

THE ASYMPTOTIC DISTANCE BETWEEN AN ULTRAFLAT UNIMODULAR POLYNOMIAL AND ITS CONJUGATE RECIPROCAL

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Dedicated to the memory of Jean-Pierre Kahane

ABSTRACT. Let

$$\mathcal{K}_n := \left\{ Q_n : Q_n(z) = \sum_{k=0}^n a_k z^k, \quad a_k \in \mathbb{C}, \quad |a_k| = 1 \right\}.$$

A sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ is called ultraflat if $(n+1)^{-1/2}|P_n(e^{it})|$ converge to 1 uniformly in $t \in \mathbb{R}$. In this paper we prove that

$$\frac{1}{2\pi} \int_0^{2\pi} |(P_n - P_n^*)(e^{it})|^q dt \sim \frac{2^q \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2} + 1\right) \sqrt{\pi}} n^{q/2}$$

for every ultraflat sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ and for every $q \in (0, \infty)$, where P_n^* is the conjugate reciprocal polynomial associated with P_n , Γ is the usual gamma function, and the \sim symbol means that the ratio of the left and right hand sides converges to 1 as $n \rightarrow \infty$. Another highlight of the paper states that

$$\frac{1}{2\pi} \int_0^{2\pi} |(P_n' - P_n^{*'}) (e^{it})|^2 dt \sim \frac{2n^3}{3}$$

for every ultraflat sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$. We prove a few other new results and reprove some interesting old results as well.

1. INTRODUCTION

Let

$$\mathcal{K}_n := \left\{ Q_n : Q_n(z) = \sum_{k=0}^n a_k z^k, \quad a_k \in \mathbb{C}, \quad |a_k| = 1 \right\}.$$

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The class \mathcal{K}_n is often called the collection of all (complex) unimodular polynomials of degree n . Let

$$\mathcal{L}_n := \left\{ Q_n : Q_n(z) = \sum_{k=0}^n a_k z^k, \quad a_k \in \{-1, 1\} \right\}.$$

The class \mathcal{L}_n is often called the collection of all (real) unimodular polynomials of degree n . By Parseval's formula,

$$\int_0^{2\pi} |P_n(e^{it})|^2 dt = 2\pi(n+1)$$

for all $P_n \in \mathcal{K}_n$. Therefore

$$\min_{t \in \mathbb{R}} |P_n(e^{it})| \leq \sqrt{n+1} \leq \max_{t \in \mathbb{R}} |P_n(e^{it})|.$$

An old problem (or rather an old theme) is the following.

Problem 1.1 (Littlewood's Flatness Problem). *How close can a polynomial $P_n \in \mathcal{K}_n$ or $P_n \in \mathcal{L}_n$ come to satisfying*

$$(1.1) \quad |P_n(e^{it})| = \sqrt{n+1}, \quad t \in \mathbb{R}?$$

Obviously (1.1) is impossible if $n \geq 1$. So one must look for less than (1.1), but then there are various ways of seeking such an "approximate situation". One way is the following. In his paper [Li1] Littlewood had suggested that, conceivably, there might exist a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ (possibly even $P_n \in \mathcal{L}_n$) such that $(n+1)^{-1/2}|P_n(e^{it})|$ converge to 1 uniformly in $t \in \mathbb{R}$. We shall call such sequences of unimodular polynomials "ultraflat". More precisely, we give the following definition.

Definition 1.2. *Given a positive number ε , we say that a polynomial $P_n \in \mathcal{K}_n$ is ε -flat if*

$$(1 - \varepsilon)\sqrt{n+1} \leq |P_n(e^{it})| \leq (1 + \varepsilon)\sqrt{n+1}, \quad t \in \mathbb{R}.$$

Definition 1.3. *Let (n_k) be an increasing sequence of positive integers. Given a sequence (ε_{n_k}) of positive numbers tending to 0, we say that a sequence (P_{n_k}) of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is (ε_{n_k}) -ultraflat if each P_{n_k} is ε_{n_k} -flat. We simply say that a sequence (P_{n_k}) of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is ultraflat if it is (ε_{n_k}) -ultraflat with a suitable sequence (ε_{n_k}) of positive numbers tending to 0.*

The existence of an ultraflat sequence of unimodular polynomials seemed very unlikely, in view of a 1957 conjecture of P. Erdős (Problem 22 in [Er]) asserting that, for all $P_n \in \mathcal{K}_n$ with $n \geq 1$,

$$(1.2) \quad \max_{t \in \mathbb{R}} |P_n(e^{it})| \geq (1 + \varepsilon)\sqrt{n+1},$$

where $\varepsilon > 0$ is an absolute constant (independent of n). Yet, refining a method of Körner [Kö], Kahane [Ka] proved that there exists a sequence (P_n) with $P_n \in \mathcal{K}_n$ which is (ε_n) -ultraflat, where $\varepsilon_n = O(n^{-1/17} \sqrt{\log n})$. (Kahane's paper contained though a slight error which was corrected in [QS2].) Thus the Erdős conjecture (1.2) was disproved for the classes \mathcal{K}_n . For the more restricted class \mathcal{L}_n the analogous Erdős conjecture is unsettled to this date. It is a common belief that the analogous Erdős conjecture for \mathcal{L}_n is true, and consequently there is no ultraflat sequence of polynomials $P_n \in \mathcal{L}_n$. An interesting result related to Kahane's breakthrough is given in [Be]. For an account of some of the work done till the mid 1960's, see Littlewood's book [Li2] and [QS2]. Littlewood polynomials with small L_4 norm have also been intensively studied, see [BM], [BC1], [BC2],[BC3], and [JKSch], for example.

If Q_n is a polynomial of degree n of the form

$$Q_n(z) = \sum_{k=0}^n a_k z^k, \quad a_k \in \mathbb{C},$$

then its conjugate polynomial is defined by

$$Q_n^*(z) := z^n \overline{Q}_n(1/z) := \sum_{k=0}^n \overline{a}_{n-k} z^k,$$

where \overline{Q}_n denotes the polynomial whose coefficients are the complex conjugates of the corresponding coefficients of the polynomial Q .

