REMEZ-TYPE INEQUALITY FOR NON-DENSE MÜNTZ SPACES WITH EXPLICIT BOUND

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Abstract. Let $\Lambda := (\lambda_k)_{k=0}^\infty$ be a sequence of distinct nonnegative real numbers with $\lambda_0 := 0$ and $\sum_{k=1}^\infty 1/\lambda_k < \infty$. Let $\varrho \in (0,1)$ and $\epsilon \in (0, 1-\varrho)$ be fixed. An earlier work of the first two authors shows that

$$C(\Lambda, \epsilon, \varrho)$$

$$:= \sup \left\{ \|p\|_{[0,\varrho]} : \quad p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}, \ \ m\{x \in [\varrho, 1] : |p(x)| \leq 1\} \geq \epsilon \right\}$$

is finite. In this paper an explicit upper bound for $C(\Lambda, \epsilon, \varrho)$ is given. In the special case $\lambda_k := k^\alpha$, $\alpha > 1$, our bounds are essentially sharp.

1. Introduction

In this paper $\Lambda := (\lambda_k)_{k=0}^\infty$ always denotes a sequence of real numbers satisfying

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots .$$

In [1] a Remez-type inequality for Müntz polynomials:

$$p(x) = \sum_{k=0}^n a_k x^{\lambda_k}$$

or equivalently for Dirichlet sums:

$$P(t) = \sum_{k=0}^n a_k e^{-\lambda_k t}$$

is established. The most common form of this inequality states that for every sequence $(\lambda_k)_{k=0}^\infty$ satisfying $\sum_{k=0}^\infty 1/\lambda_k < \infty$, there exists a constant $C(\Lambda, \epsilon)$ depending only on Λ and ϵ (and not on n, ϱ, or A) so that

$$\|p\|_{[0,\varrho]} \leq C(\Lambda, \epsilon) \|p\|_A$$

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for every Müntz polynomial p, as above, associated with the sequence $(\lambda_k)_{k=0}^\infty$, and for every set $A \subset [\varrho, 1]$ of Lebesgue measure at least $\epsilon > 0$. Throughout this paper $\|\cdot\|_A$ denotes the uniform norm on $A \subset \mathbb{R}$.

Using this Remez-type inequality, we resolved two reasonably long standing conjectures in [1]. In this paper we give an explicit upper bound for the best possible $C(\Lambda, \epsilon)$ in the above Remez-type inequality for non-dense Müntz spaces. Theorem 2.3 extends an inequality of Schwartz [4] in two directions. Theorem 2.1 offers a more explicit bound for the sequences $\Lambda := (k^\alpha)_{k=0}^\infty, \ \alpha > 1$. The sharpness of the Remez-type inequality of Theorem 2.1 is shown by Theorem 2.2.

2. Results

Theorem 2.1. *Let $\lambda_k := k^\alpha, \ k = 0, 1, \dots, \ \alpha > 1$. Let $\varrho \in (0,1)$, $\epsilon \in (0, 1-\varrho)$, and $\epsilon \leq 1/2$. There exists a constant $c_\alpha > 0$ depending only on α so that*

$$\|p\|_{[0,\varrho]} \leq \exp\left(c_\alpha \epsilon^{1/(1-\alpha)}\right) \|p\|_A$$

for every $p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ and for every set $A \subset [\varrho, 1]$ of Lebesgue measure at least $\epsilon > 0$.

The next theorem shows that the inequality of Theorem 2.1 is essentially the best possible.

Theorem 2.2. *Let $\lambda_k := k^\alpha, \ k = 0, 1, \dots, \ \alpha > 1$. For every $\alpha > 1$ and $\epsilon \in (0, 1/2]$, there exists a constant $c_\alpha > 0$ depending only on α and Müntz polynomials*

$$0 \neq p = p_{\alpha,\epsilon} \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

depending only on α and ϵ so that

$$|p(0)| \geq \exp\left(c_\alpha \epsilon^{1/(1-\alpha)}\right) \|p\|_{[1-\epsilon,1]}\,.$$

Theorem 2.1 is a special case of the following more general, but less explicit result.

