Polynomials with Littlewood-Type Coefficient Constraints

Tamás Erdélyi

Abstract. This survey paper focuses on my contributions to the area of polynomials with Littlewood-type coefficient constraints. It summarizes the main results from many of my recent papers some of which are joint with Peter Borwein.

§1. Introduction

Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D. Let

$$\mathcal{K}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^{n} a_k z^k, \;\; a_k \in \mathbb{C}\,, \;\; |a_k| = 1 \right\} .$$

The class $\mathcal{K}_n$ is often called the collection of all (complex) unimodular polynomials of degree n. Let

$$\mathcal{L}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^{n} a_k z^k, \;\; a_k \in \{-1, 1\} \right\} .$$

The class $\mathcal{L}_n$ is often called the collection of all (real) unimodular polynomials of degree n. By Parseval's formula,

$$\int_0^{2\pi} |P_n(e^{it})|^2 \, dt = 2\pi(n+1)$$

for all $P_n \in \mathcal{K}_n$. Therefore

$$\min_{z \in \partial D} |P_n(z)| \leq \sqrt{n+1} \leq \max_{z \in \partial D} |P_n(z)| \tag{1.1}$$

for all $P_n \in \mathcal{K}_n$. An old problem (or rather an old theme) is the following.

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Problem 1.1. (Littlewood's Flatness Problem). *How close can a unimodular polynomial $P_n \in \mathcal{K}_n$ or $P_n \in \mathcal{L}_n$ come to satisfying*

$$|P_n(z)| = \sqrt{n+1}\,, \qquad z \in \partial D\,? \tag{1.2}$$

Obviously (1.2) is impossible if $n \geq 1$, so one must look for less than (1.2). But then there are various ways of seeking such an "approximate situation". One way is the following. In his paper [51], Littlewood had suggested that, conceivably, there might exist a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ (possibly even $P_n \in \mathcal{L}_n$) such that $(n+1)^{-1/2}|P_n(e^{it})|$ converge to 1 uniformly in $t \in \mathbb{R}$. We shall call such sequences of unimodular polynomials "ultraflat". More precisely, we give the following definitions. In the rest of the paper, we assume that (n_k) is a strictly increasing sequence of positive integers.

Definition 1.2. *Given a positive number ε, we say that a polynomial $P_n \in \mathcal{K}_n$ is ε-*flat *if*

$$(1-\varepsilon)\sqrt{n+1} \leq |P_n(z)| \leq (1+\varepsilon)\sqrt{n+1}\,, \qquad z \in \partial D\,, \tag{1.3}$$

or equivalently

$$\max_{z \in \partial D} \big||P_n(z)| - \sqrt{n+1}\big| \leq \varepsilon\sqrt{n+1}\,.$$

Definition 1.3. *Given a sequence (ε_{n_k}) of positive numbers tending to 0, we say that a sequence (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is (ε_{n_k})-*ultraflat *if each P_{n_k} is ε_{n_k}-flat, that is*

$$(1-\varepsilon_{n_k})\sqrt{n_k+1} \leq |P_{n_k}(z)| \leq (1+\varepsilon_{n_k})\sqrt{n_k+1}\,, \qquad z \in \partial D\,. \tag{1.4}$$

We say that a sequence (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is ultraflat *if there is a sequence (ε_{n_k}) of positive numbers tending to 0 for which (P_{n_k}) is (ε_{n_k})-ultraflat.*

The existence of an ultraflat sequence of unimodular polynomials seemed very unlikely, in view of a 1957 conjecture of P. Erdős (Problem 22 in [28]) asserting that, for all $P_n \in \mathcal{K}_n$ with $n \geq 1$,

$$\max_{z \in \partial D} |P_n(z)| \geq (1+\varepsilon)\sqrt{n+1}\,, \tag{1.5}$$

where $\varepsilon > 0$ is an absolute constant (independent of n). Yet, combining some probabilistic lemmas from Körner's paper [47] with some constructive methods (Gauss polynomials, etc., which are completely unrelated to the deterministic part of Körner's paper), Kahane [43] proved that there exists a sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ which is (ε_n)-ultraflat, where

$$\varepsilon_n = O\left(n^{-1/17}\sqrt{\log n}\right)\,. \tag{1.6}$$

Thus the Erdős conjecture (1.5) was disproved for the classes $\mathcal{K}_n$. For the more restricted class $\mathcal{L}_n$, the analogous Erdős conjecture is unsettled to this date.

It is a common belief that the analogous Erdős conjecture for $\mathcal{L}_n$ is true, and consequently there is no ultraflat sequence of unimodular polynomials $P_n \in \mathcal{L}_n$. I thank H. Queffelec for providing more details about the existence of ultraflat sequences (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$. The story is roughly the following.

Littlewood [51] had constructed polynomials $P_n \in \mathcal{K}_n$ so that on one hand $|P_n(z)| \leq B\sqrt{n+1}$ for every $z \in \partial D$, and on the other hand $|P_n(z)| \geq A\sqrt{n+1}$ with an absolute constant $A > 0$ for every $z \in \partial D$ except for a small arc. In the light of this result he asked how close we can get to satisfying $|P_n(z)| = \sqrt{n+1}$ for every $z \in \partial D$ if $P_n \in \mathcal{K}_n$. The first result in this direction is due to Körner [47]. By using a result of Byrnes, he showed that there are absolute constants $0 < A < B$ such that $A\sqrt{n+1} \leq |P_n(z)| \leq B\sqrt{n+1}$ for every $z \in \partial D$. Then Kahane [43] constructed a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ for which

$$(1-\varepsilon_n)\sqrt{n+1} \leq |P_n(z)| \leq (1+\varepsilon_n)\sqrt{n+1}\,, \qquad z \in \partial D\,,$$

with a sequence (ε_n) of positive real numbers converging to 0. Such a sequence is called (ε_n)-ultraflat.

Kahane's construction seemed to indicate a very rigid behavior for the phase function α_n, where

$$P_n(e^{it}) = R_n(t)e^{i\alpha_n(t)}\,, \qquad R_n(t) = |P_n(e^{it})|\,.$$

Saffari [66] had conjectured in 1991 that for every ultraflat sequence (P_n), $\alpha_n'(t)/n$ converges in measure to the uniform distribution on $[0,1]$, that is,

$$m\,\{t \in [0,2\pi] : 0 \leq \alpha_n'(t) \leq nx\} \to 2\pi x\,, \qquad 0 \leq x \leq 1\,, \tag{1.7}$$

where m is the Lebesgue measure on the Borel subsets of $[0,2\pi)$. Since it can be seen easily that $X_n := \alpha_n'(t)/n$ is uniformly bounded, the method of moments applies and everything could be obtained from

$$\int_0^1 X_n^q(t)\,dt = \frac{1}{q+1} + o_{n,q}\,, \qquad q = 0,1,\ldots\,, \tag{1.8}$$

where the numbers $o_{n,q}$ converge to 0 for every fixed q as $n \to \infty$. This was proved by Saffari [66] for $q = 0,1,2$. Then in 1996 Queffelec and Saffari [65] used Kahane's method with a slight modification to show the existence of an ultraflat sequence (P_n) which satisfies (1.7). They also showed that (1.8) is true for $q = 3$ (and almost for $q = 4$) for any ultraflat sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$. When their work was submitted to Journal of Fourier Analysis and Applications, the editor in chief, J. Benedetto, and one of his students discovered an error in Byrnes work which, as a result, invalidated Körner's work. It was discovered that the deterministic part of Körner's [47] work was incorrect, and it was based on the incorrect "Theorem 2" of Byrnes' paper [21]. For details of the story see the forthcoming paper by J.S. Byrnes and Saffari [22].

Fortunately Kahane's work was independent of Byrnes'. It contained though an other slight error which was corrected in [65]. Ultraflat sequences (P_n) of polynomials $P_n \in \mathcal{K}_n$ do exist! It is important to note this, otherwise the work of a number of papers would be without object. In [32] we answer Saffari's Problem affirmatively, namely we show that (1.7) (or equivalently (1.8)) is true for every ultraflat sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$.

An interesting related result to Kahane's breakthrough is given by Beck [4]. He proved that for every sufficiently large integer k (he states the result for $k = 400$) there are polynomials P_n of degree n $(n = 1, 2, \ldots)$ so that each coefficient of each P_n is a k-th root of unity, and with some absolute constants $c_1, c_2 > 0$ we have

$$c_1\sqrt{n} \le |P_n(e^{it})| \le c_2\sqrt{n}\,, \qquad t \in \mathbb{R}\,, \quad n = 1, 2, \ldots\,.$$

For an account of some of the work done till the mid 1960's, see Littlewood's book [52] and [65].

§2. On the Phase Problem of Saffari

Let (P_n) be an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. We write

$$P_n(e^{it}) = R_n(t)e^{i\alpha_n(t)}\,, \qquad R_n(t) = |P_n(e^{it})|\,. \tag{2.1}$$

It is a simple exercise to show that α_n can be chosen to be an element of $C^\infty(\mathbb{R})$. This is going to be our understanding throughout this section. The following result was conjectured by Saffari [66] and proved in [32]:

Theorem 2.1. (Uniform Distribution Theorem for the Angular Speed.) *Let (P_n) be an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Then, with the notation (2.1), in the interval $[0, 2\pi]$, the distribution of the normalized angular speed $\alpha_n'(t)/n$ converges to the uniform distribution as $n \to \infty$. More precisely, we have*

$$\mathrm{m}\{t \in [0, 2\pi] : 0 \le \alpha_n'(t) \le nx\} = 2\pi x + o_n(x) \tag{2.2}$$

for every $x \in [0, 1]$, where $\lim_{n\to\infty} o_n(x) = 0$ for every $x \in [0, 1]$, As a consequence, $|P_n'(e^{it})|/n^{3/2}$ also converges to the uniform distribution as $n \to \infty$. More precisely, we have

$$\mathrm{m}\{t \in [0, 2\pi] : 0 \le |P_n'(e^{it})| \le n^{3/2}x\} = 2\pi x + o_n(x) \tag{2.3}$$

for every $x \in [0, 1]$, where $\lim_{n\to\infty} o_n(x) = 0$ for every $x \in [0, 1]$. In both statements the convergence of $o_n(x)$ is uniform on $[0, 1]$.

The basis of Saffari's conjecture was that for the special ultraflat sequences of unimodular polynomials produced by Kahane [43], (2.6) is indeed true. In Section 4 of [32] we prove this conjecture in general.

In the general case, (2.6) can, by integration, be reformulated (equivalently) in terms of the moments of the angular speed $\alpha_n'(t)$. This was observed and recorded by Saffari [66]. We present the proof of this equivalence in Section 4 of [32] and we settle Conjecture 2.1 by proving the following result.

Theorem 2.2. (Reformulation of the Uniform Distribution Conjecture.) *Let (P_n) be an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Then, for any $q > 0$ we have*

$$\frac{1}{2\pi}\int_0^{2\pi} |\alpha_n'(t)|^q \, dt = \frac{n^q}{q+1} + o_{n,q} n^q \,. \tag{2.4}$$

with suitable constants $o_{n,q}$ converging to 0 as $n \to \infty$ for every fixed $q > 0$.

An immediate consequence of (2.8) is the remarkable fact that for large values of $n \in \mathbb{N}$, the $L_q(\partial D)$ Bernstein factors

$$\frac{\int_0^{2\pi} |P_n'(e^{it})|^q \, dt}{\int_0^{2\pi} |P_n(e^{it})|^q \, dt}$$

of the elements of ultraflat sequences (P_n) of unimodular polynomials are essentially independent of the polynomials. More precisely (2.8) implies the following result.

