IMPROVED LOWER BOUND FOR THE MAHLER MEASURE OF THE FEKETE POLYNOMIALS

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Abstract. We show that there is an absolute constant $c > 1/2$ such that the Mahler measure of the Fekete polynomials f_p of the form

$$f_p(z) := \sum_{k=1}^{p-1} \left(\frac{k}{p} \right) z^k \,,$$

(where the coefficients are the usual Legendre symbols) is at least $c\sqrt{p}$ for all sufficiently large primes p. This improves the lower bound $\left(\frac{1}{2} - \varepsilon\right)\sqrt{p}$ known before for the Mahler measure of the Fekete polynomials f_p for all sufficiently large primes $p \geq c_\varepsilon$. Our approach is based on the study of the zeros of the Fekete polynomials on the unit circle.

1. Introduction and Notation

Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D. Let

$$\mathcal{K}_n := \left\{ P_n : P_n(z) = \sum_{k=0}^{n} a_k z^k, \;\; a_k \in \mathbb{C}\,, \;\; |a_k| = 1 \right\} .$$

The class $\mathcal{K}_n$ is often called the collection of all (complex) unimodular polynomials of degree n. Let

$$\mathcal{L}_n := \left\{ P_n : P_n(z) = \sum_{k=0}^{n} a_k z^k, \;\; a_k \in \{-1, 1\} \right\} .$$

The class $\mathcal{L}_n$ is often called the collection of all Littlewood polynomials of degree n. By Parseval's formula,

$$\int_0^{2\pi} |P_n(e^{it})|^2 \, dt = 2\pi(n+1)$$

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for all $P_n \in \mathcal{K}_n$. Therefore

$$\min_{z \in \partial D} |P_n(z)| < \sqrt{n+1} < \max_{z \in \partial D} |P_n(z)|$$

for all $P_n \in \mathcal{K}_n$ and $n \geq 1$. An old problem (or rather an old theme) is the following.

Let $\alpha < \beta$ be real numbers. The Mahler measure $M_0(P, [\alpha, \beta])$ is defined for bounded measurable functions $P(e^{it})$ defined on $[\alpha, \beta]$ as

$$M_0(P, [\alpha, \beta]) := \exp\left(\frac{1}{\beta - \alpha} \int_\alpha^\beta \log |P(e^{it})| \, dt\right) .$$

It is well known that

$$M_0(P, [\alpha, \beta]) = \lim_{q \to 0+} M_q(P, [\alpha, \beta]) ,$$

where, for $q > 0$,

$$M_q(P, [\alpha, \beta]) := \left(\frac{1}{\beta - \alpha} \int_\alpha^\beta \left|P(e^{it})\right|^q \, dt\right)^{1/q} .$$

It is a simple consequence of the Jensen formula that

$$M_0(P) := M_0(P, [0, 2\pi]) = |c| \prod_{k=1}^n \max\{1, |z_k|\}$$

for every polynomial of the form

$$P(z) = c \prod_{k=1}^n (z - z_k) , \qquad c, z_k \in \mathbb{C} .$$

P. Borwein and Lockhart [B-01] investigated the asymptotic behavior of the mean value of normalized L_q norms of Littlewood polynomials for arbitrary $q > 0$. Using the Lindeberg Central Limit Theorem and dominated convergence, they proved that

$$\lim_{n \to \infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} \frac{(M_q(f, [0, 2\pi]))^q}{n^{q/2}} = \Gamma\left(1 + \frac{q}{2}\right)$$

for every $q > 0$. In [C-15] we proved that

$$\lim_{n \to \infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} \frac{M_q(f, [0, 2\pi])}{n^{1/2}} = \left(\Gamma\left(1 + \frac{q}{2}\right)\right)^{1/q}$$

for every $q > 0$. We also proved analogous results for the Mahler measure. Namely, using the notation $\widehat{f}(z) := \max\{|f(z)|, n^{-1}\}$, we have

$$\lim_{n \to \infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} \log\left(\frac{M_0(\widehat{f}, [0, 2\pi])}{n^{1/2}}\right) = -\gamma/2$$

and

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{f\in\mathcal{L}_n} \frac{M_0(f,[0,2\pi])}{n^{1/2}} = e^{-\gamma/2}\,,$$

where

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 0.577215\ldots$$

is the Euler constant and $e^{-\gamma/2} = 0.749306\ldots$. These are analogues of the results proved earlier by Choi and Mossinghoff [C-11] for polynomials in $\mathcal{K}_n$.