Theorem 2.3. *Suppose $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ and $\sum_{k=0}^\infty 1/\lambda_k < \infty$. Let $\varrho \in (0,1)$ and $\epsilon \in (0, 1-\varrho)$. Let $\delta := -\frac{1}{2}\log(1-\epsilon)$. Let $N \in \mathbb{N}$ be chosen so that*

$$\sum_{k=N+1}^\infty \frac{1}{\lambda_k} \leq \frac{\delta}{3}\,.$$

Let

$$\sigma_k := A\lambda_k \qquad \text{with} \qquad A := \frac{\delta}{3N}\,.$$

Then, with $c := \|t^{-1}\sin t\|_{L_2(\mathbb{R})}$,

$$\|p\|_{[0,\varrho]} \leq \frac{3c}{\delta} \prod_{k=1}^N \left(2 + \frac{1}{\sigma_k}\right) \|p\|_A$$

for every $p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ and for every set $A \subset [\varrho, 1]$ of Lebesque measure at least $\epsilon > 0$.

3. Lemmas

Our first lemma shows that $C(\Lambda,\epsilon)$ in the Remez-type inequality is related to a much simpler (Chebyshev-type) extremal problem. This is proved in both [1] and [2].

Lemma 3.1. *Suppose* $0=\lambda_0<\lambda_1<\lambda_2<\cdots$, $\rho\in(0,1)$, *and* $\epsilon\in(0,1-\rho)$. *Then*

$$\sup\left\{\|p\|_{[0,\varrho]}:\ p\in\operatorname{span}\{x^{\lambda_0},x^{\lambda_1},\dots\},\ m\{x\in[\varrho,1]:|p(x)|\le 1\}\ge\epsilon\right\}$$
$$=\sup\left\{\frac{|p(0)|}{\|p\|_{[1-\epsilon,1]}}:\ p\in\operatorname{span}\{x^{\lambda_0},x^{\lambda_1},\dots\}\right\}.$$

Our key lemma is the following.

Lemma 3.2. *Suppose* $0=\lambda_0<\lambda_1<\lambda_2<\cdots$ *and* $\sum_{k=0}^{\infty}1/\lambda_k<\infty$. *Given* $\delta\in(0,1)$, *let* $N\in\mathbb{N}$ *be chosen so that*

$$\sum_{k=N+1}^{\infty}\frac{1}{\lambda_k}\le\frac{\delta}{3}\,.$$

Let

$$\sigma_k:=A\lambda_k\qquad\text{with}\qquad A:=\frac{\delta}{3N}\,.$$

Then

$$|P(\infty)|\le\frac{3c}{\delta}\prod_{k=1}^{N}\left(2+\frac{1}{\sigma_k}\right)\|P\|_{[-\delta,\delta]}$$

for every $P\in\operatorname{span}\{e^{-\lambda_0 t},e^{-\lambda_1 t},\dots\}$ *with* $c:=\|t^{-1}\sin t\|_{L_2(\mathbb{R})}$.

In the proof of Lemma 3.2 we will need the following observation.

Lemma 3.3. *Let* $0=\lambda_0<\lambda_1<\lambda_2<\cdots$. *Suppose*

(1) $F\in E^{\delta}\cap L_2(\mathbb{R})$;

(2) $F(i\lambda_k)=0\,,\ k=1,2,\dots$ *(i is the imaginary unit)*;

(3) $F(0)=1\,.$

Then

$$|P(\infty)|\le\|F\|_{L_2(\mathbb{R})}\,\|P\|_{L_2[-\delta,\delta]}$$

for every $P\in\operatorname{span}\{e^{-\lambda_0 t},e^{-\lambda_1 t},\dots\}$.

An entire function f is called a function of *exponential type* δ if there exists a constant c depending only on f so that

$$|f(z)|\le c\,\exp(\delta|z|)\,,\quad z\in\mathbb{C}\,.$$

The collection of all such entire functions of exponential type δ is denoted be E^{δ}. The Paley-Wiener Theorem (see, for example, [3]) characterizes the functions F which can be written as the Fourier transform of some function $f\in L_2[-\delta,\delta]$. We will need it in the proof of Lemma 3.3.

Theorem (Paley-Wiener). *Let $\delta \in (0,\infty)$. Then $f \in E^\delta \cap L_2(\mathbb{R})$ if and only if there exists an $f \in L_2[-\delta,\delta]$ so that*

$$F(z) = \int_{-\delta}^{\delta} f(t)e^{itz}\,dt\,.$$

The following comparison theorem for Müntz polynomials is proved in [2]. We will need it in the proof of Theorem 2.3.