Theorem 2.3. (The Bernstein Factors.) *Let q be an arbitrary positive real number. Let (P_n) be an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. We have*

$$\frac{\int_0^{2\pi} |P_n'(e^{it})|^q \, dt}{\int_0^{2\pi} |P_n(e^{it})|^q \, dt} = \frac{n^q}{q+1} + o_{n,q} n^q \,,$$

and as a limit case,

$$\frac{\max_{0 \le t \le 2\pi} |P_n'(e^{it})|}{\max_{0 \le t \le 2\pi} |P_n(e^{it})|} = n + o_n n \,.$$

with suitable constants $o_{n,q}$ and o_n converging to 0 as $n \to \infty$ for every fixed q.

In Section 3 of [32] we show the following result which turns out to be stronger than Theorem 2.2.

Theorem 2.4. (Negligibility Theorem for Higher Derivatives.) *Let (P_n) be an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. For every integer $r \ge 2$, we have*

$$\max_{0 \le t \le 2\pi} |\alpha_n^{(r)}(t)| \le o_{n,r} n^r$$

with suitable constants $o_{n,r} > 0$ converging to 0 for every fixed $r = 2, 3, \ldots$.

We show in Section 4 of [32] how Theorem 2.1 follows from Theorem 2.4. Finally in Section 4 of [32] we give the following extension of Theorem 2.1 (Uniform Distribution Conjecture) to higher derivatives.

Theorem 2.5. (Distribution of the Modulus of Higher Derivatives of Ultraflat Sequences of Unimodular Polynomials.) *Let (P_n) be an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Then*

$$\left(\frac{|P_n^{(r)}(e^{it})|}{n^{r+1/2}}\right)^{1/r}$$

converges to the uniform distribution as $n \to \infty$. More precisely, we have

$$\mathrm{m}\left\{t \in [0, 2\pi] : 0 \le |P_n^{(r)}(e^{it})| \le n^{r+1/2} x^r\right\} = 2\pi x + o_{r,n}(x)$$

for every $x \in [0,1]$, where $\lim_{n\to\infty} o_{r,n}(x) = 0$ for every fixed $r = 1, 2, \ldots$ and $x \in [0,1]$. The convergence of $o_n(x)$ is uniform on $[0,1]$.

Remark 2.6. Assume that (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. As before, we use notation (2.1). We denote the number of zeros of P_n inside the open unit disk D by $Z(P_n)$. We claim that

$$Z(P_n) = \frac{n}{2}(1 + o_n)\,,$$

where o_n is a sequence converging to 0 as $n \to \infty$. To see this we argue as follows. By Conjecture 2.1 (proved in [32]) we have

$$\alpha_n(2\pi) - \alpha_n(0) = \frac{1}{2}(1 + o_n)(2\pi) = (1 + o_n)n\pi$$

with constants o_n converging to 0 as $n \to \infty$. So the "Argument Principle" yields the result we stated.

For continuous functions f defined on $[0, 2\pi]$, and for $q \in (0, \infty)$, we define

$$\|f\|_q := \left(\int_0^{2\pi} |f(t)|^q \, dt\right)^{1/q} .$$

We also define

$$\|f\|_\infty := \lim_{q\to\infty} \|f\|_q = \max_{t\in[0,2\pi]} |f(t)|\,.$$

In [65] the following conjecture is made.

Conjecture 2.7. *Assume that (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$ and $f_n(t) = \mathrm{Re}(P_n(e^{it}))$. Let $q \in (0, \infty)$. Then*

$$\|f_n\|_q \sim \left(\frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}}\right)^{1/q} n^{1/2}\,,$$

and

$$\|f_n'\|_q \sim \left(\frac{\Gamma\left(\frac{q+1}{2}\right)}{(q+1)\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}}\right)^{1/q} n^{3/2}\,,$$

where Γ denotes the usual gamma function and the $\sim$ symbol means that the ratio of the left and right hand sides converges to 1 as $n \to \infty$.

The above conjecture follows from Theorems 2.1 and 2.4. The arguments will be presented in my forthcoming paper [36].

§3. On Saffari's Near Orthogonality Conjectures

The structure of ultraflat sequences of unimodular polynomials is studied in [32] and [34] where several conjectures of Saffari are proved. In [35], based on the results in [32], we proved yet another Saffari conjecture formulated in [66].

Theorem 3.1. (Saffari's Near-Orthogonality Conjecture.) *Assume that* (P_n) *is an ultraflat sequence of unimodular polynomials* $P_n \in \mathcal{K}_n$. *Let*

$$P_n(z) := \sum_{k=0}^{n} a_{k,n} z^k \,.$$

Then

$$\sum_{k=0}^{n} a_{k,n} a_{n-k,n} = o(n)\,.$$

Here, as usual, $o(n)$ *denotes a quantity for which* $\lim_{n\to\infty} o(n)/n = 0$. *The statement remains true if the ultraflat sequence* (P_n) *of unimodular polynomials* $P_n \in \mathcal{K}_n$ *is replaced by an ultraflat sequence* (P_{n_k}) *of unimodular polynomials* $P_{n_k} \in \mathcal{K}_{n_k}$, $0 < n_1 < n_2 < \cdots$.

If Q_n is a polynomial of degree n of the form $Q_n(z) = \sum_{k=0}^{n} a_k z^k$, $a_k \in \mathbb{C}$, then its conjugate reciprocal polynomial is defined by $Q_n^*(z) := z^n \overline{Q}_n(1/z) := \sum_{k=0}^{n} \overline{a}_{n-k} z^k$. In terms of the above definition Theorem 1.4 may be rewritten as

Corollary 3.2. *Assume that* (P_n) *is an ultraflat sequence of unimodular polynomials* $P_n \in \mathcal{K}_n$. *Then*

$$\int_{\partial D} |P_n(z) - P_n^*(z)|^2 \, |dz| = 2n + o(n)\,.$$

Remark 3.3. Theorem 3.1 clearly shows that there is no ultraflat sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ that are conjugate reciprocal. Otherwise, using the fact that $a_{k,n} = \overline{a}_{n-k,n}$, we would have

$$\sum_{k=0}^{n} a_{k,n} a_{n-k,n} = \sum_{k=0}^{n} |a_{k,n}|^2 = n+1,$$

which contradicts Theorem 3.1. In fact, Theorem 3.1 tells us much more. It measures how far is an ultraflat sequence of unimodular polynomials is from being conjugate reciprocal.

Remark 3.4. In [66] another "near orthogonality" relation has been conjectured. Namely it was suspected that if (P_{n_m}) is an ultraflat sequence of unimodular polynomials $P_{n_m} \in \mathcal{K}_{n_m}$ and

$$P_n(z) := \sum_{k=0}^{n} a_{k,n} z^k\,, \qquad n = n_m\,, \qquad m = 1,2,\dots\,,$$

then

$$\sum_{k=0}^{n} a_{k,n}\overline{a}_{n-k,n} = o(n)\,, \qquad n = n_m\,, \qquad m = 1,2,\dots\,,$$

where, as usual, $o(n_m)$ denotes a quantity for which $\lim_{n_m\to\infty} o(n_m)/n_m = 0$. However, it was Saffari himself, together with Queffelec [65], who showed that this could not be any farther away from being true. Namely they constructed an ultraflat sequence (P_{n_m}) of plain-reciprocal unimodular polynomials $P_{n_m} \in \mathcal{K}_{n_m}$ such that

$$P_n(z) := \sum_{k=0}^{n} a_{k,n} z^k\,, \qquad a_{k,n} = a_{n-k,n}\,, \qquad k = 0,1,2,\dots n\,,$$

and hence

$$\sum_{k=0}^{n} a_{k,n}\overline{a}_{n-k,n} = n+1$$

for the values $n = n_m\,,\ m = 1,2,\dots\,.$

Remark 3.5. One can ask how flat a conjugate reciprocal unimodular polynomial can be. We present a simple result here. Let $P_n \in \mathcal{K}_n$ be a conjugate reciprocal polynomial of degree n. Then

$$\max_{z\in\partial D} |P_n(z)| \geq (1+\varepsilon)\sqrt{n}$$

with $\varepsilon := \sqrt{\frac{4}{3}} - 1$. This is an observation made by Erdős [29] but his constant $\varepsilon > 0$ is unspecified.

To prove the statement, observe that Malik's inequality [57], p. 676 gives

$$\max_{z\in\partial D} |P_n'(z)| \leq \frac{n}{2} \max_{z\in\partial D} |P_n(z)|\,.$$

(Note that the fact that P_n is conjugate reciprocal improves the Bernstein factor on ∂D from n to $n/2$.) Using $P_n \in \mathcal{K}_n$, Parseval's formula, and Malik's inequality, we obtain

$$2\pi\,\frac{n^3}{3} \leq 2\pi\,\frac{n(n+1)(2n+1)}{6} = \int_{\partial D} |P_n'(z)|^2\,|dz| \leq 2\pi\left(\frac{n}{2}\right)^2 \max_{z\in\partial D} |P_n(z)|^2\,,$$

and

$$\max_{z\in\partial D} |P_n(z)| \geq \sqrt{4/3}\sqrt{n}$$

follows.

§4. Some Littlewood-type results

This section is essentially copied from [17]. We examine a number of problems concerning polynomials with coefficients restricted in various ways. We are particularly interested in how small such polynomials can be on the interval $[0,1]$. For example, in [17] we prove that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$\exp\left(-c_1\sqrt{n}\right) \le \inf_{0 \ne p \in \mathcal{F}_n} \|p\|_{[0,1]} \le \exp\left(-c_2\sqrt{n}\right)$$

for every $n \ge 2$, where $\mathcal{F}_n$ denotes the set of polynomials of degree at most n with coefficients from $\{-1,0,1\}$.

Littlewood considered minimization problems of this variety on the unit disk, hence, the title of the section. His most famous, now solved, conjecture (see [27] on pages 285 – 288) was that the L_1 norm of an element $f \in \mathcal{F}_n$ on the unit circle grows at least as fast as $c\log N$, where N is the number of non-zero coefficients in f and $c > 0$ is an absolute constant. This was proved by Konjagin [45] and independently by McGehee, Pigno, and Smith [56].

When the coefficients are required to be integers, the questions have a Diophantine nature and have been studied from a variety of points of view. See [2,3,8,18,39,61].

One key to the analysis is a study of the related problem of how large an order zero these restricted polynomials can have at 1. In [17] we answer this latter question precisely for the class of polynomials of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_j| \le 1\,, \quad a_j \in \mathbb{C}$$

with fixed $|a_0| \ne 0$.

Variants of these questions have attracted considerable study, though rarely have precise answers been possible to give. See in particular [1,7,38,68, 69,71,41,6]. Indeed the classical, much studied, and presumably very difficult problem of Prouhet, Tarry, and Escott rephrases as a question of this variety. (Precisely: what is the maximal vanishing at 1 of a polynomial with integer coefficients with l_1 norm $2n$? It is conjectured to be n. See [18] and [6].