Let (ε_n) be a sequence of positive numbers tending to 0. Let the sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ be (ε_n) -ultraflat. We write

$$(1.3) \quad P_n(e^{it}) = R_n(t) e^{i\alpha_n(t)}, \quad R_n(t) = |P_n(e^{it})|, \quad t \in \mathbb{R}.$$

It is simple to show that α_n can be chosen to be in $C^\infty(\mathbb{R})$. This is going to be our understanding throughout the paper. It is easy to find a formula for $\alpha_n(t)$ in terms of P_n . We have

$$(1.4) \quad \alpha_n'(t) = \operatorname{Re} \left(\frac{e^{it} P_n'(e^{it})}{P_n(e^{it})} \right),$$

see formulas (7.1) and (7.2) on p. 564 and (8.2) on p. 565 in [Sa1]. The angular function α_n^* and modulus function $R_n^* = R_n$ associated with the polynomial P_n^* are defined by

$$P_n^*(e^{it}) = R_n^*(t) e^{i\alpha_n^*(t)}, \quad R_n^*(t) = |P_n^*(e^{it})|.$$

Similarly to α_n , the angular function α_n^* can also be chosen to be in $C^\infty(\mathbb{R})$ on \mathbb{R} . By applying formula (1.4) to P_n^* , it is easy to see that

$$(1.5) \quad \alpha_n'(t) + \alpha_n^{*\prime}(t) = n, \quad t \in \mathbb{R}.$$

The structure of ultraflat sequences of unimodular polynomials is studied in [Er1], [Er2], [Er3], [Er4], and [Er5], where several conjectures of Saffari are proved. In [Er6], based on the results in [Er1], we proved yet another conjecture of Queffelec and Saffari, see (1.30) in [QS2]. Namely we proved asymptotic formulas for the L_q norms of the real part and the derivative of the real part of ultraflat unimodular polynomials on the unit circle. A recent paper of Bombieri and Bourgain [BB] is devoted to the construction of ultraflat sequences of unimodular polynomials. In particular, they obtained a much improved estimate for the error term. A major part of their paper deals also with the long-standing problem of the effective construction of ultraflat sequences of unimodular polynomials.

For $\lambda \geq 0$, let

$$\mathcal{K}_n^\lambda := \left\{ P_n : P_n(z) = \sum_{k=0}^n a_k k^\lambda z^k, \quad a_k \in \mathbb{C}, \quad |a_k| = 1 \right\}.$$

Ultraflat sequences (P_n) of polynomials $P_n \in \mathcal{K}_n^\lambda$ are defined and studied thoroughly in [EN] where various extensions of Saffari's conjectures have been proved.

In [Er2] we examined how far an ultraflat unimodular polynomial is from being conjugate reciprocal, and we proved the following three theorems.

Theorem 1.4. *Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. We have*

$$\frac{1}{2\pi} \int_0^{2\pi} (|P'_n(e^{it})| - |P_n^{*'}(e^{it})|)^2 dt = \left(\frac{1}{3} + \gamma_n \right) n^3,$$

where (γ_n) is a sequence of real numbers converging to 0.

Theorem 1.5. *Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. If the coefficients of P_n are denoted by $a_{k,n}$, that is,*

$$P_n(z) = \sum_{k=0}^n a_{k,n} z^k, \quad k = 0, 1, \dots, n, \quad n = 1, 2, \dots,$$

then

$$\sum_{k=0}^n k^2 |a_{k,n} - \bar{a}_{n-k,n}|^2 = \frac{1}{2\pi} \int_0^{2\pi} |(P'_n - P_n^{*'})(e^{it})|^2 dt \geq \left(\frac{1}{3} + h_n \right) n^3,$$

where (h_n) is a sequence of real numbers converging to 0.

Theorem 1.6. *Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Using the notation of Theorem 1.5 we have*

$$\sum_{k=0}^n |a_{k,n} - \bar{a}_{n-k,n}|^2 = \frac{1}{2\pi} \int_0^{2\pi} |(P_n - P_n^*)(e^{it})|^2 dt \geq \left(\frac{1}{3} + h_n \right) n,$$

where (h_n) is the same sequence of real numbers converging to 0 as in Theorem 1.5.

There are quite a few recent publications on or related to ultraflat sequences of unimodular polynomials. Some of them (not mentioned before) are [Bo], [Sa2], [QS1], [Od], and [Mo].

2. RESULTS

Theorems 2.1–2.4 and 2.6 are new, Theorems 2.5 and 2.7 recapture old results.

In our results below Γ denotes the usual gamma function, and the \sim symbol means that the ratio of the left and right hand sides converges to 1 as $n \rightarrow \infty$.

Theorem 2.1. *If (P_n) is an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$ and $q \in (0, \infty)$, then*

$$\frac{1}{2\pi} \int_0^{2\pi} |(P_n - P_n^*)(e^{it})|^q dt \sim \frac{2^q \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2} + 1\right) \sqrt{\pi}} n^{q/2}.$$

Our next theorem is a special case ($q = 2$) of Theorem 2.1. Compare it with Theorem 1.6.

Theorem 2.2. *Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. If the coefficients of P_n are denoted by $a_{k,n}$, that is,*

$$P_n(z) = \sum_{k=0}^n a_{k,n} z^k, \quad k = 0, 1, \dots, n, \quad n = 1, 2, \dots,$$

then

$$\sum_{k=0}^n |a_{k,n} - \bar{a}_{n-k,n}|^2 = \frac{1}{2\pi} \int_0^{2\pi} |(P_n - P_n^*)(e^{it})|^2 dt \sim 2n.$$

Our next theorem should be compared with Theorem 1.5.