Lemma 3.4. *Let $\Lambda := (\lambda_k)_{k=0}^\infty$ and $\Gamma := (\gamma_k)_{k=0}^\infty$ be increasing sequences of nonnegative real numbers with $\lambda_0 = 0$, $\gamma_0 = 0$, and $\lambda_k \le \gamma_k$ for each k. Let $0 < a < b$. Then*

$$\begin{aligned}&\max\left\{\frac{|p(0)|}{\|p\|_{[a,b]}}:\ \ p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}\right\}\\ \ge &\max\left\{\frac{|p(0)|}{\|p\|_{[a,b]}}:\ \ p \in \operatorname{span}\{x^{\gamma_0}, x^{\gamma_1}, \dots, x^{\gamma_n}\}\right\}.\end{aligned}$$

4. Proofs

Proof of Lemma 3.3. By the Paley-Wiener Theorem

$$F(z) = \int_{-\delta}^{\delta} f(t)e^{itz}\,dt$$

for some $f \in L_2[-\delta,\delta]$. Now if

$$P(t) = a_0 + \sum_{k=1}^n a_k e^{-\lambda_k t}\,,$$

then

$$\begin{aligned}\int_{-\delta}^{\delta} f(t)P(t)\,dt &= a_0 \int_{-\delta}^{\delta} f(t)\,dt + \sum_{k=1}^n a_k \int_{-\delta}^{\delta} f(t)e^{-\lambda_k t}\,dt\\ &= a_0F(0) + \sum_{k=1}^n a_k F(i\lambda_k) = a_0 = P(\infty)\,.\end{aligned}$$

Hence by the Cauchy-Schwartz Inequality and the L_2 inversion theorem of Fourier transforms, we obtain

$$|P(\infty)| \le \|f\|_{L_2[-\delta,\delta]}\,\|P\|_{L_2[-\delta,\delta]} \le \|F\|_{L_2(\mathbb{R})}\,\|P\|_{L_2[-\delta,\delta]}$$

and the lemma is proved. $\square$

Proof of Lemma 3.2. We define

$$F(z) := \frac{\sin(\delta z/3)}{\delta z/3} \prod_{k=1}^{N} \left(\left(1 - \frac{z}{i\lambda_k}\right) \frac{\sin(\sigma_k z/\lambda_k)}{\sigma_k z/\lambda_k} \right) \prod_{k=N+1}^{\infty} \left(1 - \left(\frac{\sin(z/\lambda_k)}{\sin i}\right)^4 \right),$$

where i is the imaginary unit. It follows easily that

$$F \in E^\delta\,, \quad F(0) = 1\,, \quad F(i\lambda_k) = 0\,, \quad k = 1, 2, \dots$$

and

$$|F(t)| \le \frac{\sin(\delta t/3)}{(\delta t/3} \prod_{k=1}^{N} \left(2 + \frac{1}{\sigma_k}\right)\,, \qquad t \in \mathbb{R}\,.$$

Hence Lemma 3.3 implies that

$$|P(\infty)| \le \frac{3c}{\delta} \prod_{k=1}^{N} \left(2 + \frac{1}{\sigma_k}\right) \|P\|_{[-\delta,\delta]}$$

for every $P \in \operatorname{span}\{e^{-\lambda_0 t}, e^{-\lambda_1 t}, \dots\}$ with $c := \|t^{-1}\sin t\|_{L_2(\mathbb{R})}$. □

Proof of Theorem 2.3. When $A = [1-\epsilon, 1]$, the theorem follows from Lemma 3.2 by the substitution $x = e^{-\delta}e^{-t}$. The general case follows from Lemma 3.1. □

Proof of Theorem 2.1. Let

$$\delta := -\frac{1}{2}\log(1-\epsilon)\,. \tag{4.1}$$

Observe that N in Theorem 2.1 can be chosen so that

$$N := \left\lfloor \left(\frac{\delta(\alpha-1)}{3}\right)^{1/(1-\alpha)} \right\rfloor + 1\,. \tag{4.2}$$