We introduce the following classes of polynomials. Let

$$\mathcal{P}_n^c := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{C} \right\}$$

denote the set of algebraic polynomials of degree at most n with complex coefficients. Let

$$\mathcal{P}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{R} \right\}$$

denote the set of algebraic polynomials of degree at most n with real coefficients. Let

$$\mathcal{Z}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{Z} \right\}$$

denote the set of algebraic polynomials of degree at most n with integer coefficients. Let

$$\mathcal{F}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \{-1, 0, 1\} \right\}$$

denote the set of polynomials of degree at most n with coefficients from $\{-1, 0, 1\}$. Let

$$\mathcal{A}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \{0, 1\} \right\}$$

denote the set of polynomials of degree at most n with coefficients from $\{0, 1\}$. Finally, let

$$\mathcal{L}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \{-1, 1\} \right\}$$

denote the set of polynomials of degree at most n with coefficients from $\{-1, 1\}$.

So obviously

$$\mathcal{L}_n, \ \ \mathcal{A}_n \subset \mathcal{F}_n \subset \mathcal{Z}_n \subset \mathcal{P}_n \subset \mathcal{P}_n^c \,.$$

Throughout this section the uniform norm on a set $A \subset \mathbb{R}$ is denoted by $\|.\|_A$.

In his monograph [52], Littlewood discusses the class $\mathcal{L}_n$ and its complex analogue when the coefficients are complex numbers of modulus 1. On page 25 he writes "These raise fascinating questions." It is easy to see that the L_2 norm of any polynomial of degree n with complex coefficients of modulus one on the unit circle is $\sqrt{n+1}$. (Here we have normalized so that the unit circle has length 1.) Hence the minimum supremum norm of any such polynomial on the unit circle is at least $\sqrt{n+1}$.

The Rudin-Shapiro polynomials (see [51], for example) show that there are polynomials from $\mathcal{L}_n$ with maximum modulus less than $c\sqrt{n+1}$ on the unit circle. Littlewood remarks in [52] that although it has been known for more than 50 years that $g_n(\theta) := \sum_{m=0}^{n} e^{im\log m} e^{im\theta}$ satisfies $|g_n(\theta)| < c\sqrt{n+1}$ on the real line, the existence of polynomials $p_n \in \mathcal{L}_n$ with $|p_n(z)| < c\sqrt{n+1}$ on the unit circle has only fairly recently been shown. He adds "As a matter of cold fact, many people had doubted its truth." Rudin and Shapiro had the following simple idea:

$$\begin{aligned} P_0(z) &= Q_0(z) = 1\,, \\ P_{n+1}(z) &= P_n(z) + z^{2^n} Q_n(z)\,, \\ Q_{n+1}(z) &= P_n(z) - z^{2^n} Q_n(z)\,. \end{aligned}$$

We have at once

$$\begin{aligned}|P_n|^2 + |Q_n|^2 &= 2(|P_{n-1}|^2 + |Q_{n-1}|^2)\\ &= 2^2(|P_{n-2}|^2 + |Q_{n-2}|^2)\\ &= \cdots\\ &= 2^n(|P_0|^2 + |Q_0|^2) = 2(\mu_n + 1)\end{aligned}$$

on the unit circle, where

$$\mu_n := \deg(P_n) = \deg(Q_n) = 2^n - 1\,.$$

So $P_n, Q_n \in \mathcal{L}_{\mu_n}$ and $|P_n(z)| \le \sqrt{2}\sqrt{\deg(P_n) + 1}$ on the unit circle. From this it is a routine work to construct $P_n \in \mathcal{L}_n$ such that $|P_n(z)| \le c\sqrt{n}$ for every $n = 1, 2, \ldots$ with an absolute constant $c > 0$.

However, it is not known whether or not there are such polynomials from $p_n \in \mathcal{L}_n$ with minimal modulus also at least $c\sqrt{n}$ on the unit circle, where $c > 0$ is an absolute constant. Littlewood conjectures that there are such polynomials.

Littlewood also makes the above conjecture in [51] as well as several others. In [48] he writes that the problem of finding polynomials of degree n with coefficients of modulus 1 and with modulus on the unit disk bounded below by $c\sqrt{n}$ "seems singularly elusive and intriguing."

Erdős conjectured that the maximum modulus of a polynomial from $\mathcal{L}_n$ is always at least $c\sqrt{n+1}$ with an absolute constant $c > 1$. Erdős offers \$100 for a solution to this problem in [30]. Both Littlewood's and Erdős' conjectures are still open.

In the paper [48] Littlewood also considers $\sum_{m=0}^{n-1} \omega^{m(m+1)/2} z^m$ and shows that this polynomial has almost constant modulus (in an asymptotic sense) except on a set of measure $cn^{-1/2+\delta}$. Here ω is a primitive nth root of unity. Further related results are to be found in [4, 5, 9, 19, 25, 26, 40, 43, 45, 47, 55, 60].

Carrol, Eustice, and T. Figiel [40] show that

$$\liminf \frac{\log(m(n))}{\log(n+1)} > .431\,,$$

where $m(n)$ denotes the largest value that the minimum modulus of a polynomial from $\mathcal{L}_n$ can be on the unit circle. They also prove that

$$\sup \frac{\log(m(n))}{\log(n+1)} = \lim \frac{\log(m(n))}{\log(n+1)}\,.$$

They further conjecture that $m(n)n^{-1/2}$ tends to zero (contrary to Littlewood).

The average maximum modulus is computed by Salem and Zygmund [67] who show that for all but $o(2^n)$ polynomials from $\mathcal{L}_n$ the maximum modulus on the unit disk lies between $c_1\sqrt{n\log n}$ and $c_2\sqrt{n\log n}$.

The expected L^4 norm of a polynomial $p \in \mathcal{L}_n$ is $(2n^2 - n)^{1/4}$. This is due to Newman and Byrnes [59]. They also compute the L^4 norm of the Rudin-Shapiro polynomials.

In the case of complex coefficients these problems are mostly solved. A very interesting result of Kahane [43] proves the existence of polynomials of degree n with *complex* coefficients of modulus 1 and with minimal and maximal modulus both asymptotically $\sqrt{n+1}$ on the unit circle. See also Section 2 and [32].

The study of the location of zeros of the classes $\mathcal{F}_n$, $\mathcal{L}_n$, and $\mathcal{A}_n$ begins with Bloch and Pólya [6]. They prove that the average number of real zeros of a polynomial from $\mathcal{F}_n$ is at most $c\sqrt{n}$. They also prove that a polynomial from $\mathcal{F}_n$ cannot have more than

$$\frac{cn \log\log n}{\log n}$$

real zeros. This quite weak result appears to be the first on this subject. Schur [69] and by different methods Szegő [71] and Erdős and Turán [38] improve this to $c\sqrt{n \log n}$ (see also [10]). (Their results are more general, but in this specialization not sharp.)

Our Theorem 7.2 gives the right upper bound of $c\sqrt{n}$ for the number of real zeros of polynomials from a much larger class, namely for all polynomials of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \le 1\,, \quad |a_0| = |a_n| = 1\,, \quad a_j \in \mathbb{C}\,.$$

Schur [69] claims that Schmidt gives a version of part of this theorem. However, it does not appear in the reference he gives, namely [68], and we have not been able to trace it to any other source. Also, our method is able to give $c\sqrt{n}$ as an upper bound for the number of zeros of a polynomial $p \in \mathcal{P}_n^c$ with $|a_0| = 1, |a_i| \le 1$, inside any polygon with vertices in the unit circle (of course, c depends on the polygon). This is discussed in Section 8.

Bloch and Pólya [6] also prove that there are polynomials $p \in \mathcal{F}_n$ with

$$\frac{cn^{1/4}}{\sqrt{\log n}}$$

distinct real zeros of odd multiplicity. (Schur [69] claims they do it for polynomials with coefficients only from $\{-1, 1\}$, but this appears to be incorrect.)

In a seminal paper Littlewood and Offord [53] prove that the number of real roots of a $p \in \mathcal{L}_n$, on average, lies between

$$\frac{c_1 \log n}{\log\log\log n} \qquad \text{and} \qquad c_2 \log^2 n$$

and it is proved by Boyd [20] that every $p \in \mathcal{L}_n$ has at most $c \log^2 n / \log\log n$ zeros at 1 (counting multiplicities).

Kac [42] shows that the expected number of real roots of a polynomial of degree n with random uniformly distributed coefficients is asymptotically $(2/\pi)\log n$. He writes "I have also stated that the same conclusion holds if the coefficients assume only the values 1 and -1 with equal probabilities. Upon closer examination it turns out that the proof I had in mind is inapplicable... . This situation tends to emphasize the particular interest of the discrete case, which surprisingly enough turns out to be the most difficult." In a recent related paper Solomyak [70] studies the random series $\sum_{n=0}^{\infty} \pm\lambda^n$.

§5. Number of Zeros at 1

Theorems 5.1 and 5.2 below (see [17] for the proofs) offer upper bounds for the number of zeros at 1 of certain classes of polynomials with restricted coefficients. The first result sharpens and generalizes results of Amoroso [1], Bombieri and Vaaler [7], and Hua [41], who give versions of this result for polynomials with integer coefficients.

Theorem 5.1. *There is an absolute constant $c > 0$ such that every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \le 1\,, \quad a_j \in \mathbb{C}\,,$$

has at most

$$c\left(n(1-\log|a_0|)\right)^{1/2}$$

zeros at 1.

Applying Theorem 5.1 with $q(x) := x^n p(x^{-1})$ immediately gives the following.

Theorem 5.2. *There is an absolute constant $c > 0$ such that every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \le 1\,, \quad a_j \in \mathbb{C}\,,$$

has at most

$$c\left(n(1-\log|a_n|)\right)^{1/2}$$

zeros at 1.

The sharpness of the above theorems is shown by

Theorem 5.3. *Suppose $n \in \mathbb{N}$. Then there exists a polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_j| \le 1\,, \quad a_j \in \mathbb{R}\,,$$

such that p has a zero at 1 with multiplicity at least

$$\min\left\{\frac{1}{6}((n\,(1-\log|a_0|))^{1/2}-1\,,n\right\}.$$

The following two theorems can be obtained from the results above with slightly worse constants. However, we have distinct attractive proofs of Theorems 5.4 and 5.5 below and in [17] we give them also.

Theorem 5.4. *Every polynomial p of the form*

$$p(x)=\sum_{j=0}^{n}a_jx^j\,,\qquad |a_0|=1\,,\quad |a_j|\le 1\,,\quad a_j\in\mathbb{C}\,,$$

has at most $\lfloor\frac{16}{7}\sqrt{n}\rfloor+4$ zeros at 1.

Theorem 5.5. *For every $n\in\mathbb{N}$, there exists a polynomial*

$$p_n(x)=\sum_{j=0}^{n^2-1}a_jx^j$$

such that $a_{n^2-1}=1$; $a_0,a_1,\dots,a_{n^2-2}$ are real numbers of modulus less than 1; and p_n has a zero at 1 with multiplicity at least $n-1$.

Theorem 5.5 immediately implies

Corollary 5.6. *For every $n\in\mathbb{N}$, there exists a polynomial*

$$p_n(x)=\sum_{j=0}^{n}a_jx^j\,,\qquad a_n=1\,,\quad |a_j|\le 1\,,\quad a_j\in\mathbb{R}\,,$$

and p_n has a zero at 1 with multiplicity at least $\lfloor\sqrt{n}-1\rfloor$.

The next related result (see [18]) is well known (in a variety of forms) but its proof is simple and we include it in [17].

