every fixed $\lambda > 0$, seems a natural question. For $\lambda > 0$ the answer to this question is not known, not even for $\lambda = 1$. It is possible that modifications of the techniques used by Kahane [Ka], Queffélec and Saffari [QS], and Bombieri and Bourgain [BB], give the existence of such $\{\lambda, (\varepsilon_{j_n})\}$-ultraflat sequences for every fixed $\lambda > 0$ as well. This looks an interesting but perhaps rather difficult and long project that we do not try to attempt in this paper.

In what follows, given a polynomial

$$P_n(z) = \sum_{j=0}^{n} a_j z^j \,, \qquad a_j \in \mathbb{C}\,,$$

of exact degree n, we define the *conjugate n-reverse polynomial*, or, simply the *reverse polynomial* P_n^* by

$$P_n^*(z) \stackrel{\text{def}}{=} z^n \overline{P_n(1/\overline{z})} = \sum_{j=0}^{n} \overline{a}_{n-j} z^j \tag{1.1}$$

which is of degree at most n. This is a standard definition albeit no standard terminology seems to exist for it.

Associated with a polynomial P_n of the above form we also define

$$\overline{P}_n(z) \stackrel{\text{def}}{=} \sum_{j=0}^{n} \overline{a}_j z^j \,.$$

A polynomial P_n of the above form is called conjugate reciprocal if $a_j = \overline{a}_{n-j}$ for each $j = 0, 1, \ldots, n$, that is, $P_n^* = P_n$.

A polynomial P_n of the above form is called plain reciprocal if $a_j = a_{n-j}$ for each $j = 0, 1, \ldots, n$, that is, $P_n^* = \overline{P}_n$.

A polynomial P_n of the above form is called skew reciprocal if $a_j = (-1)^j \overline{a}_{n-j}$ for each $j = 0, 1, \ldots, n$, that is, $P_n^*(z) = P_n(-z)$ for each $z \in \mathbb{C}$.

The Lebesgue measure of a measurable set $A \subset \mathbb{R}$ or $\{\cdot\}$ will be denoted by $m(A)$ or $m\{\cdot\}$, respectively.

Suppose (P_n) is a $\{\lambda, (\varepsilon_n)\}$-ultraflat sequence of polynomials. We write

$$P_n(e^{it}) = R_n(t) e^{i\alpha_n(t)} \,, \qquad R_n(t) = |P_n(e^{it})| \,. \tag{1.2}$$

It is simple to show that α_n can be chosen to be in $C^\infty(\mathbb{R})$. This is going to be our understanding throughout the paper. It is easy to find a formula for $\alpha_n(t)$ in terms of P_n. This was done by Saffari and it asserts that

$$\alpha_n'(t) = \operatorname{Re}\left(\frac{e^{it} P_n'(e^{it})}{P_n(e^{it})}\right) \,, \tag{1.3}$$

see formulas (7.1) & (7.2) on p. 564 and (8.2) on p. 565 in [S]. The angular function α_n^* and modulus function R_n^* associated with the polynomial P_n^* are defined by

$$P_n^*(e^{it}) = R_n^*(t) e^{i\alpha_n^*(t)} \,, \qquad R_n^*(t) = |P_n^*(e^{it})| \,.$$

Similarly to α_n, the angular function α_n^* can also be chosen to be in $C^\infty(\mathbb{R})$ on $\mathbb{R}$.

2. The Phase Problem: Results and Conjectures of Saffari

For the case $\lambda = 0$, Saffari conjectured the following two theorems, Theorems 2.1 and 2.2, see p. 560 in [S]; we borrowed his terminology. For the case $\lambda = 0$ Theorems 2.1 and 2.2 were proved in [E2].

Theorem 2.1 (Uniform Distribution Theorem for the Angular Speed)**.** *Let* (P_n) *be a* $\{\lambda, (\varepsilon_n)\}$*-ultraflat sequence of polynomials. Then*

$$m\,\{t \in [0, 2\pi] : 0 \leq \alpha_n'(t) \leq nx\} = 2\pi x^{2\lambda+1} + o_n(x)\,, \qquad x \in [0, 1], \tag{2.1}$$

where $\lim_{n\to\infty} o_n(x) = 0$ *uniformly in* $[0, 1]$. *In addition,*

$$m\left\{t \in [0, 2\pi] : |P_n'(e^{it})| \leq \frac{n^{\lambda+3/2}x}{(2\lambda+1)^{1/2}}\right\} = 2\pi x^{2\lambda+1} + o_n(x)\,, \qquad x \in [0, 1], \tag{2.2}$$

where $\lim_{n\to\infty} o_n(x) = 0$ *uniformly in* $[0, 1]$.

When $\lambda = 0$, Saffari's basis of conjecturing Theorem 2.1 was that for the special $\{0, (\varepsilon_n)\}$-ultraflat sequence of unimodular polynomials produced by Kahane [Ka], (2.1) is indeed true. In Section 4 we prove Theorem 2.1 for every $\lambda \geq 0$.

In the general case, by integration, (2.1) can be reformulated equivalently in terms of the moments of the angular speed $\alpha_n'(t)$. We will present the proof of this equivalence in Section 4 and will verify Theorem 2.1 by proving the following result first.

Theorem 2.2 (Reformulation of the Uniform Distribution Theorem)**.** *Let* (P_n) *be a* $\{\lambda, (\varepsilon_n)\}$*-ultraflat sequence of polynomials. Then, for every* $q > 0$, *we have*

$$\frac{1}{2\pi}\int_0^{2\pi} |\alpha_n'(t)|^q\, dt = \frac{(2\lambda+1)n^q}{q+2\lambda+1} + o_{n,q}n^q \tag{2.3}$$

with suitable constants $o_{n,q}$ *converging to* 0 *as* $n \to \infty$.

An immediate consequence of (2.3) is the remarkable fact that, for large values of $n \in \mathbb{N}$, the $L_q(\partial D)$ Bernstein-M. Riesz-Zygmund factors

$$\frac{\int_0^{2\pi} |P_n'(e^{it})|^q\, dt}{\int_0^{2\pi} |P_n(e^{it})|^q\, dt}$$

of the elements of ultraflat sequences of polynomials (P_n) are essentially independent of the polynomials. More precisely (2.3) implies the following result.