Finding polynomials with suitably restricted coefficients and maximal Mahler measure has interested many authors. Beller and Newman [B-73] constructed unimodular polynomials of degree n whose Mahler measure is at least $\sqrt{n} - c/\log n$. For a prime p the p-th Fekete polynomial is defined as

$$f_p(z) := \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) z^k\,,$$

where

$$\left(\frac{k}{p}\right) = \begin{cases} 1, & \text{if } \; x^2 \equiv k \pmod p \; \text{ for an } \; x \not\equiv 0 \pmod p\,, \\ 0, & \text{if } \; p \; \text{divides } \; k\,, \\ -1, & \text{otherwise} \end{cases}$$

is the usual Legendre symbol. Note that $g_p(z) := f_p(z)/z$ is a Littlewood polynomial of degree $p-2$, and has the same Mahler measure as f_p.

Montgomery [M-80] proved the following fundamental result.

Theorem 1.1. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1\sqrt{p}\log\log p \leq \max_{z\in\partial D} |f_p(z)| \leq c_2\sqrt{p}\log p\,.$$

In [E-07] we proved the following result.

Theorem 1.2. *For every $\varepsilon > 0$ there is a constant c_ε such that*

$$M_0(f_p,[0,2\pi]) \geq \left(\frac{1}{2} - \varepsilon\right)\sqrt{p}$$

for all primes $p \geq c_\varepsilon$.

From Jensen's inequality,

$$M_0(f_p,[0,2\pi]) \leq M_2(f_p,[0,2\pi]) = \sqrt{p-1}\,.$$

However, as it was observed in [E-07], $\frac{1}{2} - \varepsilon$ in Theorem 1.2 cannot be replaced by $1 - \varepsilon$. Indeed, if $p \geq 3$ is a prime and $p = 4m + 1$, then f_p is self-reciprocal, that is, $z^p f_p(1/z) = f_p(z)$, and hence

$$f_p(e^{2it}) = e^{ipt} \sum_{k=0}^{(p-3)/2} a_k \cos((2k+1)t), \qquad a_k \in \{-2, 2\}.$$

Therefore a result of Littlewood [L-66] implies that

$$\begin{aligned} M_0(f_p) &\leq \frac{1}{2\pi} \int_0^{2\pi} |f_p(e^{iu})|\, du = \frac{1}{2\pi} \int_0^{\pi} |f_p(e^{2it})|\, 2dt = \frac{1}{2\pi} \int_0^{2\pi} |f_p(e^{2it})|\, dt \\ &\leq (1 - \varepsilon_0)\sqrt{p-1} \end{aligned}$$

with some absolute constant $\varepsilon_0 > 0$. If $p \geq 3$ is a prime and $p = 4m + 3$, then f_p is anti-self-reciprocal, that is, $z^p f_p(1/z) = -f_p(z)$, and hence

$$if_p(e^{2it}) = e^{ipt} \sum_{k=0}^{(p-3)/2} a_k \sin((2k+1)t), \qquad a_k \in \{-2, 2\}.$$

Therefore a result of Littlewood [L-66] implies that

$$\begin{aligned} M_0(f_p, [0, 2\pi]) &\leq \frac{1}{2\pi} \int_0^{2\pi} |if_p(e^{iu})|\, du = \frac{1}{2\pi} \int_0^{\pi} |if_p(e^{2it})|\, 2dt = \frac{1}{2\pi} \int_0^{2\pi} |if_p(e^{2it})|\, dt \\ &\leq (1 - \varepsilon_0)\sqrt{p-1} \end{aligned}$$

with some absolute constant $\varepsilon_0 > 0$.

It is an interesting open question whether there is a sequence of Littlewood polynomials (f_n) such that for an arbitrary $\varepsilon > 0$, and n large enough,

$$M_0(f_n, [0, 2\pi]) \geq (1 - \varepsilon)\sqrt{n}\,.$$

In [E-11] Theorem 1.2 was extended to subarcs of the unit circle.

Theorem 1.3. *There exists an absolute constant $c_1 > 0$ such that*

$$M_0(f_p, [\alpha, \beta]) \geq c_1 p^{1/2}$$

for all primes p and for all $\alpha, \beta \in \mathbb{R}$ such that $(\log p)^{3/2} p^{-1/2} \leq \beta - \alpha \leq 2\pi$.