Theorem 5.7. *There is an absolute constant $c>0$ such that for every $n\in\mathbb{N}$ there is a $p\in\mathcal{F}_n$ having at least $c\sqrt{n/\log(n+1)}$ zeros at 1.*

Theorems 5.4 and 5.7 show that the right upper bound for the number of zeros a polynomial $p\in\mathcal{F}_n$ can have at 1 is somewhere between $c_1\sqrt{n/\log(n+1)}$ and $c_2\sqrt{n}$ with absolute constants $c_1>0$ and $c_2>0$. Completely closing the gap in this problem looks quite difficult.

Our next theorem from [17] slightly generalizes Theorem 5.1 and offers an *explicit* constant.

Theorem 5.8. *If $|a_0| \geq \exp(-L^2)$ and $|a_j| \leq 1$ for each $j = L^2+1, L^2+2, \dots, n$, then the polynomial*

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad a_j \in \mathbb{C}$$

has at most $\frac{44}{7}(L+1)\sqrt{n} + 5$ zeros at 1.

The next result from [17] is a simple observation about the maximal number of zeros a polynomial $p \in \mathcal{A}_n$ can have.

Theorem 5.9. *There is an absolute constant $c > 0$ such that every $p \in \mathcal{A}_n$ has at most $c \log n$ zeros at -1.*

Remark to Theorem 5.9. Let R_n be defined by

$$R_n(x) := \prod_{i=1}^{n} (1 + x^{a_i})\,,$$

where $a_1 := 1$ and a_{i+1} is the smallest odd integer that is greater than $\sum_{k=1}^{i} a_k$. It is tempting to speculate that R_n is the lowest degree polynomial with coefficients from $\{0,1\}$ and a zero of order n at -1. This is true for $n = 1,2,3,4,5$ but fails for $n = 6$ and hence for all larger n.

Our final result in this section shows that a polynomial $Q \in \mathcal{F}_n$ with k zeros at 1 has many other zeros on the unit circle (at certain roots of unity). It is shown in BE-99 but a version of it may also be deduced from results in [7].

Theorem 5.10. *Let $p \leq n$ be a prime. Suppose $Q \in \mathcal{F}_n$ and Q has exactly k zeros at 1 and exactly m zeros at a primitive pth root of unity. Then*

$$p(m+1) \geq k\,\frac{\log p}{\log(n+1)}\,.$$

§6. The Chebyshev Problem on $[0,1]$

If p is a polynomial of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j$$

with $a_1 = a_2 = \cdots = a_{m-1} = 0$ and $a_m \neq 0$, then we call $I(p) := a_m$ the first non-zero coefficient of p.

Our first theorem in this section (see [17] for a proof) shows how small the uniform norm of a polynomial $0 \neq p$ on $[0,1]$ can be under some restriction on its coefficients.

Theorem 6.1. *Let $\delta \in (0,1]$. There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp\left(-c_1(n(1-\log\delta))^{1/2}\right) \leq \inf_p \|p\|_{[0,1]} \leq \exp\left(-c_2(n(1-\log\delta))^{1/2}\right),$$

where the infimum is taken over all polynomials p of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_j| \leq 1\,, \quad a_j \in \mathbb{C}\,,$$

with $|I(p)| \geq \delta \geq \exp\left(\frac{1}{2}(6-n)\right)$.

The following result is a special case of Theorem 6.1.

Theorem 6.2. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp\left(-c_1\sqrt{n}\right) \leq \inf_p \|p\|_{[0,1]} \leq \exp\left(-c_2\sqrt{n}\right)$$

for every $n \geq 2$, where the infimum is taken over all polynomials p of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_j| \leq 1\,, \quad a_j \in \mathbb{C}\,,$$

with $|I(p)| = 1$.

For the class $\mathcal{F}_n$ we have

Theorem 6.3. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp\left(-c_1\sqrt{n}\right) \leq \inf_{0 \neq p \in \mathcal{F}_n} \|p\|_{[0,1]} \leq \exp\left(-c_2\sqrt{n}\right)$$

for every $n \geq 2$.

See [17] for a proof. Note that the lower bound in the above theorem is a special case of Theorem 6.2. The proof of the upper bound, however, requires new ideas.

The approximation rate in Theorems 6.2 and 6.3 should be compared with

$$\inf_p \|p\|_{[0,1]}^{1/n} = \frac{2^{1/n}}{4}\,,$$

where the infimum is taken for all monic $p \in \mathcal{P}_n$, and also with

$$\frac{1}{2.376\ldots} < \inf_{0 \neq p \in \mathcal{Z}_n} \|p\|_{[0,1]}^{1/n} < \frac{1+\varepsilon_n}{2.3605}\,, \qquad \varepsilon_n \to 0\,.$$

The first equality above is attained by the normalized Chebyshev polynomial shifted linearly to $[0,1]$ and is proved by a simple perturbation argument. The second inequality is much harder (the exact result is open) and is discussed in

[9]. It is an interesting fact that the polynomials $0 \neq p \in \mathcal{Z}_n$ with the smallest uniform norm on $[0,1]$ are very different from the usual Chebyshev polynomial of degree n. For example, they have at least 52% of their zeros at either 0 or 1. Relaxation techniques do not allow for their approximate computation.

Likewise, polynomials $0 \neq p \in \mathcal{F}_n$ with small uniform norm on $[0,1]$ are again quite different from polynomials $0 \neq p \in \mathcal{Z}_n$ with small uniform norm on $[0,1]$.

The story is roughly as follows. Polynomials $0 \neq p \in \mathcal{P}_n$ with leading coefficient 1 and with smallest possible uniform norm on $[0,1]$ are characterized by equioscillation and are given explicitly by the Chebyshev polynomials. In contrast, finding polynomials from $\mathcal{Z}_n$ with small uniform norm on $[0,1]$ is closely related to finding irreducible polynomials with all their roots in $[0,1]$.

The construction of non-zero polynomials from $\mathcal{F}_n$ with small uniform norm on $[0,1]$ is more or less governed by how many zeros such a polynomial can have at 1. Indeed, non-zero polynomials from $\mathcal{F}_n$ with minimal uniform norm on $[0,1]$ are forced to have close to the maximal possible number of zeros at 1.

This problem of the maximum order of a zero at 1 for a polynomial in $\mathcal{F}_n$, and closely related problems for polynomials of small height have attracted considerable attention but there is still a gap in what is known (see Theorems 5.4 and 5.7).

For the class $\mathcal{A}_n$ we have the following Chebyshev-type theorem. This result should be compared with Theorem 6.3. See Theorem [17] for a proof.

Theorem 6.4. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp\left(-c_1 \log^2(n+1)\right) \leq \inf_{0 \neq p \in \mathcal{A}_n} \|p(-x)\|_{[0,1]} \leq \exp\left(-c_2 \log^2(n+1)\right)$$

for every $n \geq 2$.

Our last theorem in this section is a sharp Chebyshev-type inequality for $\mathcal{F} := \cup_{n=1}^{\infty} \mathcal{F}_n$ and $\mathcal{S}$, where $\mathcal{S}$ denotes the collection of all analytic functions f on the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ that satisfy

$$|f(z)| \leq \frac{1}{1-|z|}\,, \qquad z \in D\,.$$

See [17] for a proof.

Theorem 6.5. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp(-c_1/a) \leq \inf_{p \in \mathcal{S},\, |p(0)|=1} \|p\|_{[1-a,1]} \leq \inf_{p \in \mathcal{F},\, |p(0)|=1} \|p\|_{[1-a,1]} \leq \exp(-c_2/a)$$

for every $a \in (0,1)$.

§7. More on the Number of Real Zeros

Theorems 7.2 and 7.3 below give upper bounds for the number of *real* zeros of polynomials p when their coefficients are restricted in various ways.

The prototype for these theorems is given below. It was apparently first proved, at least up to the correct constant, by Schmidt in the early thirties. His complicated proof was not published – the first published proof is due to Schur [69]. Later new and simpler proofs and generalizations were published by Szegő [71] and Erdős and Turán [38] and others. A version of the approach of Erdős and Turán is presented in [8].

Theorem 7.1. *Suppose*

$$p(z) := \sum_{j=0}^{n} a_j z^j\,, \qquad a_j \in \mathbb{C}\,,$$

has m positive real roots. Then

$$m^2 \le 2n \log\left(\frac{|a_0| + |a_1| + \cdots + |a_n|}{\sqrt{|a_0 a_n|}}\right).$$

Our Theorem 7.2 below (see [17] for a proof) improves the above bound of $c\sqrt{n\log n}$ in the cases we are interested in where the coefficients are of similar size. Up to the constant c it is the best possible result.

Theorem 7.2. *There is an absolute constant $c > 0$ such that every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \le 1\,, \quad |a_0| = 1\,, \quad a_j \in \mathbb{C}\,,$$

has at most $c\sqrt{n}$ zeros in $[-1,1]$.

There is an absolute constant $c > 0$ such that every polynomial p of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \le 1\,, \quad |a_n| = 1\,, \quad a_j \in \mathbb{C}\,,$$

has at most $c\sqrt{n}$ zeros in $\mathbb{R} \setminus (-1,1)$.

There is an absolute constant $c > 0$ such that every polynomial p of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \le 1\,, \quad |a_0| = |a_n| = 1\,, \quad a_j \in \mathbb{C}\,,$$

has at most $c\sqrt{n}$ real zeros.

In [17] we also prove

Theorem 7.3. *There is an absolute constant $c > 0$ such that every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j , \qquad |a_j| \le 1 , \quad |a_0| = 1 , \quad a_j \in \mathbb{C} , \tag{7.1}$$

has at most c/a zeros in $[-1+a, 1-a]$ whenever $a \in (0,1)$.

Theorem 7.3 is sharp up to the constant. It is possible to construct a polynomial (of degree $n \le ck^2$) of the form (4.1) with a zero of order k in the interval $(0, 1-1/k]$. This is discussed in [3].

The next theorem from [13] gives an upper bound for the number of zeros of a polynomial p lying on a subarc of the unit circle when the coefficients of p are restricted as in the first statement of Theorem 7.2.

Theorem 7.4. *There is an absolute constant $c > 0$ such that every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j , \qquad |a_j| \le 1 , \quad |a_0| = 1 , \quad a_j \in \mathbb{C} ,$$

has at most $cn\alpha$ zeros on a subarc I_α of length α of the unit circle if $\alpha \ge n^{-1/2}$, while it has at most $c\sqrt{n}$ zeros on I_α if $\alpha \le n^{-1/2}$. The polynomial $p(z) := z^n - 1$ ($\alpha \ge n^{-1/2}$) and Theorem 5.4 ($\alpha \le n^{-1/2}$) show that these bounds are essentially sharp.

One can observe that Jensen's inequality implies that every function f analytic in the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ and satisfying the growth condition

$$|f(0)| = 1 , \qquad |f(z)| \le \frac{1}{1-|z|} , \qquad z \in D$$

has at most $(c/a)\log(1/a)$ zeros in the disk $D_a := \{z \in \mathbb{C} : |z| < 1-a\}$, where $0 < a < 1$ and $c > 0$ is an absolute constant. This observation plays a crucial role in the next section.

§8. Further Results on the Zeros

There is a huge literature on the zeros of polynomials with restricted coefficients. See, for example, [1, 6, 3, 7, 41, 38, 8, 13, 17, 52, 57, 61, 68, 69, 71].

In [13] we prove the three essentially sharp theorems below.

Theorem 8.1. *Every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j , \qquad |a_0| = 1 , \quad |a_j| \le 1 , \quad a_j \in \mathbb{C} ,$$

has at most $c\sqrt{n}$ zeros inside any polygon with vertices on the unit circle, where the constant $c > 0$ depends only on the polygon.