Theorem 2.3. *Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Using the notation in Theorem 2.2 we have*

$$\sum_{k=0}^n k^2 |a_{k,n} - \bar{a}_{n-k,n}|^2 = \frac{1}{2\pi} \int_0^{2\pi} |(P_n' - P_n^{*'}) (e^{it})|^2 dt \sim \frac{2n^3}{3}.$$

We also prove the following result.

Theorem 2.4. *If (P_n) is an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$ and $q \in (0, \infty)$, then*

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{d}{dt} |(P_n - P_n^*)(e^{it})| \right|^q dt \sim \frac{\Gamma\left(\frac{q+1}{2}\right)}{(q+1)\Gamma\left(\frac{q}{2} + 1\right) \sqrt{\pi}} n^{3q/2}.$$

As a Corollary of Theorem 2.2 we can recapture Saffari's "near orthogonality conjecture" raised in [Sa] and proved first in [Er4].

Theorem 2.5. *Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Using the notation in Theorem 2.2 we have*

$$\sum_{k=0}^n a_{k,n} a_{n-k,n} = o(n).$$

As a Corollary of Theorem 2.3 we can easily prove a new "near orthogonality" formula.

Theorem 2.6. *Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Using the notation in Theorem 2.2 we have*

$$\sum_{k=0}^n k^2 a_{k,n} a_{n-k,n} = o(n^3).$$

Finally we recapture the asymptotic formulas for the real part and the derivative of the real part of ultraflat unimodular polynomials proved in [Er6] first.

Theorem 2.7. *If (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$, and $q \in (0, \infty)$, then for $f_n(t) := \operatorname{Re}(P_n(e^{it}))$ we have*

$$\frac{1}{2\pi} \int_0^{2\pi} |f_n(t)|^q dt \sim \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2} + 1\right) \sqrt{\pi}} n^{q/2}$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} |f'_n(t)|^q dt \sim \frac{\Gamma\left(\frac{q+1}{2}\right)}{(q+1)\Gamma\left(\frac{q}{2} + 1\right) \sqrt{\pi}} n^{3q/2}.$$

We remark that trivial modifications of the proof of Theorem 2.1–2.7 yield that the statement of the above theorem remains true if the ultraflat sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ is replaced by an ultraflat sequence (P_{n_k}) of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$, where (n_k) is an increasing sequence of positive integers.

3. LEMMAS

To prove Theorems 2.1 and 2.2 we need a few lemmas. The first two are from [Er1].

Lemma 3.1 (Uniform Distribution Theorem for the Angular Speed). *Suppose (P_n) is an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Then, with the notation (1.3), in the interval $[0, 2\pi]$, the distribution of the normalized angular speed $\alpha'_n(t)/n$ converges to the uniform distribution as $n \rightarrow \infty$. More precisely, we have*

$$\operatorname{meas}(\{t \in [0, 2\pi] : 0 \leq \alpha'_n(t) \leq nx\}) = 2\pi x + \gamma_n(x)$$

for every $x \in [0, 1]$, where $\lim_{n \rightarrow \infty} \max_{x \in [0, 1]} |\gamma_n(x)| = 0$.

Our next lemma is a simple observation of Saffari [Sa1], which follows from (1.4), Bernstein's inequality, and the ultraflatness property given by Definition 1.3.

Lemma 3.2. *Suppose (P_n) is an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Then, with the notation (1.3), we have*

$$(3.1) \quad o_n n \leq \alpha'_n(t) \leq n - o_n n, \quad t \in \mathbb{R},$$

with real numbers o_n converging to 0.

Lemma 3.3 (Negligibility Theorem for Higher Derivatives). *Suppose (P_n) is an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Then, with the notation (1.3), for every integer $r \geq 2$, we have*

$$\max_{0 \leq t \leq 2\pi} |\alpha_n^{(r)}(t)| \leq \gamma_{n,r} n^r$$

with real numbers $\gamma_{n,r} > 0$ converging to 0 for every fixed $r = 2, 3, \dots$

Our next lemma is a special case of Lemma 4.2 from [Er1].

Lemma 3.4. *Suppose (P_n) is an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Using notation (1.3), we have*

$$\max_{0 \leq t \leq 2\pi} |R'_n(t)| = \varphi_n n^{3/2}, \quad n = 1, 2, \dots,$$

with real numbers φ_n converging to 0.

The next lemma follows simply from the ultraflatness property given by Definition 1.3.

Lemma 3.5. *Let $q \in (0, \infty)$. Using the notation (1.3) and*

$$\beta_n(t) := \frac{1}{2}(\alpha_n(t) - \alpha_n^*(t)) = \alpha_n(t) - \frac{nt}{2} - t_0$$

we have

$$\frac{1}{2\pi} \int_0^{2\pi} |(P_n - P_n^*)(e^{it})|^q dt = \int_0^{2\pi} \left| n^{1/2}(1 + \delta_n(t))2 \sin(\beta_n(t)) \right|^q dt$$

with real numbers $\delta_n(t)$ converging to 0 uniformly on $[0, 2\pi]$.

Lemma 3.5*. *Let (P_n) be an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$, $q \in (0, \infty)$, and $f_n(t) := \operatorname{Re}(P_n(e^{it}))$. Using the notation (1.3) we have*

$$\frac{1}{2\pi} \int_0^{2\pi} |f_n(t)|^q dt = \int_0^{2\pi} \left| n^{1/2}(1 + \delta_n(t)) \cos(\alpha_n(t)) \right|^q dt$$

with real numbers $\delta_n(t)$ converging to 0 uniformly on $[0, 2\pi]$.

The next lemma follows simply from the ultraflatness property given by Definition 1.3 and Lemma 3.4.