Also, σ_k in Lemma 3.2 is of the form

$$\sigma_k = \frac{\delta k^\alpha}{3N}\,.$$

Let $M+1$ be the smallest value of $k \in \mathbb{N}$ for which

$$\frac{1}{\sigma_k} < 1\,, \qquad \text{that is}\,, \qquad \frac{3N}{k^\alpha \delta} \le 1\,.$$

Note that

$$M := \left\lfloor \left(\frac{3N}{\delta}\right)^{1/\alpha} \right\rfloor\,.$$

If $0 < M < N$, then

$$\begin{aligned}\prod_{k=1}^{N}\left(2+\frac{1}{\sigma_k}\right) &= \prod_{k=1}^{N}\left(2+\frac{3N}{\delta k^\alpha}\right)\\
&\leq \left(\prod_{k=1}^{M}\frac{9N}{\delta k^\alpha}\right)\left(\prod_{k=M+1}^{N}3\right) = \left(\frac{9N}{\delta}\right)^M\left(\frac{M}{e}\right)^{-\alpha M}3^{N-M}\\
&= \left(\frac{9e^\alpha N}{\delta}\right)^M M^{-\alpha M}3^{N-M}\\
&\leq \left(\frac{9e^\alpha N}{\delta}\right)^M\left(\frac{1}{2}\left(\frac{3N}{\delta}\right)^{1/\alpha}\right)^{-\alpha M}3^{N-M}\\
&\leq (3(2e)^\alpha)^M 3^{N-M} \leq (3(2e)^\alpha)^N\,,\end{aligned}$$

and the theorem follows by (4.1), (4.2), and Theorem 2.1.

If $N \leq M$, then

$$\begin{aligned}\prod_{k=1}^{N}\left(2+\frac{1}{\sigma_k}\right) &= \prod_{k=1}^{N}\left(2+\frac{3N}{\delta k^\alpha}\right)\\
&\leq \left(\prod_{k=1}^{N}\frac{9N}{\delta k^\alpha}\right) = \left(\frac{9N}{\delta}\right)^N\left(\frac{N}{e}\right)^{-\alpha N}\\
&= \left(\frac{9e^\alpha N^{(1-\alpha)}}{\delta}\right)^N \leq \left(\frac{9e^\alpha}{\delta}\right)^N\left(\left(\frac{\delta(\alpha-1)}{3}\right)^{1/(1-\alpha)}\right)^{(1-\alpha)N}\\
&\leq \left(\frac{9e^\alpha}{\delta}\right)^N\left(\frac{\delta(\alpha-1)}{3}\right)^N \leq (3e^\alpha(\alpha-1))^N\,,\end{aligned}$$

and the theorem follows by (4.1), (4.2), and Theorem 2.1.

If $M = 0$, then

$$\prod_{k=1}^{N}\left(2+\frac{1}{\sigma_k}\right) \leq \prod_{k=1}^{N}3 = 3^N\,,$$

and the theorem follows by (4.1), (4.2), and Theorem 2.1. $\square$

Proof of Theorem 2.2. Let $n \in \mathbb{N}$ be a fixed. We define $\gamma_k := kn^{\alpha-1},\ \ k = 0.1,\dots$. Let $T_n(x) := \left(\frac{1}{2}(x-1)\right)^n$ and

$$Q_n(x) := T_n\left(\frac{2x^{n^{\alpha-1}}}{1-(1-\epsilon)^{n^{\alpha-1}}} - \frac{1+(1-\epsilon)^{n^{\alpha-1}}}{1-(1-\epsilon)^{n^{\alpha-1}}}\right)^n \in \operatorname{span}\{x^{\gamma_0}, x^{\gamma_1}, \dots x^{\gamma_n}\}.$$

Then, by Lemma 3.4,

$$\begin{aligned}\sup\left\{\frac{|p(0)|}{\|p\|_{[1-\epsilon,1]}}:\ \ p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}\right\} &\geq \frac{|Q_n(0)|}{\|Q_n\|_{[1-\epsilon,1]}} = |Q_n(0)|\\
&= \left(\frac{1}{1-(1-\epsilon)^{n^{\alpha-1}}}\right)^n.\end{aligned}$$

Now let n be the smallest integer satisfying $n^{\alpha-1} \geq \epsilon^{-1}$. Since $(1-\epsilon)^{1/\epsilon}$ is bounded away from 0 on $(0, 1/2]$, the result follows. $\square$

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