Theorem 8.2. *There is an absolute constant $c > 0$ such that every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j \,, \qquad |a_0| = |a_n| = 1 \,, \quad |a_j| \le 1 \,, \quad a_j \in \mathbb{C} \,,$$

has at most $c(n\alpha + \sqrt{n})$ zeros in the strip

$$\{z \in \mathbb{C} : \ |\mathrm{Im}(z)| \le \alpha\} \,,$$

and in the sector

$$\{z \in \mathbb{C} : \ |\arg(z)| \le \alpha\} \,.$$

Theorem 8.3. *Let $\alpha \in (0,1)$. Every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j \,, \qquad |a_0| = 1 \,, \quad |a_j| \le 1 \,, \quad a_j \in \mathbb{C} \,,$$

has at most c/α zeros inside any polygon with vertices on the circle

$$\{z \in \mathbb{C} : \ |z| = 1 - \alpha\} \,,$$

where the constant $c > 0$ depends only on the number of the vertices of the polygon.

For $z_0 \in \mathbb{C}$ and $r > 0$, let

$$D(z_0, r) := \{z \in \mathbb{C} : \ |z - z_0| < r\} \,.$$

In [33] we show that a polynomial p of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j \,, \qquad |a_0| = 1 \,, \quad |a_j| \le 1 \,, \quad a_j \in \mathbb{C} \,,$$

has at most $(c_1/\alpha)\log(1/\alpha)$ zeros in the disk $D(0, 1-\alpha)$ for every $\alpha \in (0,1)$, where $c_1 > 0$ is an absolute constant. This is a simple consequence of Jensen's formula. However it is not so simple to show that this estimate for the number of zeros in $D(0, 1-\alpha)$ is sharp. In [33] we present two examples to show the existence of polynomials p_α ($\alpha \in (0,1)$) of the form (1.1) (with a suitable $n \in \mathbb{N}$ depending on α) with at least $\lfloor (c_2/\alpha)\log(1/\alpha) \rfloor$ zeros in $D(0, 1-\alpha)$ ($c_2 > 0$ is an absolute constant). In fact, we show the existence of such polynomials from much smaller classes with more restrictions on the coefficients. Our first example has probabilistic background and shows the existence of polynomials p_α ($\alpha \in (0,1)$) with *complex* coefficients of modulus *exactly* 1 and with at least

$\lfloor (c_2/\alpha)\log(1/\alpha)\rfloor$ zeros in $D(0,1-\alpha)$ ($c_2>0$ is an absolute constant). Our second example is constructive and defines polynomials p_α ($\alpha\in(0,1)$) with *real* coefficients of modulus *at most* 1, with constant term 1, and with at least $\lfloor (c_2/\alpha)\log(1/\alpha)\rfloor$ zeros in $D(0,1-\alpha)$ ($c_2>0$ is an absolute constant). So, in particular, the constant in Theorem 1.3 cannot be made independent of the number of vertices of the polygon.

Some other observations on polynomials with restricted coefficients are also formulated in [33]. More precisely in [33] we prove Theorems 8.4 - 8.9, 8.11 and 8.13 below.

Theorem 8.4. *Let $\alpha\in(0,1)$. Every polynomial of the form*

$$p(x)=\sum_{j=0}^{n} a_j x^j\,,\qquad |a_0|=1\,,\quad |a_j|\le 1\,,\quad a_j\in\mathbb{C}\,,$$

has at most $(2/\alpha)\log(1/\alpha)$ zeros in the disk $D(0,1-\alpha)$.

Theorem 8.5. *For every $\alpha\in(0,1)$ there is a polynomial $Q:=Q_\alpha$ of the form*

$$Q_\alpha(x)=\sum_{j=0}^{n} a_{j,\alpha} x^j\,,\qquad |a_{j,\alpha}|=1\,,\quad a_{j,\alpha}\in\mathbb{C}\,,$$

such that Q_α has at least $\lfloor (c_2/\alpha)\log(1/\alpha)\rfloor$ zeros in the disk $D(0,1-\alpha)$, where $c_2>0$ is an absolute constant.

Theorem 8.5 follows from

Theorem 8.6. *For every $n\in\mathbb{N}$ there is a polynomial p_n of the form*

$$p_n(x)=\sum_{j=0}^{n} a_{j,n} x^j\,,\qquad |a_{j,n}|=1\,,\quad a_{j,n}\in\mathbb{C}\,,$$

such that p_n has no zeros in the annulus

$$\left\{z\in\mathbb{C}: 1-\frac{c_3\log n}{n}<|z|<1+\frac{c_3\log n}{n}\right\}\,,$$

where $c_3>0$ is an absolute constant.

To formulate some interesting corollaries of Theorems 8.4 and 8.6 we introduce some notation. Let $\mathcal{E}_n$ be the collection of polynomials of the form

$$p(x)=\sum_{j=0}^{n} a_j x^j\,,\qquad |a_0|=|a_n|=1\,,\quad a_j\in[-1,1]\,.$$

Let $\mathcal{E}_n^c$ be the collection of polynomials of the form

$$p(x)=\sum_{j=0}^{n} a_j x^j\,,\qquad |a_0|=|a_n|=1\,,\quad a_j\in\mathbb{C}\,,\quad |a_j|\le 1\,.$$

As before, let $\mathcal{L}_n$ be the collection of polynomials of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j \,, \qquad a_j \in \{-1, 1\} \,.$$

Finally let $\mathcal{K}_n$ be the collection of polynomials of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j \,, \qquad a_j \in \mathbb{C} \,, \quad |a_j| = 1 \,.$$

For a polynomial p, let

$$d(p) := \min\{|1 - |z|| : \ z \in \mathbb{C} \,, \ p(z) = 0\} \,.$$

For a class of polynomials $\mathcal{A}$ we define

$$\gamma(\mathcal{A}) := \sup\{d(p) : \ p \in \mathcal{A}\}$$

Theorem 8.7. *There are absolute constants* $c_4 > 0$ *and* $c_5 > 0$ *such that*

$$\frac{c_4 \log n}{n} \leq \gamma(\mathcal{K}_n^c) \leq \gamma(\mathcal{E}_n^c) \leq \frac{c_5 \log n}{n} \,.$$

Theorem 8.8. *There is an absolute constant* $c_6 > 0$ *such that*

$$\gamma(\mathcal{L}_n) \leq \gamma(\mathcal{E}_n) \leq \frac{c_6 \log n}{n} \,.$$

There is an absolute constant $c_7 > 0$ *such that for infinitely many positive integer values of* n *we have*

$$\frac{c_7}{n} \leq \gamma(\mathcal{L}_n) \leq \gamma(\mathcal{K}_n) \,.$$

Theorem 8.9. *For every* $\alpha \in (0,1)$ *there is a polynomial* $P := P_\alpha$ *of the form*

$$P(x) = \sum_{j=0}^{n} a_{j,\alpha} x^j \,, \qquad a_{0,\alpha} = 1 \,, \quad a_{j,\alpha} \in [-1, 1] \,,$$

that has at least $\lfloor (c_8/\alpha) \log(1/\alpha) \rfloor$ *zeros in the disk* $D(0, 1-\alpha)$*, where* $c_8 > 0$ *is an absolute constant.*

Conjecture 8.10. *Every polynomial* $p \in \mathcal{L}_n$ *has at least one zero in the annulus*

$$\left\{ z \in \mathbb{C} : 1 - \frac{c_9}{n} < |z| < 1 + \frac{c_9}{n} \right\} \,,$$

where $c_9 > 0$ *is an absolute constant.*

In the case when a polynomial $p \in \mathcal{L}_n$ is self-reciprocal, we can prove more than the conclusion of Conjecture 8.10. Namely

Theorem 8.11. *Every self-reciprocal polynomial $p \in \mathcal{L}_n$ has at least one zero on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.*

In [33] we also show that Conjecture 8.10 implies the conjecture below.

Conjecture 8.12. *There is no ultraflat sequence $(p_{n_m})_{m=1}^{\infty}$ of polynomials $p_{n_m} \in \mathcal{L}_{m_n}$ satisfying*

$$(1 - \varepsilon_{n_m})(n_m + 1)^{1/2} \le |p_n(z)| \le (1 + \varepsilon_{n_m})(n_m + 1)^{1/2}$$

for all $z \in \mathbb{C}$ with $|z| = 1$ and for all $m \in \mathbb{N}$, where $(\varepsilon_{n_m})_{m=1}^{\infty}$ is a sequence of positive numbers converging to 0.

Theorem 8.13. *Conjecture 8.10 implies Conjecture 8.12.*

§9. Littlewood-Type Problems on Subarcs of the Unit Circle

Littlewood's well-known and now resolved conjecture of around 1948 concerns polynomials of the form

$$p(z) := \sum_{j=1}^{n} a_j z^{k_j},$$

where the coefficients a_j are complex numbers of modulus at least 1 and the exponents k_j are distinct non-negative integers. It states that such polynomials have L_1 norms on the unit circle

$$\partial D := \{z \in \mathbb{C} : |z| = 1\}$$

that grow at least like $c \log n$ with an absolute constant $c > 0$. This was proved by Konjagin [45] and independently by McGehee, Pigno, and Smith [56].

Pichorides, who contributed essentially to the proof of the Littlewood conjecture, observed in [62] that the original Littlewood conjecture (when all the coefficients are from $\{0, 1\}$ would follow from a result on the L_1 norm of such polynomials on sets $E \subset \partial D$ of measure π. Namely if

$$\int_E \left| \sum_{j=0}^{n} z^{k_j} \right| |dz| \ge c$$

for any subset $E \subset \partial D$ of measure π with an absolute constant $c > 0$, then the original Littlewood conjecture holds. Throughout this section the measure of a set $E \subset \partial D$ is the linear Lebesgue measure of the set

$$\{t \in [-\pi, \pi) : e^{it} \in E\}.$$

Konjagin [46] gives a lovely probabilistic proof that this hypothesis fails. He does however conjecture the following: for any *fixed* set $E \subset \partial D$ of positive measure there exists a constant $c = c(E) > 0$ depending only on E such that

$$\int_E \left| \sum_{j=0}^{n} z^{k_j} \right| |dz| \ge c(E).$$

In other words the sets $E_\varepsilon \subset \partial D$ of measure π in his example where

$$\int_{E_\varepsilon} \Big| \sum_{j=0}^{n} z^{k_j} \Big| \, |dz| < \varepsilon$$

must vary with $\varepsilon > 0$.

In [11] we show, among other things, that Konjagin's conjecture holds on subarcs of the unit circle ∂D.

Additional material on Littlewood's conjecture and related problems concerning the growth of polynomials with unimodular coefficients in various norms on the unit disk is to be found, for example, in [19, 4, 44, 52, 55, 59, 61, 70].

All the results of [11] concern how small polynomials of the above and related forms can be in the L_p norms on subarcs of the unit disk. For $1 \le p \le \infty$ the results are sharp, at least up to a constant in the exponent.

An interesting related result is due to Nazarov [58]. One of its simpler versions states that there is an absolute constant $c > 0$ such that

$$\max_{z \in I} |p(z)| \le \left(\frac{c\, m(I)}{m(A)} \right)^n \max_{z \in A} |p(z)|$$

for every polynomial p of the form $p(z) = \sum_{j=0}^n a_j z^{k_j}$ with $k_j \in \mathbb{N}$ and $a_j \in \mathbb{C}$ and for every $A \subset I$, where I is a subarc of ∂D with length $m(I)$, and A is measurable with Lebesgue measure $m(A)$. This extends a result of Turán [72] called *Turán's Lemma*, where $I = \partial D$ and A is a subarc.