Lemma 3.6. *Let $q \in (0, \infty)$. Using the notation (1.3) and*

$$\beta_n(t) := \frac{1}{2}(\alpha_n(t) - \alpha_n^*(t)) = \alpha_n(t) - \frac{nt}{2} - t_0$$

we have

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{d}{dt} |(P_n - P_n^*)(e^{it})| \right|^q dt \\ &= \int_0^{2\pi} \left| n^{1/2}(1 + \delta_n(t))2 \cos(\beta_n(t))\beta'_n(t) + \eta_n(t)n^{3/2} \right|^q dt \end{aligned}$$

with real numbers $\delta_n(t)$ and $\eta_n(t)$ converging to 0 uniformly on $[0, 2\pi]$.

Lemma 3.6*. *Let (P_n) be an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$, $q \in (0, \infty)$, and $f_n(t) := \operatorname{Re}(P_n(e^{it}))$. Using the notation (1.3) we have*

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} |f'_n(t)|^q dt \\ &= \int_0^{2\pi} \left| n^{1/2}(1 + \delta_n(t)) \sin(\alpha_n(t)) \alpha'_n(t) + \eta_n(t) n^{3/2} \right|^q dt \end{aligned}$$

with real numbers $\delta_n(t)$ converging to 0 uniformly on $[0, 2\pi]$.

To prove Lemmas 3.8 and 3.9 we need the technical lemma below that follows by a simple calculation using formulas (6.2.1), (6.2.2), and (6.1.8) on pages 258 and 255 in [AS].

Lemma 3.7. *Assume that $A, B \in \mathbb{R}$, $B \neq 0$, $q > 0$, and $I \subset [0, 2\pi]$ is an interval. Then*

$$\int_I |\cos(Bt + A)|^q dt = K(q) \operatorname{meas}(I) + \delta_1(A, B, q)$$

and

$$\int_I |\sin(Bt + A)|^q dt = K(q) \operatorname{meas}(I) + \delta_2(A, B, q),$$

where

$$K(q) := \frac{1}{2\pi} \int_0^{2\pi} |\sin t|^q dt = \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2} + 1\right) \sqrt{\pi}}$$

and

$$|\delta_1(A, B, q)| \leq \pi |B|^{-1} \quad \text{and} \quad |\delta_2(A, B, q)| \leq \pi |B|^{-1}.$$

Our next lemma takes care of the most difficult part of the proof of Theorem 2.1.

Lemma 3.8. *Suppose that β_n , $n = 1, 2, \dots$, are real-valued functions defined on $[0, 2\pi]$ such that their second derivatives β_n'' are continuous on $[0, 2\pi]$. Suppose also that*

$$(3.2) \quad \operatorname{meas}(\{t \in [0, 2\pi] : |2\beta'_n(t)| \leq nx\}) = \gamma(x) + \gamma_n(x), \quad x \in [0, 1],$$

where

$$(3.3) \quad \lim_{x \rightarrow 0^+} \gamma(x) = \lim_{n \rightarrow \infty} \max_{x \in [0, 1]} |\gamma_n(x)| = 0,$$

and

$$(3.4) \quad \max_{0 \leq t \leq 2\pi} |\beta_n''(t)| \leq \gamma_{n,2} n^2$$

with real numbers $\gamma_{n,2} > 0$ converging to 0. Then

$$(3.5) \quad \frac{1}{2\pi} \int_0^{2\pi} |\sin(\beta_n(t))|^q dt \sim K(q) := \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2} + 1\right) \sqrt{\pi}}.$$

Proof of Lemma 3.8. Let $\varepsilon > 0$ be fixed. Let $L_n := \gamma_{n,2}^{-1/4}$. We divide the interval $[0, 2\pi)$ into subintervals

$$I_m := [a_{m-1}, a_m) := \left[\frac{(m-1)L_n}{n}, \frac{mL_n}{n} \right), \quad m = 1, 2, \dots, N-1 := \left\lfloor \frac{2\pi n}{L_n} \right\rfloor,$$

and

$$I_N := [a_{N-1}, a_N) := \left[\frac{(N-1)L_n}{n}, 2\pi \right).$$

For the sake of brevity let

$$A_{m-1} := \beta_n(a_{m-1}), \quad m = 1, 2, \dots, N,$$

and

$$B_{m-1} := \beta'_n(a_{m-1}), \quad m = 1, 2, \dots, N.$$

Using Taylor's Theorem and assumption (3.4) we obtain that

$$|\beta_n(t) - (A_{m-1} + B_{m-1}(t - a_{m-1}))| \leq \gamma_{n,2} n^2 (L_n/n)^2 \leq \gamma_{n,2} \gamma_{n,2}^{-1/2} \leq \gamma_{n,2}^{1/2}$$

for every $t \in I_m$, where $\lim_{n \rightarrow \infty} \gamma_{n,2}^{1/2} = 0$. Hence the functions

$$G_{n,q}(t) := \begin{cases} |\sin(A_0 + B_0(t - a_0))|^q, & t \in I_1, \\ |\sin(A_1 + B_1(t - a_1))|^q, & t \in I_2, \\ \vdots & \vdots \\ |\sin(A_{N-1} + B_{N-1}(t - a_{N-1}))|^q, & t \in I_N, \end{cases}$$

and

$$F_{n,q}(t) := |\sin(\beta_n(t))|^q$$

satisfy

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, 2\pi)} |G_{n,q}(t) - F_{n,q}(t)| = 0.$$

Therefore, in order to prove (3.5), it is sufficient to prove that

$$(3.6) \quad \int_0^{2\pi} G_{n,q}(t) dt \sim 2\pi K(q).$$

If $|B_{m-1}| \geq n\varepsilon$, then Lemma 3.7 gives

$$\left| \int_{I_m} G_{n,q}(t) dt - K(q) \text{meas}(I_m) \right| \leq \frac{\pi}{n\varepsilon}.$$