Theorem 2.3 (Bernstein-M. Riesz-Zygmund Factors)**.** *Let (P_n) be a $\{\lambda, (\varepsilon_n)\}$-ultraflat sequence of polynomials. Then, for every fixed $q > 0$, we have*

$$\frac{\int_0^{2\pi} |P_n'(e^{it})|^q \, dt}{\int_0^{2\pi} |P_n(e^{it})|^q \, dt} = \frac{(2\lambda+1)n^q}{q+2\lambda+1} + o_{n,q} n^q \, ,$$

and, as a limiting case,

$$\frac{\max_{0 \le t \le 2\pi} |P_n'(e^{it})|}{\max_{0 \le t \le 2\pi} |P_n(e^{it})|} = n + o_n n \, .$$

with suitable constants $o_{n,q}$ and o_n that converge to 0 as $n \to \infty$.

In Section 3 we will show the following result which turns out to be stronger than Theorem 2.2.

Theorem 2.4 (Negligibility Theorem for Higher Derivatives)**.** *Let (P_n) be a $\{\lambda, (\varepsilon_n)\}$-ultraflat sequence of polynomials. Then, for every fixed integer $r \ge 2$, we have*

$$\max_{0 \le t \le 2\pi} |\alpha_n^{(r)}(t)| \le o_{n,r} n^r$$

with suitable constants $o_{n,r}$ that converge to 0 as $n \to \infty$.

We will show in Section 4 how Theorem 2.1 follows from Theorem 2.4.

We also give an extension of Saffari's Uniform Distribution Conjecture to higher derivatives; the proof will be in Section 4 as well.

Theorem 2.5. *Let (P_n) be a $\{\lambda, (\varepsilon_n)\}$-ultraflat sequence of polynomials. Then*

$$m\left\{ t \in [0, 2\pi] : |P_n^{(r)}(e^{it})| \le \frac{n^{r+\lambda+1/2} x^r}{(2\lambda+1)^{1/2}} \right\} = 2\pi x^{2\lambda+1} + o_{r,n}(x) \, , \qquad x \in [0,1],$$

where, for every fixed $r \in \mathbb{N}$, we have $\lim_{n \to \infty} o_{r,n}(x) = 0$ uniformly in $[0,1]$.

As a consequence of Theorems 2.1 and 2.4 we obtain the following theorem.

Theorem 2.6. *Let (P_n') be a $\{\lambda+1, (\varepsilon_n)\}$-ultraflat sequence of polynomials. Then*

$$\lim_{n \to \infty} n^{-(\lambda+1/2)} \max_{t \in \mathbb{R}} |P_n(e^{it})| = \infty \, .$$

Theorem 2.6 together with Lemma 4.5 implies

Theorem 2.7. *Let (P_{j_n}) be a sequence of polynomials $P_{j_n} \in \mathcal{K}^0_{j_n}$ so that*

$$\max_{z\in\partial D} |P^{*}_{j_n}{}'(z)| = \max_{z\in\partial D} |P'_{j_n}(z)|\,, \qquad n \in \mathbb{N}_0\,.$$

Then (Q_{j_n}) with $Q_{j_n}(z) \stackrel{\text{def}}{=} zP'_{j_n}(z)+1$ is not a $\{1,(\varepsilon_{j_n})\}$-ultraflat sequence of polynomials.

Corollary 2.8. *Let (j_n) be an increasing sequence of nonnegative integers. There are no sequences (P_{j_n}) of either conjugate, or plain, or skew reciprocal unimodular polynomials $P_{j_n} \in \mathcal{K}^0_{j_n}$ such that (Q_{j_n}) with $Q_{j_n}(z) \stackrel{\text{def}}{=} zP'_{j_n}(z)+1$ is a $\{1,(\varepsilon_{j_n})\}$-ultraflat sequence of polynomials.*

Remark 2.9 Theorems 2.1–2.6 remain true if n is replaced by j_n, where (j_n) is an increasing sequence of nonnegative integers, and the classes

$$\mathcal{K}^\lambda_n \stackrel{\text{def}}{=} \left\{ P_n : P_n(z) = \sum_{k=0}^{n} a_k k^\lambda z^k,\;\; a_k \in \mathbb{C}\,,\;\; |a_k| = 1 \right\}$$

are replaced by

$$\mathcal{K}^\lambda_{j_n}(\Gamma_{j_n}) \stackrel{\text{def}}{=} \left\{ P_{j_n} : P_{j_n}(z) = \sum_{k=0}^{j_n} a_k k^\lambda z^k,\;\; a_k \in \mathbb{C}\,,\;\; |a_k| = \gamma_{k,j_n} \right\}\,,$$

where

$$\Gamma_{j_n} := \{\gamma_{0,j_n}, \gamma_{1,j_n}, \ldots, \gamma_{j_n,j_n}\} \subset \mathbb{R}$$

and for every $\varepsilon > 0$ there is an N such that

$$|\gamma_{k,j_n} - 1| < \varepsilon\,, \quad N \le k \le j_n,\;\; , n \in \mathbb{N}_0\,.$$

The above remark is needed to prove Theorem 2.7 as a consequence of Theorem 2.6. The reader will easily see that the proofs of Theorems 2.1–2.6 in Section 4 can be modified in a straightforward fashion (by simply replacing n by j_n) to conclude Remark 2.9. We will use the terminology that P_{j_n} are asymptotically in $\mathcal{K}^\lambda_{j_n}$ if $P_{j_n} \in \mathcal{K}^\lambda_{j_n}(\Gamma_{j_n})$ with some Γ_{j_n}, $n \in \mathbb{N}_0$, satisfying the assumptions given in Remark 2.9.

3. Proof of Theorem 2.4

To prove Theorem 2.4, we need a few lemmas. The first one is a standard polynomial inequality sometimes attributed to Bernstein. Its proof is a simple exercise in complex analysis consisting of a straightforward application of the Maximum Principle; it may be found in a number of books, see e.g., [BE, p. 390]. We will use the notation

$$D(z_0,R) \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z - z_0| < R\}\,, \qquad \partial D(z_0,R) \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z - z_0| = R\}\,,$$

and $\|f\|_A \stackrel{\text{def}}{=} \sup_{z\in A} |f(z)|$ for a complex-valued function f defined on a set A. As before, let $D \stackrel{\text{def}}{=} D(0,1)$ and $\partial D \stackrel{\text{def}}{=} \partial D(0,1)$.

Lemma 3.1. *We have*

$$|P(z)| \le |z|^n \|P\|_{\partial D}, \qquad |z| > 1\,,$$

for every polynomial P of degree at most n with complex coefficients. In addition, we have

$$|T(t)| \le e^{n|\mathrm{Im}(t)|}\|T\|_{\mathbb{R}}, \qquad t \in \mathbb{C},$$

for every trigonometric polynomial T of the form

$$T(t) = \sum_{k=-n}^{n} a_k e^{ikt}\,, \qquad a_k \in \mathbb{C}\,.$$

Note that the second inequality of Lemma 3.1 follows from the first one applied to a polynomial P of degree at most $2n$ with complex coefficients defined by $P(e^{it}) \stackrel{\text{def}}{=} e^{int}T(t)$.[1]

The main tool to prove Theorem 2.4 is Jensen's Formula; for a proof, see, for example, E.10 c] of Section 4.2 in[BE].