We introduce some notation. For $M > 0$ and $\mu \ge 0$, let $\mathcal{S}_M^\mu$ denote the collection of all analytic functions f on the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ that satisfy

$$|f(z)| \le \frac{M}{(1 - |z|)^\mu}\,, \qquad z \in D\,.$$

We define the following subsets of $\mathcal{S}_1^1$. Let

$$\mathcal{F}_n := \left\{ f : f(z) = \sum_{j=0}^n a_j z^j\,, \;\; a_j \in \{-1, 0, 1\} \right\},$$

and denote the set of all polynomials with coefficients from the set $\{-1, 0, 1\}$ by

$$\mathcal{F} := \bigcup_{n=0}^\infty \mathcal{F}_n\,.$$

More generally we define the following classes of polynomials. For $M > 0$ and $\mu \ge 0$ let

$$\mathcal{K}_M^\mu := \left\{ f : f(z) = \sum_{j=0}^n a_j z^j\,, \;\; a_j \in \mathbb{C}\,, \;\; |a_j| \le M j^\mu\,, \;\; |a_0| = 1\,, \;\; n \in \mathbb{N} \right\}.$$

On occasion we let $\mathcal{S} := \mathcal{S}_1^1$, $\mathcal{S}_M := \mathcal{S}_M^1$, and $\mathcal{K}_M := \mathcal{K}_M^0$.

We also employ the following standard notations. We denote by $\mathcal{P}_\backslash$ the set of all polynomials of degree at most n with real coefficients. We denote by $\mathcal{P}_\backslash^\rfloor$ the set of all polynomials of degree at most n with complex coefficients. The height of a polynomial

$$p_n(z) := \sum_{j=0}^{n} a_j z^j\,, \quad a_j \in \mathbb{C}\,, \quad a_n \neq 0\,,$$

is defined by

$$H(p_n) := \max\left\{\frac{|a_j|}{|a_n|} : \ j = 0, 1, \ldots, n\right\}.$$

Also,

$$\|p\|_A := \sup_{z \in A} |p(z)|$$

and

$$\|p\|_{L_q(A)} := \left(\int_A |p(z)|^q \,|dz|\right)^{1/q}$$

are used throughout this section for measurable functions (in this section usually polynomials) p defined on a measurable subset of the unit circle or the real line, and for $q \in (0, \infty)$.

Theorems 9.1 - 9.5, Corrolaries 9.6 and 9.7, and Theorem 9.8 below are proved in [11].

Theorem 9.1. *Let $0 < a < 2\pi$ and $M \geq 1$. Let A be a subarc of the unit circle with length $m(A) = a$. Then there is an absolute constant $c_1 > 0$ such that*

$$\|f\|_A \geq \exp\left(\frac{-c_1(1 + \log M)}{a}\right)$$

for every $f \in \mathcal{S}_M (:= \mathcal{S}_M^1)$ that is continuous on the closed unit disk and satisfies $|f(z_0)| \geq \frac{1}{2}$ for every $z_0 \in \mathbb{C}$ with $|z_0| = \frac{1}{4M}$.

Corollary 9.2. *Let $0 < a < 2\pi$ and $M \geq 1$. Let A be a subarc of the unit circle with length $m(A) = a$. Then there is an absolute constant $c_1 > 0$ such that*

$$\|f\|_A \geq \exp\left(\frac{-c_1(1 + \log M)}{a}\right)$$

for every $f \in \mathcal{K}_M (:= \mathcal{K}_M^1)$.

The next two results from [11] show that the previous results are, up to constants, sharp.

Theorem 9.3. *Let $0 < a < 2\pi$. Let A be the subarc of the unit circle with length $m(A) = a$. Then there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\inf_{0 \neq f \in \mathcal{F}} \|f\|_A \le \exp\left(\frac{-c_1}{a}\right)$$

whenever $m(A) = a \le c_2$.

Theorem 9.4. *Let $0 < a < 2\pi$ and $M \ge 1$. Let A be the subarc of the unit circle with length $m(A) = a$. Then there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\inf_{0 \neq f \in \mathcal{K}_M} \|f\|_A \le \exp\left(\frac{-c_1(1 + \log M)}{a}\right)$$

whenever $m(A) = a \le c_2$.

The next two results from [11] extend the first two results to the L_1 norm (and hence to all L_p norms with $p \ge 1$).

Theorem 9.5. *Let $0 < a < 2\pi$, $M \ge 1$, and $\mu = 1, 2, \ldots$. Let A be a subarc of the unit circle with length $m(A) = a$. Then there is an absolute constant $c_1 > 0$ such that*

$$\|f\|_{L_1(A)} \ge \exp\left(\frac{-c_1(\mu + \log M)}{a}\right)$$

for every $f \in \mathcal{S}_M^\mu$ that is continuous on the closed unit disk and satisfies $|f(z_0)| \ge \frac{1}{2}$ for every $z_0 \in \mathbb{C}$ with $|z_0| \le \frac{1}{4M2^\mu}$.

Corollary 9.6. *Let $0 < a < 2\pi$, $M \ge 1$, and $\mu = 1, 2, \ldots$. Let A be a subarc of the unit circle with length $m(A) = a$. Then there is an absolute constant $c_1 > 0$ such that*

$$\|f\|_{L_1(A)} \ge \exp\left(\frac{-c_1(1 + \mu \log \mu + \log M)}{a}\right)$$

for every $f \in \mathcal{K}_M^\mu$.

The following is an interesting consequence of the preceding results.

Corollary 9.7. *Let A be a subarc of the unit circle with length $m(A) = a$. If (p_k) is a sequence of monic polynomials that tends to 0 in $L_1(A)$, then the sequence $H(p_k)$ of heights tends to ∞.*

The final result of this section shows that the theory does not extend to arbitrary sets of positive measure. This is shown in [11] as well.

Theorem 9.8. *For every $\varepsilon > 0$ there is a polynomial $p \in \mathcal{K}_1$ such that $|p(z)| < \varepsilon$ everywhere on the unit circle except possibly in a set of linear measure at most ε.*

§10. Markov- and Bernstein-type inequalities

Erdős studied and raised many questions about polynomials with restricted coefficients. Both Erdős and Littlewood showed particular fascination about the class $\mathcal{L}_n$, where, as before, $\mathcal{L}_n$ denotes the set of all polynomials of degree n with each of their coefficients in $\{-1,1\}$. A related class of polynomials is $\mathcal{F}_n$ that, as before, denotes the set of all polynomials of degree at most n with each of their coefficients in $\{-1,0,1\}$. Another related class is $\mathcal{G}_n$, that is the collection of all polynomials p of the form

$$p(x) = \sum_{j=m}^{n} a_j x^j \,, \qquad |a_m| = 1\,, \quad |a_j| \le 1\,,$$

where m is an unspecified nonnegative integer not greater than n. For the sake of brevity, let

$$\|p\|_A := \sup_{z\in A} |p(z)|$$

for a complex-valued function p defined on A.

In [12] and [14] we establish the right Markov-type inequalities for the classes $\mathcal{F}_n$ and $\mathcal{G}_n$ on $[0,1]$. Namely there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 n \log(n+1) \le \max_{0\ne p\in\mathcal{F}_n} \frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}} \le c_2 n \log(n+1)$$

and

$$c_1 n^{3/2} \le \max_{0\ne p\in\mathcal{G}_n} \frac{|p'(1)|}{\|p\|_{[0,1]}} \le \max_{0\ne p\in\mathcal{G}_n} \frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}} \le c_2 n^{3/2}\,.$$

It is quite remarkable that the right Markov factor for $\mathcal{G}_n$ is much larger than the right Markov factor for $\mathcal{F}_n$. In [12] and [14] we also show that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 n \log(n+1) \le \max_{0\ne p\in\mathcal{L}_n} \frac{|p'(1)|}{\|p\|_{[0,1]}} \le \max_{0\ne p\in\mathcal{L}_n} \frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}} \le c_2 n \log(n+1)$$

for every $p \in \mathcal{L}_n$. For polynomials $p \in \mathcal{F} := \bigcup_{n=0}^{\infty} \mathcal{F}_n$ with $|p(0)| = 1$ and for $y \in [0,1)$ the Bernstein-type inequality

$$\frac{c_1 \log\left(\frac{2}{1-y}\right)}{1-y} \le \max_{\substack{p\in\mathcal{F}\\ |p(0)|=1}} \frac{\|p'\|_{[0,y]}}{\|p\|_{[0,1]}} \le \frac{c_2 \log\left(\frac{2}{1-y}\right)}{1-y}$$

is also proved with absolute constants $c_1 > 0$ and $c_2 > 0$.

For continuous functions p defined on the complex unit circle, and for $q \in (0,\infty)$, we define

$$\|p\|_q := \left(\int_0^{2\pi} |p(e^{it})|^q \, dt\right)^{1/q}\,.$$

We also define

$$\|p\|_\infty := \lim_{q\to\infty} \|p\|_q = \max_{t\in[0,2\pi]} |p(e^{it})| \,.$$

Based on the ideas of F. Nazarov, Qeffelec and Saffari [65] showed that

$$\sup_{p\in\mathcal{L}_n} \frac{\|p'\|_q}{\|p\|_q} = \gamma_{n,q} n \,, \qquad \lim_{n\to\infty} \gamma_{n,q} = 1 \,,$$

for every $q \in (0,\infty]$, $q \neq 2$ (when $q=2$, $\lim_{n\to\infty} \gamma_{n,q} = 3^{-1/2}$ by the Parseval Formula). It is interesting to compare this result with Theorem 2.3. It shows that Bernstein's classical inequality (extended by Arestov for all $q \in (0,\infty]$) stating that

$$\|p'\|_q \leq n\|p'\|_q$$

for all polynomials of degree at most n with complex coefficients, cannot be essentially improved for the class $\mathcal{L}_n$, except the trivial $q=2$ case.

§11. Trigonometric Polynomials with Many Real Zeros

As before, let

$$\mathcal{L}_n := \left\{ p : p(z) = \sum_{j=0}^{n} a_j z^j \,, \;\; a_j \in \{-1,1\} \right\}.$$

Let D denote the closed unit disk of the complex plane. Let ∂D denote the unit circle of the complex plane. Littlewood made the following conjecture about $\mathcal{L}_n$ in the fifties.

Conjecture 11.1. *(Littlewood). There are at least infinitely many values of $n \in \mathbb{N}$ for which there are polynomials $p_n \in \mathcal{L}_n$ so that*

$$c_1(n+1)^{1/2} \leq |p_n(z)| \leq c_2(n+1)^{1/2}$$

for all $z \in \partial D$. Here the constants c_1 and c_2 are independent of n.

There is a related conjecture of Erdős [29].

Conjecture 11.2. *(Erdős). There is a constant $\varepsilon > 0$ (independent of n) so that*

$$\max_{z\in\partial D} |p_n(z)| \geq (1+\varepsilon)(n+1)^{1/2}$$

for every $p_n \in \mathcal{L}_n$ and $n \in \mathbb{N}$. That is, the constant C_2 in Conjecture 1.1 must be bounded away from 1 (independently of n).

This conjecture is also open. One of our results in [15] is formulated by Corollary 11.6. Littlewood gives a proof of this in [48] and explores related issues in [49, 50, 51]. The approach is via Theorem 11.3 which estimates the measure of the set where a real trigonometric polynomial of degree at most n with at least k zeros in $K := \mathbb{R} \pmod{2\pi}$ is small. There are two reasons for doing this. First the approach is, we believe, easier and secondly it leads to explicit constants.