By assumption (3.4) we have $\lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} \gamma_{n,2}^{-1/4} = \infty$, and hence

$$(3.7) \quad \left| \sum_m \int_{I_m} G_{n,q}(t) dt - K(q) \sum_m \text{meas}(I_m) \right| \leq N \frac{\pi}{n\varepsilon} \leq \left(\frac{2\pi n}{L_n} + 1 \right) \frac{\pi}{n\varepsilon} \leq \eta_n(\varepsilon),$$

where the summation is taken over all $m = 1, 2, \dots, N$ for which $|B_{m-1}| \geq n\varepsilon$, and where $(\eta_n(\varepsilon))$ is a sequence of real numbers tending to 0. Now let

$$E_{n,\varepsilon} := \bigcup_{m: |B_{m-1}| \leq n\varepsilon} I_m.$$

If $|B_{m-1}| \leq n\varepsilon$, then by assumption (3.4) we have

$$|\beta'_n(t)| \leq |B_{m-1}| + \frac{L_n}{n} \max_{t \in I_m} |\beta''_n(t)| \leq |B_{m-1}| + \frac{\gamma_{n,2}^{-1/4}}{n} \gamma_{n,2} n^2 \leq 2n\varepsilon$$

for every $t \in I_m$ if n is sufficiently large. So

$$E_{n,\varepsilon} \subset \{t \in [0, 2\pi] : |\beta'_n(t)| \leq 2n\varepsilon\}$$

for every sufficiently large n . Hence, by assumptions (3.2) we have

$$\text{meas}(E_{n,\varepsilon}) \leq \gamma(4\varepsilon) + \gamma_n(4\varepsilon)$$

for every sufficiently large n . Combining this with $0 \leq G_{n,q}(t) \leq 1$, $t \in [0, 2\pi]$, we obtain

$$(3.8) \quad \left| \sum_m \int_{I_m} G_{n,q}(t) dt - K(q) \sum_m \text{meas}(I_m) \right| \leq (\gamma(4\varepsilon) + \gamma_n(4\varepsilon))(1 + K(q)),$$

for every sufficiently large n , where the summation is taken over all $m = 1, 2, \dots, N$ for which $|B_{m-1}| < n\varepsilon$, and where $\lim_{\varepsilon \rightarrow 0^+} \gamma(4\varepsilon) = 0$ and $\lim_{n \rightarrow \infty} \gamma_n(4\varepsilon) = 0$ by assumption (3.3). Since $\varepsilon > 0$ is arbitrary, (3.6) follows from (3.7) and (3.8). As we have already pointed out (3.5) follows from (3.6). \square

Our final lemma takes care of the most difficult part of the proof of Theorem 2.4.

Lemma 3.9. *Suppose that β_n , $n = 1, 2, \dots$, are real-valued functions defined on $[0, 2\pi]$ such that their second derivatives β''_n are continuous on $[0, 2\pi]$ and*

$$(3.9) \quad \text{meas}(\{t \in [0, 2\pi] : |2\beta'_n(t)| \leq nx\}) = 2\pi x + \gamma_n(x), \quad x \in [0, 1],$$

where (3.3) holds (with $\gamma(x) := 2\pi x$). Suppose also that (3.4) holds and

$$(3.10) \quad \max_{t \in [0, 2\pi]} |\beta'_n(t)| \leq cn$$

with an absolute constant $c > 0$. Then

$$(3.11) \quad \frac{1}{2\pi} \int_0^{2\pi} |\cos(\beta_n(t))n^{-1}\beta'_n(t)|^q dt \sim \frac{K(q)}{2^q(q+1)} := \frac{\Gamma(\frac{q+1}{2})}{2^q(q+1)\Gamma(\frac{q}{2}+1)\sqrt{\pi}}.$$

We note that conditions (3.9), (3.3), and (3.10) imply in a standard fashion that

$$(3.12) \quad \frac{1}{2\pi} \int_0^{2\pi} |2\beta'_n(t)|^q dt = \frac{n^q}{q+1} + \delta_{n,q}n^q$$

with real numbers $\delta_{n,q}$ converging to 0 for every fixed $q > 0$.

Proof of Lemma 3.9. Let $\varepsilon > 0$ be fixed. Let $L_n := \gamma_{n,2}^{-1/4}$ be the same as in Lemma 3.8. Let the intervals $I_m := [a_{m-1}, a_m)$ and the numbers A_{m-1} and B_{m-1} , $m = 1, 2, \dots, N$, be the same as in the proof of Lemma 3.8. We define

$$(3.13) \quad \begin{aligned} F_{n,q}(t) &:= |\cos(\beta_n(t))|^q, \\ \tilde{F}_{n,q}(t) &:= F_{n,q}(t) |n^{-1}\beta'_n(t)|^q, \\ G_{n,q}(t) &:= \begin{cases} |\cos(A_0 + B_0(t - a_0))|^q, & t \in I_1, \\ |\cos(A_1 + B_1(t - a_1))|^q, & t \in I_2, \\ \vdots & \vdots \\ |\cos(A_{N-1} + B_{N-1}(t - a_{N-1}))|^q, & t \in I_N, \end{cases} \\ H_{n,q}(t) &:= \begin{cases} |n^{-1}B_0|^q, & t \in I_1, \\ |n^{-1}B_1|^q, & t \in I_2, \\ \vdots & \vdots \\ |n^{-1}B_{N-1}|^q, & t \in I_N, \end{cases} \end{aligned}$$

and

$$(3.14) \quad \tilde{G}_{n,q}(t) := G_{n,q}(t)H_{n,q}(t).$$

Similarly to the corresponding argument in the proof of Lemma 3.8, we obtain

$$(3.15) \quad \lim_{n \rightarrow \infty} \sup_{t \in [0, 2\pi)} |G_{n,q}(t) - F_{n,q}(t)| = 0.$$