Lemma 3.2 (Jensen's Formula). *Suppose h is a nonnegative integer and*

$$f(z) = \sum_{k=h}^{\infty} c_k z^k\,, \qquad c_h \neq 0\,,$$

is analytic in a disk $D(0,R_*)$ with some $R_* > R$. Suppose that the zeros of f in $D(0,R)\backslash\{0\}$ are $a_1, a_2, \dots, a_m$, where each zero is listed as many times as its multiplicity. Then

$$\log|c_h| + h\log R + \sum_{k=1}^{m} \log\frac{R}{|a_k|} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta\,.$$

Our next two lemmas are straightforward extensions of the corresponding lemmas used in [E2] and their proofs are quite similar to them.

Lemma 3.3. *Suppose (ε_n) is a sequence of numbers from $(0,1/3)$ tending to 0 as $n \to \infty$. Let (P_n) be a $\{\lambda,(\varepsilon_n)\}$-ultraflat sequence of polynomials. Then each P_n has no zeros in the open annulus*

$$\left\{z \in \mathbb{C} : 1 - \frac{1}{2n\delta_n} < |z| < 1 + \frac{1}{2n\delta_n}\right\}\,,$$

where the positive numbers

$$\delta_n \stackrel{\text{def}}{=} \max\left\{\frac{2}{-\log(3\,\varepsilon_n)}\,, \frac{1}{n}\right\}$$

[1]The second inequality of Lemma 3.1 is stated as Lemma 3.1 in [E2] with a typo as $\mathrm{Im}(t)$ in the exponent should be replaced with $|\mathrm{Im}(t)|$.

tend to 0 *as* $n \to \infty$.

Proof of Lemma 3.3. Let (P_n) be a $\{\lambda, (\varepsilon_n)\}$-ultraflat sequence of polynomials, that is, $P_n \in \mathcal{K}_n^\lambda$ satisfies

$$(1-\varepsilon_n)\frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}} < |P_n(z)| < (1+\varepsilon_n)\frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}}, \qquad z \in \partial D.$$

Then

$$(1-\varepsilon_n)^2\frac{n^{2\lambda+1}}{2\lambda+1} < z^{-n}P_n(z)P_n^*(z) = |P_n(z)|^2 < (1+\varepsilon_n)^2\frac{n^{2\lambda+1}}{2\lambda+1}, \qquad z \in \partial D.$$

We define

$$Q_n(z) \stackrel{\text{def}}{=} P_n(z)P_n^*(z) - \frac{n^{2\lambda+1}}{2\lambda+1}\, z^n\,. \tag{3.2}$$

Then Q_n is a polynomial of degree $2n$ and

$$-3\,\varepsilon_n\,\frac{n^{2\lambda+1}}{2\lambda+1} < z^{-n}Q_n(z) = |P_n(z)|^2 - \frac{n^{2\lambda+1}}{2\lambda+1} < 3\,\varepsilon_n\,\frac{n^{2\lambda+1}}{2\lambda+1}, \qquad z \in \partial D.$$

From this we conclude that

$$|Q_n(z)| < 3\,\varepsilon_n\,\frac{n^{2\lambda+1}}{2\lambda+1} \qquad z \in \partial D. \tag{3.3}$$

Using Lemma 3.1 and (3.3), we obtain that

$$|Q_n(z)| \le |z|^{2n}3\varepsilon_n\,\frac{n^{2\lambda+1}}{2\lambda+1} < \frac{n^{2\lambda+1}}{2\lambda+1} \tag{3.4}$$

for every $z \in \mathbb{C}$ that satisfies

$$1 \le |z| < 1 + \frac{1}{n\delta_n}\,,$$

where δ_n is defined in the lemma. Suppose that P_n has a zero in the annulus

$$\left\{z \in \mathbb{C} : 1 - \frac{1}{2n\delta_n} < |z| < 1 + \frac{1}{2n\delta_n}\right\}\,.$$

Then $P_nP_n^*$ has a zero z_0 in the annulus

$$\left\{z \in \mathbb{C} : 1 \le |z| < 1 + \frac{1}{n\delta_n}\right\}\,.$$

Hence, by (3.2), we have

$$|Q_n(z_0)| = \left|P_n(z_0)P_n^*(z_0) - \frac{n^{2\lambda+1}}{2\lambda+1}\,z_0^n\right| = \frac{n^{2\lambda+1}}{2\lambda+1}\,|z_0|^n \ge \frac{n^{2\lambda+1}}{2\lambda+1}\,,$$

which is impossible by (3.4). $\square$

Lemma 3.4. *Let $\varepsilon \in (0,1/3)$, $n \in \mathbb{N}$, $1/n \le R$, and let $z_0 \in \partial D$. If P is a $\{\lambda, n, \varepsilon\}$-flat polynomial, then P has at most $5nR$ zeros in the disk $D(z_0, R)$.*

Proof. We use Jensen's formula on the disk $D(z_0, 2R)$. Note that since P is $\{\lambda, n, \varepsilon\}$-flat, we have

$$\begin{aligned}\log|P(z_0)| &\ge \log(1-\varepsilon) + (\lambda + 1/2)\log n - \frac{1}{2}\log(2\lambda+1) \\ &\ge -\frac{1}{2} + (\lambda+1/2)\log n - \frac{1}{2}\log(2\lambda+1)\,,\end{aligned}$$

and then Lemma 3.1 yields

$$|P(z)| \le (1+\varepsilon)\frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}}(1+2R)^n\,, \qquad z \in \partial D(z_0, 2R)\,,$$

that is,

$$\log|P(z)| \le \frac{1}{3} + (\lambda+1/2)\log n - \frac{1}{2}\log(2\lambda+1) + n(2R)\,, \qquad z \in \partial D(z_0, 2R)\,.$$

Now, if m denotes the number of zeros of P in $D(z_0, R)$, then by Jensen's formula

$$\begin{aligned}&-\frac{1}{2} + (\lambda+1/2)\log n - \frac{1}{2}\log(2\lambda+1) + m\log 2 \\ &\le \frac{1}{3} + (\lambda+1/2)\log n - \frac{1}{2}\log(2\lambda+1) + 2nR\,,\end{aligned}$$

whence

$$m \le \frac{3nR}{\log 2} \le 5nR\,,$$

and, thus, the lemma has been proved. □

Our next lemma is a well-known inequality in approximation theory; see [Ne] for the reasons we attribute it to M. Riesz even if until recently it was associated with Bernstein's name.