Let $K := \mathbb{R} \pmod{2\pi}$. For the sake of brevity the uniform norm of a continuous function p on K will be denoted by $\|p\|_K := \|p\|_{L_\infty(K)}$. Let $\mathcal{T}_n$ denote the set of all real trigonometric polynomials of degree at most n, and let $\mathcal{T}_{n,k}$ denote the subset of those elements of $\mathcal{T}_n$ that have at least k zeros in K (counting multiplicities). In [15] we prove Theorems 11.3 – 11.5 and Corollary 11.6 below.

Theorem 11.3. *Suppose $p \in \mathcal{T}_n$ has at least k zeros in K (counting multiplicities). Let $\alpha \in (0,1)$. Then*

$$m\{t \in K : |p(t)| \le \alpha \|p\|_K\} \ge \frac{\alpha}{e}\frac{k}{n},$$

where $m(A)$ denotes the one-dimensional Lebesgue measure of $A \subset K$.

Theorem 11.4. *We have*

$$2\pi\left(1 - \frac{c_2 k}{n}\right) \le \sup_{p \in \mathcal{T}_{n,k}} \frac{\|p\|_{L_1(K)}}{\|p\|_{L_\infty(K)}} \le 2\pi\left(1 - \frac{c_1 k}{n}\right)$$

for some absolute constants $0 < c_1 < c_2$.

Theorem 11.5. *Assume that $p \in \mathcal{T}_n$ satisfies*

$$\|p\|_{L_2(K)} \le A n^{1/2} \tag{11.1}$$

and

$$\|p'\|_{L_2(K)} \ge B n^{3/2}. \tag{11.2}$$

Then there is a constant $\varepsilon > 0$ depending only on A and B such that

$$\|p\|_K^2 \ge (2\pi - \varepsilon)^{-1} \|p\|_{L_2(K)}^2. \tag{11.3}$$

Here

$$\varepsilon = \frac{\pi^3}{1024e}\frac{B^6}{A^6}$$

works.

Corollary 11.6. *Let $p \in \mathcal{T}_n$ be of the form*

$$p(t) = \sum_{k=1}^{n} a_k \cos(kt - \gamma_k), \qquad a_k = \pm 1, \quad \gamma_k \in \mathbb{R}, \quad k = 1, 2, \ldots, n.$$

Then there is a constant $\varepsilon > 0$ such that

$$\|p\|_K^2 \ge (2\pi - \varepsilon)^{-1} \|p\|_{L_2(K)}^2.$$

Here

$$\varepsilon := \frac{\pi^3}{1024e}\frac{1}{27}$$

works.

§12. On the flatness of trigonometric polynomials

In [15] we give short and elegant proofs of some of the main results from Littlewood's papers [48, 49, 50, 51, 52]. There are two reasons for doing this. First our approaches are, we believe, much easier, and secondly they lead to explicit constants. Littlewood himself remarks that his methods were "extremely indirect".

We use the notation $K := \mathbb{R} \pmod{2\pi}$. Let

$$\|p\|_{L_\lambda(K)} := \left(\int_K |p(t)|^\lambda \, dt \right)^{1/\lambda}$$

and

$$M_\lambda(p) := \left(\frac{1}{2\pi} \int_K |p(t)|^\lambda \, dt \right)^{1/\lambda} .$$

In [16] we prove

Theorem 12.1. *Assume that p is a trigonometric polynomial of degree at most n with real coefficients that satisfies*

$$\|p\|_{L_2(K)} \le A n^{1/2} \tag{12.1}$$

and

$$\|p'\|_{L_2(K)} \ge B n^{3/2} . \tag{12.2}$$

Then there exists a constant $\varepsilon > 0$ so that

$$M_4(p) - M_2(p) \ge \varepsilon M_2(p)$$

where

$$\varepsilon := \left(\frac{1}{221} \right) \left(\frac{B}{A} \right)^{12} .$$

Let the Littlewood class $\mathcal{H}_n$ be the collection of all trigonometric polynomials of the form

$$p(t) := p_n(t) := \sum_{j=1}^{n} a_j \cos(jt + \alpha_j) , \qquad a_j = \pm 1 , \quad \alpha_j \in \mathbb{R} .$$

Note that for the Littlewood class $\mathcal{H}_n$ we have

$$\left(\frac{B}{A} \right)^{12} = 3^{-6} .$$

Corollary 12.2. *We have*

$$M_4(p) - M_2(p) \ge \frac{M_2(p)}{160874}$$

for every $p \in \mathcal{A}_n$. The merit factor

$$\left(\frac{M_4^4(p)}{M_2^4(p)} - 1\right)^{-1}$$

is bounded above by 20110 *for every $p \in \mathcal{H}_n$.*

If Q_n is a polynomial of degree n of the form

$$Q_n(z) = \sum_{k=0}^{n} a_k z^k \,, \qquad a_k \in \mathbb{C}\,,$$

and the coefficients a_k of Q_n satisfy

$$a_k = \overline{a}_{n-k}\,, \qquad k = 0, 1, \ldots n\,,$$

then we call Q_n a conjugate-reciprocal polynomial of degree n. We say that the polynomial Q_n is unimodular if $|a_k| = 1$ for each $k = 0, 1, 2, \ldots, n$. Note that if $p \in \mathcal{A}_n$, then

$$1 + p(t) = e^{int} Q_{2n}(e^{it})$$

with a conjugate-reciprocal unimodular polynomial Q_{2n} of degree at most $2n$. One can ask how flat a conjugate reciprocal unimodular polynomial can be. The inequality of Remark 3.5 implies the result below.

Theorem 12.3. *For every $p \in \mathcal{H}_n$,*

$$M_\infty(1+p) - M_2(1+p) \geq (\sqrt{4/3} - 1) M_2(1+p).$$

This improves the unspecified constant in a result of Erdős [29].

In [16] we give a numerical value of an unspecified constant in one of the main results of [50].

Theorem 12.4. *For every $p \in \mathcal{A}_n$,*

$$M_2(p) - M_1(p) \geq 10^{-31} M_2(p).$$

Based on the fact that for a fixed trigonometric polynomial p the function

$$\lambda \to \lambda \log(M_\lambda(p))$$

is an increasing convex function on $[0, \infty)$, we can state explicit numerical values of certain unspecified constants in some other related Littlewood results. For example, as a consequence of Theorem 12.4, we have

Theorem 12.5. *For every $p \in \mathcal{H}_n$ and $\lambda > 2$,*

$$\log(M_\lambda(p)) - \log(M_2(p)) \geq \frac{\lambda - 2}{\lambda} \log\left(\frac{1}{1 - 10^{-31}}\right), \qquad \lambda > 2\,,$$

and

$$\log(M_2(p)) - \log(M_\lambda(p)) \geq \frac{2 - \lambda}{\lambda} \log\left(\frac{1}{1 - 10^{-31}}\right), \qquad 1 \leq \lambda < 2.$$

§13. On the norm of the polynomial truncation operator

Let D and ∂D denote the open unit disk and the unit circle of the complex plane, respectively. We denote the set of all polynomials of degree at most n with real coefficients by $\mathcal{P}_n$. We denote the set of all polynomials of degree at most n with complex coefficients by $\mathcal{P}_n^c$. We define the truncation operator S_n for polynomials $P_n \in \mathcal{P}_n^c$ of the form

$$P_n(z) := \sum_{j=0}^n a_j z^j\,, \qquad a_j \in \mathbb{C}\,,$$

by

$$S_n(P_n)(z) := \sum_{j=0}^n \widetilde{a}_j z^j\,, \qquad \widetilde{a}_j := (a_j/|a_j|)\min\{|a_j|, 1\} \tag{13.1}$$

(here $0/0$ is interpreted as 1). In other words, we take the coefficients $a_j \in \mathbb{C}$ of a polynomial P_n of degree at most n, and we truncate them. That is, we leave a coefficient a_j unchanged if $|a_j| < 1$, while we replace it by $a_j/|a_j|$ if $|a_j| \geq 1$. We form the new polynomial with the new coefficients $\widetilde{a}_j$ defined by (13.1), and we denote this new polynomial by $S_n(P_n)$. We define the norms of the truncation operators by

$$\|S_n\|_{\infty,\partial D}^{real} := \sup_{P_n \in \mathcal{P}_n} \frac{\max_{z\in\partial D}|S_n(P_n)(z)|}{\max_{z\in\partial D}|P_n(z)|}$$

and

$$\|S_n\|_{\infty,\partial D}^{comp} := \sup_{P_n \in \mathcal{P}_n^c} \frac{\max_{z\in\partial D}|S_n(P_n)(z)|}{\max_{z\in\partial D}|P_n(z)|}\,.$$

Our main theorem in [37] establishes the right order of magnitude of the norms of the operators S_n. This settles a question asked by S. Kwapien.

Theorem 13.1. *With the notation introduced above there is an absolute constant $c_1 > 0$ such that*

$$c_1\sqrt{2n+1} \leq \|S_n\|_{\infty,\partial D}^{real} \leq \|S_n\|_{\infty,\partial D}^{comp} \leq \sqrt{2n+1}\,.$$

In fact we are able to establish an $L_p(\partial D)$ analogue of this as follows. For $p \in (0,\infty)$, let

$$\|S_n\|_{p,\partial D}^{real} := \sup_{P_n \in \mathcal{P}_n} \frac{\|S_n(P_n)\|_{L_p(\partial D)}}{\|P_n\|_{L_p(\partial D)}}$$

and

$$\|S_n\|_{p,\partial D}^{comp} := \sup_{S_n \in \mathcal{P}_n^c} \frac{\|S_n(P_n)\|_{L_p(\partial D)}}{\|P_n\|_{L_p(\partial D)}}\,.$$

Theorem 13.2. *With the notation introduced above there is an absolute constant $c_1 > 0$ such that*

$$c_1(2n+1)^{1/2-1/p} \leq \|S_n\|_{p,\partial D}^{real} \leq \|S_n\|_{p,\partial D}^{comp} \leq (2n+1)^{1/2-1/p}$$

for every $p \in [2, \infty)$.

Note that it remains open what is the right order of magnitude of $\|S_n\|_{p,\partial D}^{real}$ and $\|S_n\|_{p,\partial D}^{comp}$, respectively when $0 < p < 2$. In particular, it would be interesting to see if $\|S_n\|_{p,\partial D}^{comp} \leq c$ is possible for any $0 < p < 2$ with an absolute constant c. We record the following observation, due to S. Kwapien (see also [36]), in this direction.

Theorem 13.3. *There is an absolute constant $c > 0$ such that*

$$\|S_n\|_{1,\partial D}^{real} \geq c\sqrt{\log n}\,.$$

If the unit circle ∂D is replaced by the interval $[-1, 1]$, we get a completely different order of magnitude of the polynomial truncation projector. In this case the norms of the truncation operators S_n are defined in the usual way. That is, let

$$\|S_n\|_{\infty,[-1,1]}^{real} := \sup_{P_n \in \mathcal{P}_n} \frac{\max_{x \in [-1,1]} |S_n(P_n)(x)|}{\max_{x \in [-1,1]} |P_n(x)|}$$

and

$$\|S_n\|_{\infty,[-1,1]}^{comp} := \sup_{P_n \in \mathcal{P}_n^c} \frac{\max_{x \in [-1,1]} |S_n(P_n)(x)|}{\max_{x \in [-1,1]} |P_n(x)|}\,.$$

In [Er-05] we prove the following result.