It follows from assumption (3.4) that

$$\begin{aligned} \left| |n^{-1}\beta'_n(t)| - |n^{-1}B_{m-1}| \right| &= \left| |n^{-1}\beta'_n(t)| - |n^{-1}\beta'_n(a_{m-1})| \right| \\ &\leq |n^{-1}\beta'_n(t) - n^{-1}\beta'_n(a_{m-1})| \\ &\leq \frac{L_n}{n} \max_{t \in I_m} |n^{-1}\beta''_n(t)| \leq \frac{\gamma_{n,2}^{-1/4}}{n} n^{-1} \gamma_{n,2} n^2 = \gamma_{n,2}^{3/4} \end{aligned}$$

for every $t \in I_m$, where $\lim_{n \rightarrow \infty} \gamma_{n,2}^{3/4} = 0$. Hence

$$(3.16) \quad \lim_{n \rightarrow \infty} \sup_{t \in [0, 2\pi)} \left| H_{n,q}(t) - |n^{-1} \beta'_n(t)|^q \right| = 0.$$

Observe that

$$(3.17) \quad \sup_{t \in [0, 2\pi)} |\cos(\beta_n(t))|^q \leq 1,$$

and by assumption (3.10) we have

$$(3.18) \quad \sup_{t \in [0, 2\pi)} |n^{-1} \beta'_n(t)|^q \leq c^q.$$

Now (3.13)–(3.18) imply

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, 2\pi)} |\tilde{G}_{n,q}(t) - \tilde{F}_{n,q}(t)| = 0.$$

Therefore, in order to prove (3.11), it is sufficient to prove that

$$(3.19) \quad \int_0^{2\pi} \tilde{G}_{n,q}(t) dt \sim \frac{2\pi K(q)}{q+1}.$$

As a special case of (3.18), we have

$$(3.20) \quad |n^{-1} B_{m-1}|^q \leq c^q, \quad m = 1, 2, \dots, N.$$

If $|B_{m-1}| \geq n\varepsilon$, then (3.14), (3.20), and Lemma 3.7 give that

$$\left| \int_{I_m} \tilde{G}_{n,q}(t) dt - K(q) \text{meas}(I_m) |n^{-1} B_{m-1}|^q \right| \leq c^q \frac{\pi}{n\varepsilon}.$$

It follows from assumption (3.4) that $\lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} \gamma_{n,2}^{-1/4} = \infty$, and hence

$$(3.21) \quad \begin{aligned} \left| \sum_m \int_{I_m} \tilde{G}_{n,q}(t) dt - K(q) \sum_m \text{meas}(I_m) |n^{-1} B_{m-1}|^q \right| &\leq N c^q \frac{\pi}{n\varepsilon} \\ &\leq c^q \left(\frac{2\pi n}{L_n} + 1 \right) \frac{\pi}{n\varepsilon} \\ &\leq \eta_n(\varepsilon, q), \end{aligned}$$

where the summation is taken over all $m = 1, 2, \dots, N$ for which $|B_{m-1}| \geq n\varepsilon$, and where $(\eta_n(\varepsilon, q))$ is a sequence of real numbers tending to 0 for every fixed $\varepsilon > 0$ and $q > 0$. Now let

$$E_{n,\varepsilon} := \bigcup_{m: |B_{m-1}| \leq n\varepsilon} I_m.$$

As in the proof of Lemma 3.8 we have

$$\text{meas}(E_{n,\varepsilon}) \leq 8\pi\varepsilon + \gamma_n(4\varepsilon),$$

for every sufficiently large n . Combining this with $0 \leq G_{n,q}(t) \leq 1$, $t \in [0, 2\pi)$, (3.14), and (3.20), we obtain

$$(3.22) \quad \left| \sum_m \int_{I_m} \tilde{G}_{n,q}(t) dt - K(q) \sum_m \text{meas}(I_m) |n^{-1}B_{m-1}|^q \right| \leq (8\pi\varepsilon + \gamma_n(4\varepsilon))c^q(1+K(q))$$

for every sufficiently large n , where the summation is taken over all $m = 1, 2, \dots, N$ for which $|B_{m-1}| < n\varepsilon$, and where

$$\lim_{n \rightarrow \infty} \gamma_n(4\varepsilon) = 0$$

by assumption (3.3). Since $\varepsilon > 0$ is arbitrary, from (3.21) and (3.22) we obtain that

$$(3.23) \quad \int_0^{2\pi} \tilde{G}_{n,q}(t) dt \sim K(q) \int_0^{2\pi} H_{n,q}(t) dt.$$

However, (3.16) and (3.12) imply that

$$(3.24) \quad \int_0^{2\pi} H_{n,q}(t) dt \sim n^{-q} \int_0^{2\pi} |\beta'_n(t)|^q dt \sim \frac{2\pi}{2^q(q+1)}.$$

The statement under (3.19) now follows by combining (3.23), and (3.24). As we have remarked before, (3.19) implies (3.11). \square

4. PROOFS OF THEOREMS 2.1–2.7

Proof of Theorem 2.1. Using the notation (1.3) observe that (1.5) implies that the functions

$$(4.1) \quad \beta_n(t) := \frac{1}{2}(\alpha_n(t) - \alpha_n^*(t)) = \alpha_n(t) - \frac{nt}{2} - t_0$$

satisfy

$$\beta'_n(t) = \alpha'_n(t) - n/2, \quad t \in \mathbb{R},$$

and

$$\beta''_n(t) = \alpha''_n(t), \quad t \in \mathbb{R},$$

and hence Lemmas 3.1, 3.2, and 3.3 imply that the functions β_n satisfy assumptions (3.2), (3.3), and (3.4) of Lemma 3.8. Hence the theorem follows from Lemmas 3.5 and 3.8. \square

Proof of Theorem 2.3. Using the notation (1.3) let

$$P_n(e^{it}) = R_n(t)e^{i\alpha_n(t)} \quad \text{and} \quad P_n^*(e^{it}) = R_n^*(t)e^{i\alpha_n^*(t)},$$

where

$$R_n(t) = |P_n(e^{it})| = |P_n^*(e^{it})| = R_n^*(t),$$

and both R_n and α_n are in $C^\infty(\mathbb{R})$. Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. We have

$$(4.2) \quad R_n(t)^2 = n(1 + \delta_n(t)), \quad t \in \mathbb{R}, \quad \lim_{n \rightarrow \infty} \max_{t \in [0, 2\pi]} |\delta_n(t)| = 0.$$