Lemma 3.5 (M. Riesz's Inequality). *We have*

$$\|P'\|_{\partial D} \le n\|P\|_{\partial D}$$

for every $P \in \mathcal{P}_n^c$.

Now we are ready for the proof of Theorem 2.4.

Proof of Theorem 2.4. Observe that if (z_j) denote the zeros of P_n, then

$$\frac{zP_n'(z)}{P_n(z)} = \sum_{j=1}^{n} \frac{z}{z-z_j} = \sum_{j=1}^{n} \left(1 + \frac{z_j}{z-z_j}\right).$$

Since $P_n \in \mathcal{K}_n^\lambda$, we have

$$|z_1|, |z_2|, \dots, |z_n| < 2\,. \tag{3.5}$$

To see this, let

$$P_n(z) = \sum_{j=0}^{n} j^\lambda a_j z^j, \qquad a_j \in \mathbb{C}\,, \quad |a_j| = 1\,.$$

Now if $z_0 \in \mathbb{C}$ and $|z_0| \geq 2$, then

$$\begin{aligned} \left|\sum_{j=0}^{n} j^\lambda a_j z_0^j\right| &\geq n^\lambda |z_0|^n - n^\lambda (|z_0|^{n-1} + |z_0|^{n-2} + \cdots + |z_0|^1 + |z_0|^0) \\ &= n^\lambda \left(|z_0|^n - \frac{|z_0|^n - 1}{|z_0| - 1}\right) > 0\,, \end{aligned}$$

so that $P(z_0) \neq 0$. Using (1.3) and (3.5) and substituting $z_0 = e^{it_0}$, we can estimate $|\alpha_n^{(r)}(t_0)|$ as follows.

(3.6)

$$\begin{aligned} |\alpha_n^{(r)}(t_0)| &= \left|\frac{d^{r-1}}{dt^{r-1}} \left(\mathrm{Re}\left(\frac{e^{it}P_n'(e^{it})}{P_n(e^{it})}\right)\right)(t_0)\right| \leq \left|\frac{d^{r-1}}{dt^{r-1}} \left(\frac{e^{it}P_n'(e^{it})}{P_n(e^{it})}\right)(t_0)\right| \\ &= \left|\sum_{m=0}^{r-1} A_m \frac{d^m}{dz^m} \left(\frac{zP_n'(z)}{P_n(z)}\right)(z_0)\, e^{imt_0}\right| \\ &= \left|\sum_{m=0}^{r-1} A_m \frac{d^m}{dz^m} \left(\sum_{k=1}^{n} \left(1 + \frac{z_k}{z - z_k}\right)\right)(z_0)\, e^{imt_0}\right| \\ &\leq \left|A_0 \frac{z_0 P_n'(z_0)}{P_n(z_0)}\right| + \sum_{m=1}^{r-1} |A_m| m! \sum_{k=1}^{n} |z_k| |z_0 - z_k|^{-(m+1)} \\ &\leq \left|A_0 \frac{z_0 P_n'(z_0)}{P_n(z_0)}\right| + 2 \sum_{m=1}^{r-1} |A_m| m! \sum_{k=1}^{n} |z_0 - z_k|^{-(m+1)}\,, \end{aligned}$$

where the constants A_m depend only on m. Now we define the annulus

$$E_\mu = D(z_0, 2^\mu (2n\delta_n)^{-1}) \setminus D(z_0, 2^{\mu-1}(2n\delta_n)^{-1})\,, \qquad \mu \in \mathbb{N}\,,$$

where $\delta_n \stackrel{\text{def}}{=} \max\{2/(-\log(3\varepsilon_n)), 1/n\}$ as in Lemma 3.3. We denote the number of zeros of P_n in E_μ by m_μ. By Lemma 3.4, we have $m_\mu \le 5n2^\mu/(2n\delta_n)$. Combining this with (3.6) and Lemmas 3.5 & 3.3, we obtain

$$\begin{aligned}
|\alpha_n^{(r)}(t)| &\le C_0 \frac{n(1+\varepsilon_n)(2\lambda+1)^{-1/2}n^{\lambda+1/2}}{(1-\varepsilon_n)(2\lambda+1)^{-1/2}n^{\lambda+1/2}} + C_r \sum_{m=1}^{r-1}\sum_{k=1}^{n} |z_0 - z_k|^{-(m+1)} \\
&\le 2C_0 n + C_r \sum_{m=1}^{r-1}\sum_{\mu=1}^{\infty} m_\mu \left(\frac{2^{\mu-1}}{2n\delta_n}\right)^{-(m+1)} \\
&\le 2C_0 n + C_r \sum_{m=1}^{r-1}\sum_{\mu=1}^{\infty} \frac{5n2^\mu}{2n\delta_n} \left(\frac{2^{\mu-1}}{2n\delta_n}\right)^{-(m+1)} \\
&\le 2C_0 n + C_r \sum_{m=1}^{r-1}\sum_{\mu=1}^{\infty} 2\cdot 2^{-(\mu-1)m} 5n (2n\delta_n)^m \\
&\le 2C_0 n + C_r^* n^r \delta_n \le C_r^{**} n^r \delta_n\,,
\end{aligned}$$

where C_r, C_r^*, and C_r^{**} are positive constants depending only on r. Since

$$\delta_n \stackrel{\text{def}}{=} \max\{2/(-\log(3\varepsilon_n)), 1/n\}$$

tends to 0 together with $\varepsilon_n > 0$ as $n \to \infty$, the theorem is proved. $\square$

4. Proof of Theorems 2.1, 2.2, 2.3, 2.5, and 2.6

First we prove Theorem 2.2 for $q \in \mathbb{N}$. To this end we need the following lemmas.

Lemma 4.1 (Pólya's Companion to Dini's Theorem). *If (f_n) is a sequence of increasing functions on a closed interval Δ such that $f(x) \stackrel{\text{def}}{=} \lim_{n\to\infty} f_n(x)$ exists for every $x \in \Delta$ and f is continuous on Δ, then the convergence of (f_n) is uniform in Δ.*

For Lemma 4.1, see [PSz, Problem 127, §3, Part II, Chap. 3, p. 81] and [Bo, Sec. 17, p. 113]. Sometimes, this is called "Pólya's extension of Dini's theorem" which is, of course, misleading.

Lemma 4.2 (Bernstein Inequality for Trigonometric Polynomials). *We have*

$$\max_{0\le t\le 2\pi} |T^{(m)}(t)| \le n^m \max_{0\le t\le 2\pi} |T(t)|\,, \qquad m \in \mathbb{N}\,,$$

for every trigonometric polynomial T of degree at most n with complex coefficients.