Theorem 13.4. *With the notation introduced above we have*

$$2^{n/2-1} \leq \|S_n\|_{\infty,[-1,1]}^{real} \leq \|S_n\|_{\infty,[-1,1]}^{comp} \leq \sqrt{2n+1} \cdot 8^{n/2}\,.$$

In [36] we base the proof of the lower bound of Theorems 13.1 and 13.2 on the following lemma from [54].

Lemma 13.5. (Lovász, Spencer, Vesztergombi). *Let $a_{j,k}, j = 1, 2, \ldots, n_1$, $k = 1, 2, \ldots, n_2$ be such that $|a_{j,k}| \leq 1$. Let also $p_1, p_2, \ldots, p_{n_2} \in [0, 1]$. Then there are choices*

$$\varepsilon_k \in \{-p_k, 1 - p_k\}, \qquad k = 1, 2, \ldots, n_2\,,$$

such that for all j,

$$\left| \sum_{k=1}^{n_2} \varepsilon_k a_{j,k} \right| \leq c\sqrt{n_1}$$

with an absolute constant c.

§14. Problems

We use the notation introduced in Section 4. Some of the problems below are closely related to each other. Some of them have been mentioned before. Some of them have already been formulated in [10]. Here we summarize the simplest looking but most challenging ones.

Problem 14.1. (Erdős). Is it true that

$$\max_{z \in \partial D} |P_n(z)| \geq (1+\varepsilon)\sqrt{n+1}$$

for every $P_n \in \mathcal{L}_n$ with $n \geq 1$, where $\varepsilon > 0$ is an absolute constant (independent of n)?

As a matter of fact in Problem 14.1

$$\max_{z \in \partial D} |P_n(z)| \geq \sqrt{n+1} + \varepsilon$$

with an absolute constant $\varepsilon > 0$ would already be remarkable to prove. A stronger version of Problem 14.1 is the following.

Problem 14.2. Is it true that

$$\|P_n\|_{L_4(\partial D)} \geq (1+\varepsilon)\sqrt{n+1}$$

for every $P_n \in \mathcal{L}_\backslash$ with $n \geq 1$, where $\varepsilon > 0$ is an absolute constant (independent of n)?

In Problem 14.2 even

$$\|P_n\|_{L_4(\partial D)} \geq \sqrt{n+1} + \varepsilon$$

with an absolute constant $\varepsilon > 0$ would already be remarkable to prove. Problem 14.2 can be reformulated as follows.

Problem 14.3. Suppose $n \geq 1$ and $a_0 = \pm 1, a_1 = \pm 1, \dots, a_n = \pm 1$. Let

$$b_k := \sum_{j=0}^{n-k} a_j a_{j+k}, \qquad k = 1, 2, \dots, n$$

$$b_{-k} := \sum_{j=k}^{n} a_j a_{j-k}, \qquad k = 1, 2, \dots, n$$

Is it true that

$$\sum_{k=1}^{n} \left(b_k^2 + b_{-k}^2\right) > \varepsilon(n+1)^2$$

with an absolute constant $\varepsilon > 0$ (independent of n)?

In Problem 14.3 even

$$\sum_{k=1}^{n} \left(b_k^2 + b_{-k}^2\right) > \varepsilon (n+1)^{3/2}$$

with an absolute constant $\varepsilon > 0$ would already be remarkable to prove.

Problem 14.4. Is there an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{L}_n$ (or at least $P_{n_k} \in \mathcal{L}_{n_k}$)?

A polynomial $P_n \in \mathcal{P}_n$ is called skew-reciprocal if

$$P_n(z) = z^n P_n(-1/z)\,, \qquad z \in \mathbb{C}\,,\ \ z \neq 0.$$

Problem 14.5. Is there an ultraflat sequence of skew-reciprocal unimodular polynomials $P_{n_k} \in \mathcal{L}_{n_k}$ where each n_k is a multiple of 4.

Problem 14.6. Is there a sequence of unimodular polynomials $P_n \in \mathcal{L}_n$ (or at least $P_{n_k} \in \mathcal{L}_{n_k}$)? for which

$$|P_n(z)| > c\sqrt{n+1}\,, \qquad z \in \partial D?$$

Problem 14.7. Is there a sequence of unimodular polynomials $P_n \in \mathcal{K}_n$ (or at least $P_{n_k} \in \mathcal{K}_{n_k}$) for which the derivative sequence (P_n') is ultraflat, that is

$$\lim_{n\to\infty} \frac{\max_{z\in\partial D} |P_n'(z)|}{\min_{z\in\partial D} |P_n'(z)|} = 1\,?$$

If the answer is yes, what can one say about the case when $P_n \in \mathcal{L}_n$?

Theorems 5.4 and 5.7 show that the right upper bound for the number of zeros a polynomial $p \in \mathcal{F}_n$ can have at 1 is somewhere between $c_1\sqrt{n/\log(n+1)}$ and $c_2\sqrt{n}$ with absolute constants $c_1 > 0$ and $c_2 > 0$.

Problem 14.8. How many zeros can a polynomial $p \in \mathcal{F}_n$ have at 1? Close the gap between Theorems 1.1 and 1.3. Any improvements would be interesting.

Problem 14.9. How many *distinct* zeros can a polynomial p_n of the form

$$p_n(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_0| = 1\,, \quad |a_j| \leq 1\,, \quad a_j \in \mathbb{C} \qquad (14.1)4.1$$

(or $p_n \in \mathcal{F}_n$) have in $[-1,1]$? In particular, is it possible to give a sequence (p_n) of polynomials of the form (14.1) (or maybe $(p_n) \subset \mathcal{F}_n$) so that p_n has $c\sqrt{n}$ *distinct* zeros in $[-1,1]$, where $c > 0$ is an absolute constant? If not, what is the sharp analogue of Theorem 7.2 for *distinct* zeros in $[-1,1]$?

Problem 14.10. How many *distinct* zeros can a polynomial $p_n \in \mathcal{F}_n$ have in the interval $[-1+a, 1-a]$, $a \in (0,1)$? In particular, is it possible to give a sequence (p_n) of polynomials of the form (14.1) (or maybe $(p_n) \subset \mathcal{F}_n$) so that p_n has c/a *distinct* zeros in $[-1+a, 1-a]$, where $c > 0$ is an absolute constant and $a \in [n^{-1/2}, 1)$? If not, what is the sharp analogue of Theorem 7.3 for *distinct real* zeros?

It is easy to prove (see [17]) that a polynomial $p \in \mathcal{A}_n$ can have at most $\log_2 n$ zeros at -1.

Problem 14.11. Is it true that there is an absolute constant $c > 0$ such that every $p \in \mathcal{A}_n$ with $p(0) = 1$ has at most $c \log n$ *real* zeros? If not, what is the best possible upper bound for the number of *real* zeros of polynomials $p \in \mathcal{A}_n$? What is the best possible upper bound for the number of *distinct real* zeros of polynomials $p \in \mathcal{A}_n$?

Odlyzko asked the next question after observing computationally that no $p \in \mathcal{A}_n$ with $n \le 25$ had a repeat root of modulus greater than one.

Problem 14.12. Prove or disprove that a polynomial $p \in \mathcal{A}_n$ has all its repeated zeros at 0 or on the unit circle.

One can show, not completely trivially, that there are polynomials $p \in \mathcal{F}_n$ with repeated zeros in $(0,1)$ up to multiplicity 4.

Problem 14.13. Prove or disprove that a polynomial $p \in \mathcal{A}_n$ has all its repeated zeros at 0 or on the unit circle.

One can show, not completely trivially, that there are polynomials $p \in \mathcal{F}_n$ with repeated zeros in $(0,1)$ up to multiplicity 4.

Problem 14.14. Can the multiplicity of a zero of a $p \in \cup_{n=1}^{\infty} \mathcal{F}_n$ in

$$\{z \in \mathbb{C} : 0 < |z| < 1\}$$

be arbitrarily large?

A negative answer to the above question would resolve an old conjecture of Lehmer concerning Mahler's measure. (See [3].)

Boyd [20] shows that there is an absolute constant c such that every $p \in \mathcal{L}_n$ can have at most $c \log^2 n / \log \log n$ zeros at 1. Is is easy to give polynomials $p \in \mathcal{L}_n$ with $c \log n$ zeros at 1.

Problem 14.15. Prove or disprove that there is an absolute constant c such that every polynomial $p \in \mathcal{L}_n$ can have at most $c \log n$ zeros at 1. Prove or disprove that there is an absolute constant c such that every polynomial $p \in \mathcal{L}_n$ can have at most $c \log n$ zeros in $[-1, 1]$.

Problem 14.16. Can Boyd's result be extended to the class $\mathcal{K}_n$? In other words, is there an absolute constant c such that every $p \in \mathcal{K}_n$ can have at most $c \log^2 n / \log \log n$ zeros at 1? (It is easy to give polynomials $p \in \mathcal{L}_n$ with $c \log n$ zeros at 1.)

Problem 14.17. Prove or disprove that every polynomial $p \in \mathcal{L}_n$ has at least one zero in the annulus

$$\left\{ z \in \mathbb{C} : 1 - \frac{c}{n} < |z| < 1 + \frac{c}{n} \right\},$$

where $c > 0$ is an absolute constant.

Problem 14.18. Prove or disprove that $N(p_n) \to \infty$, where $N(p_n)$ denotes the number of real zeros of

$$p_n(t) := \sum_{k=0}^{n} a_{k,n} \cos kt \,, \qquad a_{k,n} = \pm 1, \ \ k = 0, 1, 2, \dots, n \,,$$

in the period $[0, 2\pi)$.

The next question is a version of an old and hard unsolved problem known as the Tarry-Escott Problem.

Problem 14.19. Let $N \in \mathbb{N}$ be fixed. Let $a(N)$ be the smallest value of k for which there is a polynomial $p \in \cup_{n=1}^{\infty} \mathcal{F}_n$ with exactly k nonzero terms in it and with a zero at 1 with multiplicity at least N. Prove or disprove that $a(N) = 2N$.

To prove that $a(N) \geq 2N$ is simple. The fact that $a(N) \leq 2N$ is known for $N = 1, 2, \dots, 10$, but the problem is open for every $N \geq 11$. The best known upper bound for $a(N)$ in general seems to be $a(N) \leq cN^2 \log N$ with an absolute constant $c > 0$. See [17]. Even improving this (like dropping the factor $\log N$) would be a significant achievement.

Problem 14.20. It would be interesting to see if $\|S_n\|_{p,\partial D}^{comp} \leq c$ is possible for any $0 < p < 2$ with an absolute constant c, where S_n is the polynomial truncation operator defined in section 13.

In the light of Theorems 6.3 and 6.4 we ask the following questions.

Problem 14.21. Does

$$\lim_{n \to \infty} \frac{\log \left(\inf_{0 \neq p \in \mathcal{F}_n} \|p\|_{[0,1]} \right)}{\sqrt{n}}$$

exist? If it does, what is it? Does

$$\lim_{n \to \infty} \frac{\log \left(\inf_{0 \neq p \in \mathcal{A}_n} \|p(-x)\|_{[0,1]} \right)}{\log^2 (n+1)}$$

exist? If it does, what is it?

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Tamás Erdélyi,
Department of Mathematics,
Texas A&M University,
College Station, Texas 77843,
USA
`terdelyi@math.tamu.edu`