Let $\beta_n(t)$ be defined by (4.1). Let $\varepsilon > 0$ be fixed. Using the cosine theorem for triangles, Lemmas 3.2, and 3.4, and (4.1), (4.2), and (1.5), we obtain

$$(4.3) \quad \begin{aligned} & |(P'_n - P_n^{*'})'(e^{it})|^2 - |P'_n(e^{it})|^2 - |P_n^{*'}(e^{it})|^2 \\ &= 2((R_n(t)\alpha'_n(t))^2 + (R'_n(t))^2)^{1/2}(R_n(t)(\alpha_n^{*'}(t))^2 + (R'_n(t))^2)^{1/2} \cos(2\beta_n(t)) \\ &= 2(R_n(t)\alpha'_n(t))(R_n(t)\alpha_n^{*'}(t)) \cos(2\beta_n(t)) + \eta_n(t)n^3 \\ &= 2(R_n(t)^2(n\alpha'_n(t) - \alpha'_n(t)^2) \cos(2\beta_n(t)) + \eta_n(t)n^3 \\ &= 2(n + \delta_n(t))(n\alpha'_n(t) - \alpha'_n(t)^2) \cos(2\beta_n(t)) + \eta_n(t)n^3 \\ &= 2n(n\alpha'_n(t) - \alpha'_n(t)^2) \cos(2\beta_n(t)) + \varphi_n(t)n^3 + \eta_n(t)n^3 \\ &= 2n \left(\frac{n^2}{4} - \beta'_n(t)^2 \right) \cos(2\beta_n(t)) + \varphi_n(t)n^3 + \eta_n(t)n^3 \end{aligned}$$

with some real numbers $\eta_n(t)$ and $\varphi_n(t)$ satisfying

$$\max_{t \in [0, 2\pi]} |\varphi_n(t) + \eta_n(t)| < \varepsilon$$

for every sufficiently large n . Observe that

$$(4.4) \quad \int_0^{2\pi} \left(|P'_n(e^{it})|^2 + |P_n^{*'}(e^{it})|^2 \right) dt = 4\pi \frac{n(n+1)(2n+1)}{6}$$

by the Parseval formula. The integration by parts formula and Lemma 3.3 give that

$$(4.5) \quad \begin{aligned} & \left| \int_0^{2\pi} \beta'_n(t)^2 \cos(2\beta_n(t)) dt \right| \\ &= \left| \left[\frac{1}{2} (\sin(2\beta_n(t)) \beta'_n(t)) \right]_0^{2\pi} - \int_0^{2\pi} \beta''_n(t) \sin(2\beta_n(t)) dt \right| \\ &= \left| \int_0^{2\pi} \beta''_n(t) \sin(2\beta_n(t)) dt \right| \leq \int_0^{2\pi} |\beta''_n(t)| dt = \int_0^{2\pi} |\alpha''_n(t)| dt \\ &\leq 2\pi \gamma_{n,2} n^2 < \varepsilon n^2 \end{aligned}$$

for every sufficiently large n . Observe also that Lemma 3.8 gives that

$$(4.6) \quad \int_0^{2\pi} \cos(2\beta_n(t)) dt = \int_0^{2\pi} (2\sin^2(\beta_n(t)) - 1) dt = 2\pi(2K(2) + h_n - 1) \\ = 2\pi \left(2 \frac{\Gamma(\frac{3}{2})}{\Gamma(2)\sqrt{\pi}} + h_n - 1 \right) = 2\pi h_n$$

with a sequence (h_n) converging to 0. Combining (4.3)–(4.6) we conclude

$$\left| \frac{1}{2\pi} \int_0^{2\pi} |(P'_n - P_n^{*'}) (e^{it})|^2 dt - \frac{2n^3}{3} \right| \leq \varepsilon n^3$$

for every sufficiently large n . As $\varepsilon > 0$ is arbitrary, this finishes the proof. \square

Proof of Theorem 2.4. Using the notation (1.3) observe that (1.5) implies that the functions β_n defined by (4.1) satisfy

$$\beta'_n(t) = \alpha'_n(t) - n/2, \quad t \in \mathbb{R},$$

and

$$\beta''_n(t) = \alpha''_n(t), \quad t \in \mathbb{R},$$

and hence Lemmas 3.1, 3.2, and 3.3 imply that the functions β_n satisfy assumptions (3.9), (3.3), (3.4), and (3.10) of Lemma 3.9. Hence the theorem follows from Lemmas 3.6 and 3.9. \square

Proof of Theorem 2.5. Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Theorem 2.2 gives that

$$\sum_{k=0}^n |a_{k,n} - \bar{a}_{n-k,n}|^2 \sim 2n,$$

which is equivalent to

$$2\operatorname{Re} \left(\sum_{k=0}^n a_{k,n} a_{n-k,n} \right) = o(n).$$

Now let $c_n \in \mathbb{C}$, $|c_n| = 1$, and let Q_n be defined by $Q_n(z) = P_n(c_n z)$. Observe that (Q_n) is an ultraflat sequence of polynomials $Q_n \in \mathcal{K}_n$ and hence

$$\operatorname{Re} \left(\sum_{k=0}^n c_n^n a_{k,n} a_{n-k,n} \right) = o(n),$$

and hence

$$\left| \sum_{k=0}^n a_{k,n} a_{n-k,n} \right| = o(n).$$

\square

Proof of Theorem 2.6. Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Theorem 2.3 gives that

$$\sum_{k=0}^n k^2 |a_{k,n} - \bar{a}_{n-k,n}|^2 \sim \frac{2n^3}{3},$$

which is equivalent to

$$2\operatorname{Re} \left(\sum_{k=0}^n k^2 a_{k,n} a_{n-k,n} \right) = o(n^3).$$