Although there is an intimate relationship between Lemmas 3.5 and 4.2, we stated them separately for the sake of historical accuracy; see [N1] and [N2].

Our next lemma appeared in [S] first and then used in [E2] too. For the sake of brevity we give a short proof of it here as well.

Lemma 4.3. *Suppose (P_n) is a $\{\lambda, (\varepsilon_n)\}$-ultraflat sequence of polynomials. Using the notation (1.2) defining α_n, we have*

$$-o_n \leq \frac{\alpha_n'(t)}{n} \leq 1 + o_n\,, \qquad t \in \mathbb{R}\,,$$

for some real numbers $o_n > 0$ tending to 0 as $n \to \infty$.

Proof of Lemma 4.3. Combining (1.3) with Lemma 3.5 and the ultraflatness of (P_n), we obtain the upper bound of the lemma. In addition to the $\{\lambda, (\varepsilon_n)\}$-ultraflat sequence (P_n), we also look at the $\{\lambda, (\varepsilon_n)\}$-ultraflat sequence (P_n^*) of the corresponding reverse polynomials. By applying formula (1.3) to P_n^*, it is easy to see that

$$\alpha_n'(t) + {\alpha_n^*}'(t) = n, \qquad t \in \mathbb{R}\,.$$

Since the upper bound of the lemma is valid for α_n^* as well, the lower bound in the lemma follows from the latter. $\square$

Lemma 4.4. *Suppose (ε_n) is a sequence of numbers from $(0, 1/3)$ tending to 0 as $n \to \infty$. Suppose (P_n) is a $\{\lambda, (\varepsilon_n)\}$-ultraflat sequence of polynomials. Using the notation (1.2) defining R_n, we have*

$$\max_{0 \leq t \leq 2\pi} |R_n^{(m)}(t)| = o_{n,m} n^{m+\lambda+1/2}\,, \qquad m \in \mathbb{N}\,, \tag{4.1}$$

with suitable constants $o_{n,m}$ converging to 0 as $n \to \infty$ for every $m \in \mathbb{N}$.

Proof of Lemma 4.4. The proof is very similar to that of Lemma 3.3. Let

$$\delta_n \stackrel{\text{def}}{=} \max\left\{\frac{2}{-\log(3\,\varepsilon_n)}\,, \frac{1}{n}\right\}$$

as in the proof Lemma 3.3. Let (P_n) be a $\{\lambda, (\varepsilon_n)\}$-ultraflat sequence of polynomials, that is, let $P_n \in \mathcal{K}_n^\lambda$ satisfy[2]

$$(1 - \varepsilon_n)\frac{n^{\lambda+1/2}}{(2\lambda+1)^{1/2}} < |P_n(z)| < (1 + \varepsilon_n)\frac{n^{\lambda+1/2}}{(2\lambda+1)^{1/2}}\,, \qquad z \in \partial D\,. \tag{4.2}$$

Step 1. By Lemma 3.3,

$$T_n(t) \stackrel{\text{def}}{=} e^{-int} P_n(e^{it}) P_n^*(e^{it}) \tag{4.3}$$

[2] In fact, in this proof we don't need that $P_n \in \mathcal{K}_n^\lambda$, we will use only that P_n's are polynomials of degree n with complex coefficients that satisfy (4.2).

has no zeros in the strip

$$\mathcal{E}_n \stackrel{\text{def}}{=} \left\{ z \in \mathbb{C} : |\mathrm{Im}(z)| \leq \frac{1}{4n\delta_n} \right\} .$$

Therefore,

$$\widetilde{T}_n(t) \stackrel{\text{def}}{=} \sqrt{T_n(t)} = \sqrt{e^{-int} P_n(e^{it}) P_n^*(e^{it})}$$

is a well-defined analytic function in the strip $\mathcal{E}_n$. Note also that T_n is a trigonometric polynomial of degree n with real coefficients and $T_n(t) \geq 0$ for all $t \in \mathbb{R}$.

Step 2. We show that

$$|\widetilde{T}_n'(t)| \leq o_n n^{\lambda+3/2} , \qquad t \in \mathbb{R} ,$$

with suitable constants o_n converging to 0 as $n \to \infty$. Note that (4.2) and the fact that $T_n(t) \geq 0$ for all $t \in \mathbb{R}$ imply that

$$-3\,\varepsilon_n \frac{n^{2\lambda+1}}{2\lambda+1} < T_n(t) - \frac{n^{2\lambda+1}}{2\lambda+1} < 3\,\varepsilon_n \frac{n^{2\lambda+1}}{2\lambda+1} , \qquad t \in \mathbb{R} . \tag{4.4}$$

Combining this with Lemma 4.2, we obtain

$$\begin{aligned} \max_{0 \leq t \leq 2\pi} |T_n'(t)| &= \max_{0 \leq t \leq 2\pi} \left| \frac{d}{dt} \left(T_n(t) - \frac{n^{2\lambda+1}}{2\lambda+1} \right) \right| \\ &\leq n \max_{0 \leq t \leq 2\pi} \left| T_n(t) - \frac{n^{2\lambda+1}}{2\lambda+1} \right| \leq n\,3\,\varepsilon_n \frac{n^{2\lambda+1}}{2\lambda+1} \leq \frac{3\,\varepsilon_n}{2\lambda+1} n^{2\lambda+2} . \end{aligned} \tag{4.5}$$

Now

$$\begin{aligned} |\widetilde{T}_n'(t)| = \left| \frac{T_n'(t)}{2\sqrt{T_n(t)}} \right| &\leq \frac{3\,\varepsilon_n}{2\lambda+1} \times \frac{n^{2\lambda+2}}{2(1-\varepsilon_n)(2\lambda+1)^{-1/2} n^{\lambda+1/2}} \\ &\leq \frac{3\varepsilon_n}{2(1-\varepsilon_n)} n^{\lambda+3/2} \leq \frac{9}{4}\,\varepsilon_n n^{\lambda+3/2} = o_n n^{\lambda+3/2} , \qquad t \in \mathbb{R} , \end{aligned}$$

with suitable constants o_n converging to 0.