Now let $c_n \in \mathbb{C}$, $|c_n| = 1$, and let Q_n be defined by $Q_n(z) = P_n(c_n z)$. Observe that (Q_n) is an ultraflat sequence of polynomials $Q_n \in \mathcal{K}_n$ and hence

$$\operatorname{Re} \left(\sum_{k=0}^n c_n^n k^2 a_{k,n} a_{n-k,n} \right) = o(n^3),$$

and hence

$$\left| \sum_{k=0}^n k^2 a_{k,n} a_{n-k,n} \right| = o(n^3).$$

□

Proof of Theorem 2.7. Using Lemma 3.5* and then applying Lemma 3.8 with β_{2n} defined by $\beta_{2n}(t) := \alpha_n(t) + \pi/2$ we obtain the first asymptotic formula of the theorem. Using Lemma 3.6* and then applying Lemma 3.9 with β_{2n} defined by $\beta_{2n}(t) := \alpha_n(t) + \pi/2$ we obtain the second asymptotic formula of the theorem. □

REFERENCES

- [AS] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, Inc., New York, 1972.
- [Be] J. Beck, “Flat” polynomials on the unit circle – note on a problem of Littlewood, Bull. London Math. Soc. (1991), 269–277.
- [BB] E. Bombieri and J. Bourgain, *On Kahane’s ultraflat polynomials*, J. Eur. Math. Soc. **11** (2009, 3), 627–703.
- [Bo] P. Borwein, *Computational Excursions in Analysis and Number Theory*, Springer, New York, 2002.
- [BC1] P. Borwein and K.-K.S. Choi, *Merit factors of character polynomials*, J. London Math. Soc. (2) **61** (2000), no. 3, 706–720.
- [BC2] P. Borwein and K.-K.S. Choi, *Merit factors of polynomials formed by Jacobi symbols*, Canad. J. Math. **53** (2001), no. 1, 33–50.
- [BC3] P. Borwein and K.-K.S. Choi, *Explicit merit factor formulae for Fekete and Turyn polynomials*, Trans. Amer. Math. Soc. **354** (2002), no. 1, 219–234.
- [BM] P. Borwein and M. Mossinghoff, *Rudin-Shapiro-like polynomials in L_4* , Math. Comp. **69** (2000), no. 231, 1157–1166.

- [Er1] T. Erdélyi, *The phase problem of ultraflat unimodular polynomials: the resolution of the conjecture of Saffari*, Math. Ann. **300** (2000), 39–60.
- [Er2] T. Erdélyi, *How far is a sequence of ultraflat unimodular polynomials from being conjugate reciprocal*, Michigan Math. J. **49** (2001), 259–264.
- [Er3] T. Erdélyi, *The resolution of Saffari’s Phase Problem*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), 803–808.
- [Er4] T. Erdélyi, *Proof of Saffari’s near-orthogonality conjecture for ultraflat sequences of unimodular polynomials*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), 623–628.
- [Er5] T. Erdélyi, *Polynomials with Littlewood-type coefficient constraints*, in Approximation Theory X: Abstract and Classical Analysis, Charles K. Chui, Larry L. Schumaker, and Joachim Stöckler (Eds.) (2002), Vanderbilt University Press, Nashville, TN, 153–196.
- [Er6] T. Erdélyi, *On the real part of ultraflat sequences of unimodular polynomials*, Math. Ann. **326** (2003), 489–498.
- [EN] T. Erdélyi and P. Nevai, *On the derivatives of unimodular polynomials (Russian)*, Mat. Sbornik **207** (2016), no. 4, 123–142, translation in Sbornik Math. 207 (2016), no. 3–4, 590–609.
- [Er] P. Erdős, *Some unsolved problems*, Michigan Math. J. **4** (1957), 291–300.
- [JKSch] J. Jedwab, D. Katz, and K.-U. Schmidt, *Littlewood polynomials with small L_4 norm*, Adv. Math. **241** (2013), 127–136.
- [Ka] J.P. Kahane, *Sur les polynomes a coefficient unimodulaires*, Bull. London Math. Soc. **12** (1980), 321–342.
- [Kö] T. Körner, *On a polynomial of J.S. Byrnes*, Bull. London Math. Soc. **12** (1980), 219–224.
- [Li1] J.E. Littlewood, *On polynomials $\sum \pm z^m, \sum \exp(\alpha_m i) z^m, z = e^{i\theta}$* , J. London Math. Soc. **41**, 367–376, yr 1966.
- [Li2] J.E. Littlewood, *Some Problems in Real and Complex Analysis*, Heath Mathematical Monographs, Lexington, Massachusetts, 1968.
- [Mo] H.L. Montgomery, *Littlewood polynomials*, in Analytic Number Theory, Modular Forms and q-Hypergeometric Series, George Andrews and Frank Garvan (Eds.) (2017), Springer Proc. Math. Stat., vol. 221, Springer, Cham, 533–553.
- [Od] A. Odlyzko, *Search for ultraflat polynomials with plus and minus one coefficients*, in Connections in Discrete Mathematics, Steve Butler, Joshua Cooper, Glenn Hurlbert (Eds.) (2018), Cambridge Univ. Press, Cambridge, 39–55.
- [QS1] H. Queffelec and B. Saffari, *Unimodular polynomials and Bernstein’s inequalities*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995, 3), 313–318.
- [QS2] H. Queffelec and B. Saffari, *On Bernstein’s inequality and Kahane’s ultraflat polynomials*, J. Fourier Anal. Appl. **2** (1996, 6), 519–582.
- [Sa1] B. Saffari, *The phase behavior of ultraflat unimodular polynomials*, in Probabilistic and Stochastic Methods in Analysis, with Applications (1992), Kluwer Academic Publishers, Dordrecht, 555–572.
- [Sa2] B. Saffari, *Some polynomial extremal problems which emerged in the twentieth century*, in Twentieth Century Harmonic Analysis – A Celebration, James S. Byrnes (Ed.) (2001), Kluwer Academic Publishers, Dordrecht, 201–233.

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