Step 3. Let

$$\mathcal{F}_{c,n} \stackrel{\text{def}}{=} \left\{ z \in \mathbb{C} : |\mathrm{Im}(z)| \leq \frac{c}{n} \right\} .$$

We will show that there is a sufficiently small absolute constant $c > 0$ such that

$$|\widetilde{T}_n'(t)| \leq o_n n^{\lambda+3/2} , \qquad t \in \mathcal{F}_{c,n} ,$$

with suitable constants o_n converging to 0 as $n \to \infty$. To see this, first note that

$$|\widetilde{T}_n'(t)| = \left| \frac{T_n'(t)}{2\sqrt{T_n(t)}} \right| , \tag{4.6}$$

where T_n is defined by (4.3). Using (4.5) and Lemma 3.1 we obtain that

$$|T_n'(t)| \le o_n^* n^{2\lambda+2} e^{n(c/n)} = o_n n^{2\lambda+2}\,, \qquad t \in \mathcal{F}_{c,n}, \tag{4.7}$$

with suitable constants o_n^* and o_n converging to 0 as $n \to \infty$ and with a sufficiently small absolute constant $c > 0$. Similarly, (4.4), $\varepsilon_n \in (0, 1/3)$, and Lemma 3.1 give

$$\left| T_n(t) - \frac{n^{2\lambda+1}}{2\lambda+1} \right| < 3\,\varepsilon_n \frac{n^{2\lambda+1}}{2\lambda+1} e^{nc/n} \le 3\,\varepsilon_n \frac{n^{2\lambda+1}}{2\lambda+1} e^c\,, \qquad t \in \mathbb{R}\,,$$

and hence

$$|T_n(t)| \ge \frac{n^{2\lambda+1}}{2(2\lambda+1)}\,, \qquad t \in \mathcal{F}_{c,n}\,, \tag{4.8}$$

for a sufficiently small absolute constant $c > 0$. Now (4.6), (4.7), and (4.8) imply that

$$|\widetilde{T}_n'(t)| \le o_n n^{\lambda+3/2}\,, \qquad t \in \mathcal{F}_{c,n}\,,$$

with suitable constants o_n converging to 0 as $n \to \infty$ and with a sufficiently small absolute constant $c > 0$.

Step 4. From Step 3 we conclude by the Cauchy Integral Formula that

$$\begin{aligned} |\widetilde{T}_n^{(m)}(t)| &= \frac{(m-1)!}{2\pi} \left| \int_{\partial D(t,c/n)} \frac{\widetilde{T}_n'(\xi)\,d\xi}{(\xi - t)^m} \right| \\ &\le \frac{c}{n}\,(m-1)!\, o_{n,1} n^{\lambda+3/2} \left(\frac{c}{n}\right)^{-m} = o_{n,m} n^{m+\lambda+1/2}\,, \end{aligned}$$

with suitable constants $o_{n,m}$ converging to 0 as $n \to \infty$ for every fixed $m \in \mathbb{N}$.

Step 5. Note that for $t \in \mathbb{R}$ we have

$$R_n(t) = |P_n(e^{it})| = \sqrt{e^{-int} P_n(e^{it}) P_n^*(e^{it})} = \widetilde{T}_n(t)\,, \tag{4.9}$$

and, thus, by Step 4,

$$\max_{0 \le t \le 2\pi} |R_n^{(m)}(t)| = o_{n,m} n^{m+\lambda+1/2}$$

with suitable constants $o_{n,m}$ converging to 0 as $n \to \infty$ for every fixed $m \in \mathbb{N}$. This proves the lemma. □

Now we are ready to prove Theorem 2.2 for $q \in \mathbb{N}$.

Proof of Theorem 2.2 for $q \in \mathbb{N}$. Let (P_n) be a $\{\lambda, (\varepsilon_n)\}$-ultraflat sequence of polynomials. We define

$$S_n(t) \stackrel{\text{def}}{=} P_n(e^{it}) = \sum_{k=0}^{n} k^\lambda a_{k,n} e^{ikt}\,, \qquad |a_{k,n}| = 1\,. \tag{4.10}$$

We will evaluate

$$\frac{1}{2\pi}\int_0^{2\pi} S_n^{(q)}(t)\overline{S_n(t)}\,dt$$

in two different ways. On one hand, using orthogonality, we have

$$\frac{1}{2\pi}\int_0^{2\pi} S_n^{(q)}(t)\overline{S_n(t)}\,dt = i^q \sum_{k=0}^{n} k^{q+2\lambda}|a_{k,n}|^2 = i^q \frac{n^{q+2\lambda+1}}{q+2\lambda+1} + o_{n,q}n^{q+2\lambda+1}\,, \tag{4.11}$$

with suitable constants $o_{n,q}$ converging to 0 as $n \to \infty$ for every fixed nonnegative integer q.

On the other hand, with the notation for R_n and α_n introduced in (1.2), Theorem 2.4 and Lemmas 4.2, 4.3, and 4.4 yield

(4.12)

$$\begin{aligned}
S_n^{(q)}(t) &= \sum_{k=0}^{q} \binom{q}{k} \frac{d^k}{dt^k}\left(e^{i\alpha_n(t)}\right) R_n^{(q-k)}(t) \\
&= \frac{d^q}{dt^q}\left(e^{i\alpha_n(t)}\right) R_n(t) + \sum_{k=0}^{q-1} \binom{q}{k} \frac{d^k}{dt^k}\left(e^{i\alpha_n(t)}\right) R_n^{(q-k)}(t) \\
&= \frac{d^q}{dt^q}\left(e^{i\alpha_n(t)}\right) R_n(t) + \sum_{k=0}^{q-1} \binom{q}{k} c_{n,k}(t) n^k o_{n,q-k}(t) n^{q-k+\lambda+1/2} \\
&= \left(e^{i\alpha_n(t)}\alpha_n'(t)^q i^q + o_{n,q}^*(t)n^q\right) R_n(t) + o_{n,q}^{**}(t) n^{q+\lambda+1/2}
\end{aligned}$$

with suitable numbers $o_{n,q-k}(t)$, $c_{n,k}(t)$, $o_{n,q}^*(t)$, and $o_{n,q}^{**}(t)$, where

$$\lim_{n\to\infty} \max_{0\le t\le 2\pi} |o_{n,q-k}(t)| = 0$$

for every fixed q and $k = 0, 1, \dots, q-1$,

$$\sup_{n\in\mathbb{N}} \max_{0\le t\le 2\pi} |c_{n,k}(t)| < \infty$$

for every fixed $k = 0, 1, \dots, q-1$, and

$$\lim_{n\to\infty} \max_{0\le t\le 2\pi} |o_{n,q}^*(t)| = 0 \qquad \& \qquad \lim_{n\to\infty} \max_{0\le t\le 2\pi} |o_{n,q}^{**}(t)| = 0$$

for every fixed q. Now (4.2), (4.9), (4.10), and (4.12) yield

(4.13)

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} S_n^{(q)}(t)\overline{S_n(t)}\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(\left(e^{i\alpha_n(t)}\alpha_n'(t)^q i^q + o_{n,q}^*(t)n^q\right) R_n(t) + o_{n,q}^{**}(t) n^{q+\lambda+1/2}\right) R_n(t) e^{-i\alpha_n(t)}\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{n^{2\lambda+1}}{2\lambda+1}(1-o_n(t))\left(\alpha_n'(t)^q i^q + o_{n,q}^*(t)n^q\right) + o_{n,q}^{***}(t) n^{q+\lambda+1/2+\lambda+1/2}\right) dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} i^q \left(\frac{n^{2\lambda+1}}{2\lambda+1}(1-o_n(t))\right) \alpha_n'(t)^q\,dt + o_{n,q}^{****} n^{q+2\lambda+1}\,,
\end{aligned}$$

with suitable functions $o_n(t)$, $o_{n,q}^*(t)$, $o_{n,q}^{**}(t)$, and $o_{n,q}^{***}(t)$, and numbers $o_{n,q}^{****}$, where all

$$\max_{0\le t\le 2\pi}|o_n(t)| \quad \& \quad \max_{0\le t\le 2\pi}|o_{n,q}^*(t)| \quad \& \quad \max_{0\le t\le 2\pi}|o_{n,q}^{**}(t)| \quad \& \quad \max_{0\le t\le 2\pi}|o_{n,q}^{***}(t)| \quad \& \quad o_{n,q}^{****}$$

converge to 0 as $n\to\infty$ for every fixed q.

Now (4.11) and (4.13) give the statement of the theorem for integers $q\in\mathbb{N}$. $\square$

One may simply note that the fact that Theorem 2.1 as well as the case $q>0$ in Theorem 2.2 follow from the case $q\in\mathbb{N}$ in Theorem 2.2 is well known. Nevertheless, for the sake of completeness, we present detailed proofs of these as well.

Proof of Theorem 2.1. We introduce the normalized distribution functions

$$F_n(x) \stackrel{\text{def}}{=} m\{t\in[0,2\pi]: 0\le \alpha_n'(t)\le nx\}\,, \qquad x\in[0,1]\,. \tag{4.14}$$

Each F_n is continuous and nondecreasing on $[0,1]$, and

$$0\le F_n(x)\le 2\pi\,, \qquad x\in[0,1]\,.$$

Suppose (2.1) is not true. Then we can find a subsequence (F_{n_k}) of (F_n) and numbers $y\in[0,1]$ & $\varepsilon>0$ such that

$$|F_{n_k}(y)-2\pi y^{2\lambda+1}|\ge\varepsilon\,, \qquad k\in\mathbb{N}\,. \tag{4.15}$$

Then by Helly's Selection Theorem, there is a subsequence (m_k) of (n_k) such that

$$F(x) \stackrel{\text{def}}{=} \lim_{k\to\infty} F_{m_k}(x) \tag{4.16}$$

exists for every $x\in[0,1]$. Using Theorem 2.2, (4.14), Lemma 4.3, and Lebesgue's Dominated Convergence Theorem we obtain that

$$\int_0^1 x^q\,dF(x) = \frac{2\pi(2\lambda+1)}{q+2\lambda+1}\,, \qquad q\in\mathbb{N}\,.$$

Hence, all the corresponding moments of the measures $dF(x)$ and $dG(x)$ with $G(x)\stackrel{\text{def}}{=}2\pi x^{2\lambda+1}$ are the same on $[0,1]$. Therefore, using the uniqueness part of the Riesz Representation Theorem describing all continuous linear functionals on $C([0,1])$, we obtain that $F(x)\equiv 2\pi x^{2\lambda+1}$ for all $x\in[0,1]$. However, this contradicts (4.15) and (4.16). Consequently,

$$m\{t\in[0,2\pi]: 0\le\alpha_n'(t)\le nx\} = 2\pi x^{2\lambda+1}+o_n(x)$$

for every $x\in[0,1]$, where $\lim_{n\to\infty}o_n(x)=0$ for every $x\in[0,1]$. By Lemma 4.1, the convergence is uniform in $[0,1]$.

To see the second statement of the theorem, we argue as follows. Using notation (1.2) and Lemma 4.4 we have $R_n'(t) = o_n(t)n^{\lambda+3/2}$ with a constant $o_n(t)$ tending to 0 as $n \to \infty$ for every $t \in \mathbb{R}$. Therefore,

$$
\begin{aligned}
|P_n'(e^{it})| &= |R_n'(t)e^{i\alpha_n(t)} + i\alpha_n'(t)e^{i\alpha_n(t)}R_n(t)| \\
&= o_n(t)n^{\lambda+3/2} + |\alpha_n'(t)|(1+\varepsilon_n)\frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}}\,,
\end{aligned}
\tag{4.17}
$$

where $o_n(t)$ and ε_n tend to 0 as $n \to \infty$ for every $t \in \mathbb{R}$. Now (2.2) follows from (2.1) and the convergence in $[0,1]$ is uniform again by Lemma 4.1. □

Proof of Theorem 2.2 for all real $q > 0$. This follows from the already proved Theorem 2.1 in the following way. Let $F_n(x)$ be defined for $x \in [0,\infty)$ by (4.14). Let $o_n(x)$ be the same as in Theorem 2.1. Using integration by parts, Lemma 4.3, and Theorem 2.1, we obtain

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\alpha_n'(t)}{n}\right|^q dt =& \frac{1}{2\pi}\int_{-\infty}^{\infty}|x|^q\,dF_n(x) = \frac{1}{2\pi}\int_0^1 x^q\,dF_n(x) + o_{n,q}^* \\
=& \frac{1}{2\pi}\left[x^qF_n(x)\right]_0^1 - \frac{1}{2\pi}\int_0^1 qx^{q-1}F_n(x)\,dx + o_{n,q}^* \\
=& 1 - \frac{1}{2\pi}\int_0^1 qx^{q-1}\left(2\pi x^{2\lambda+1} + o_n(x)\right)\,dx + o_{n,q}^* \\
=& 1 - \int_0^1 \left(qx^{q+2\lambda} + (2\pi)^{-1}\,qx^{q-1}o_n(x)\right)dx + o_{n,q}^* \\
=& 1 - \frac{q}{q+2\lambda+1} + o_{n,q}^{**} = \frac{2\lambda+1}{q+2\lambda+1} + o_{n,q}^{**}
\end{aligned}
$$

with numbers $o_{n,q}^*$ and $o_{n,q}^{**}$ tending to 0 as $n \to \infty$. □

Proof of Theorem 2.3. This follows immediately from Theorem 2.2 and (4.17). □

Proof of Theorem 2.5. Write, as in (1.2),

$$P_n(e^{it}) = R_n(t)e^{i\alpha_n(t)}\,,$$

where, as before, $R_n(t) = |P_n(e^{it})|$. Then

$$P_n^{(r)}(z) = \sum_{k=0}^{r}\binom{r}{k}R_n^{(k)}(t)\frac{d^{(r-k)}}{dt^{r-k}}\left(e^{i\alpha_n(t)}\right)$$

Now the pointwise convergence in the theorem follows from (1.2), Theorem 2.4, Lemma 4.4, and Theorem 2.1 for every $x \in [0,1]$, and then the convergence in $[0,1]$ is uniform by Lemma 4.1. □

The next proof of Theorem 2.6 is based on Theorems 2.1 and 2.4.

Proof of Theorem 2.6. Let (P_n') be a $\{\lambda+1,(\varepsilon_n)\}$-ultraflat sequence of polynomials. Similarly to (1.2), we write

$$P_n'(e^{it}) = R_n(t)e^{i\alpha_n(t)} \qquad \text{where} \qquad R_n(t) = |P_n'(e^{it})|\,.$$

As mentioned after (1.2), we may assume that all α_n's are in $C^\infty(\mathbb{R})$. We have

$$(1-\varepsilon_n)\frac{n^{\lambda+3/2}}{\sqrt{2\lambda+1}} \le |R_n(t)| = |P_n(e^{it})| \le (1+\varepsilon_n)\frac{n^{\lambda+3/2}}{\sqrt{2\lambda+1}}\,, \qquad t\in\mathbb{R}\ \&\ n\in\mathbb{N}\,.$$

Let $\varepsilon>0$. For each sufficiently large n one can use Theorem 2.1 to pick $t_0 = t_{0,n}\in\mathbb{R}$ such that

$$|\alpha_n'(t_0)| \le \varepsilon n\,,$$

then define t_1 by $t_1 \stackrel{\text{def}}{=} t_0 + (2\,\varepsilon n)^{-1}$, and then apply the Mean Value Theorem and Theorem 2.4 to conclude that

$$|\alpha_n'(t)| \le 2\varepsilon n\,, \qquad t\in[t_0,t_1]\,, \tag{4.18}$$

for all sufficiently large n, namely for all n for which $\varepsilon/o_{n,2} \ge (2\,\varepsilon)^{-1}$ holds.[3] Combining (4.18) and the Mean Value Theorem again, we obtain

$$|\alpha_n(t)-\alpha_n(t_0)| \le |t-t_0| \max_{\xi\in[t_0,t]} |\alpha_n'(\xi)| \le\ (2\varepsilon n)^{-1} 2\,\varepsilon n \le 1\,, \qquad t\in[t_0,t_1]\,,$$

for all sufficiently large n. Hence

$$\begin{aligned}
|P_n(e^{it_1})-P_n(e^{it_0})| &= \left|\int_{t_0}^{t_1} P_n'(e^{it})e^{it} i\,dt\right| = \left|\int_{t_0}^{t_1} R_n(t)e^{i\alpha_n(t)}e^{it} i\,dt\right| \\
&\ge \left|(t_1-t_0)\frac{n^{\lambda+3/2}}{\sqrt{2\lambda+1}} e^{i\alpha_n(t_0)}e^{it_0} i\right| \\
&\quad - \int_{t_0}^{t_1}\left|e^{i\alpha_n(t_0)}e^{it_0}i\left(R_n(t)-\frac{n^{\lambda+3/2}}{\sqrt{2\lambda+1}}\right)\right| dt - \int_{t_0}^{t_1}\left|R_n(t)(e^{i\alpha_n(t)}e^{it}i - e^{i\alpha_n(t_0)}e^{it_0}i)\right| dt \\
&\ge (t_1-t_0)\frac{n^{\lambda+3/2}}{\sqrt{2\lambda+1}} - \varepsilon_n(t_1-t_0)\frac{n^{\lambda+3/2}}{\sqrt{2\lambda+1}} - (t_1-t_0)(1+\varepsilon_n)\frac{n^{\lambda+3/2}}{\sqrt{2\lambda+1}}\,2\sin\left(\frac{1}{2}+\frac{t_1-t_0}{2}\right) \\
&\ge (t_1-t_0)\frac{n^{\lambda+3/2}}{\sqrt{2\lambda+1}}\left(1-\varepsilon_n - 2\sin\left(\frac{1}{2}+\frac{t_1-t_0}{2}\right)\right)
\end{aligned}$$

---

[3]We can assume that $o_{n,2}>0$.

$$\geq c(t_1 - t_0)\frac{n^{\lambda+3/2}}{\sqrt{2\lambda+1}} \geq \frac{c}{2\varepsilon}\,\frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}}$$

for all sufficiently large n, where $c > 0$ is an absolute constant. Comparing the first and the last terms in this chain of inequalities, we see that

$$\lim_{n\to\infty} n^{-(\lambda+1/2)} \max_{t\in\mathbb{R}} |P_n(e^{it})| = \infty\,.$$

□

In order to prove Theorem 2.7, we need the following statement that is stated and proved as Theorem 3.1.27 on page 689 of [MMR].

Lemma 4.5. *We have*

$$\max_{z\in\partial D}\left(|P'(z)| + |P^{*\prime}(z)|\right) = n \max_{z\in\partial D} |P(z)|$$

for every $P \in \mathcal{P}_n \setminus \mathcal{P}_{n-1}$.

Proof of Theorem 2.7. The theorem follows easily from Theorem 2.6, Lemma 4.5 and the definition of a $\{1,(\varepsilon_{j_n})\}$-ultraflat sequence of polynomials. Assume that $Q_{j_n} \overset{\text{def}}{=} P'_{j_n}$ is a $\{1,(\varepsilon_{j_n-1})\}$-ultraflat sequence of polynomials. Here we have that $Q_{j_n} \overset{\text{def}}{=} P'_{j_n}$ are asymptotically in $\mathcal{K}_{j_n-1}$, see Remark 2.9). Observe that the assumptions of the theorem together with Lemma 4.5 imply that

$$\max_{z\in\partial D} |P'_{j_n}(z)| \geq \frac{j_n}{2} \max_{z\in\partial D} |P_{j_n}(z)|\,,$$

which contradicts the extension of Theorem 2.6 according to Remark 2.9. □

Proof of Corollary 2.8. The corollary follows easily from Theorem 2.7 and the definitions. Note that the assumptions of Theorem 2.7 are satisfied for sequences of conjugate, or plain, or skew reciprocal polynomials $P_{j_n} \in \mathcal{K}^0_{j_n}$. □

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