In [E-12] we gave an upper bound for the average value of $|f_p(z)|^q$ over any subarc I of the unit circle, valid for all sufficiently large primes p and all exponents $q > 0$.

Theorem 1.4. *There exists a constant $c_2(q,\varepsilon)$ depending only on $q>0$ and $\varepsilon>0$ such that*

$$M_q(f_p,[\alpha,\beta]) \le c_2(q,\varepsilon)p^{1/2}\,,$$

for all primes p and for all $\alpha,\beta\in\mathbb{R}$ such that $2p^{-1/2+\varepsilon} \le \beta-\alpha \le 2\pi$.

We remark that a combination of Theorems 1.3 and 1.4 shows that there is an absolute constant $c_1>0$ and a constant $c_2(q,\varepsilon)>0$ depending only on $q>0$ and $\varepsilon>0$ such that

$$c_1 p^{1/2} \le M_q(f_p,[\alpha,\beta]) \le c_2(q,\varepsilon)p^{1/2}$$

for all primes p and for all $\alpha,\beta\in\mathbb{R}$ such that $(\log p)^{3/2}p^{-1/2} \le 2p^{-1/2+\varepsilon} \le \beta-\alpha \le 2\pi$.

The L_q norm of polynomials related to Fekete polynomials were studied in several recent papers. See [B-01b], [B-02], [B-04], [G-16], [J-13a], and [J-13b], for example. An interesting extremal property of the Fekete polynomials is proved in [B-01c].

Fekete might have been the first one to study analytic properties of the Fekete polynomials. He had an idea of proving non-existence of Siegel zeros (that is, real zeros "especially close to 1") of Dirichlet L-functions from the positivity of Fekete polynomials on the interval $(0,1)$, where the positivity of Fekete polynomials is often referred to as the Fekete Hypothesis. There were many mathematicians trying to understand the zeros of Fekete polynomials including Fekete and Pólya [F-12], Pólya [P-19], Chowla [C-35], Heilbronn [H-37], Montgomery [M-80], Baker and Montgomery [B-90], and Jung and Shen [J-16].

Baker and Montgomery [B-90] proved that f_p has a large number of zeros in $(0,1)$ for almost all primes p, that is, the number of zeros of f_p in $(0,1)$ tends to ∞ as p tends to ∞, and it seems likely that there are, in fact, about $\log\log p$ such zeros.

Conrey, Granville, Poonen, and Soundararajan [C-00] showed that f_p has asymptotically κp zeros on the unit circle, where $0.500668 < \kappa < 0.500813$.

An interesting recent paper [B-17] studies power series approximations to Fekete polynomials.

It is conjectured, see [B-02] for instance, that there are sequences of flat Littlewood polynomials $P_n \in \mathcal{L}_n$ satisfying

$$c_1\sqrt{n+1} \le |P_n(z)| \le c_2\sqrt{n+1}\,, \qquad z\in\partial D\,,$$

with absolute constants $c_1>0$ and $c_2>0$. However, the lower bound part of this conjecture, by itself, seems hard, and no sequence is known that satisfies just the lower bound. A sequence of Littlewood polynomials satisfying just the upper bound is given by the Rudin-Shapiro polynomials. They appear in Harold Shapiro's 1951 thesis [S-51] at MIT and are sometimes called just Shapiro polynomials. They also arise independently in a paper by Golay (1951). They are remarkably simple to construct and are a rich source of counterexamples to possible conjectures. The Rudin-Shapiro polynomials are defined recursively as follows:

$$\begin{aligned} P_0(z) &:= 1\,, \qquad Q_0(z) := 1\,, \\ P_{n+1}(z) &:= P_n(z) + z^{2^n}Q_n(z)\,, \\ Q_{n+1}(z) &:= P_n(z) - z^{2^n}Q_n(z)\,, \qquad n = 0,1,2,\ldots\,. \end{aligned}$$

Note that both P_n and Q_n are polynomials of degree $N-1$ with $N := 2^n$ having each of their coefficients in $\{-1,1\}$. In [E-16] we showed that the Mahler measure and the maximum norm of the Rudin-Shapiro polynomials on the unit circle of the complex plane have the same size.

Theorem 1.5. *Let P_n and Q_n be the n-th Rudin-Shapiro polynomials defined in Section 1. There is an absolute constant $c_1 > 0$ such that*

$$M_0(P_n, [0, 2\pi]) = M_0(Q_n, [0, 2\pi]) \geq c_1 \sqrt{N}\,,$$

where

$$N := 2^n = \deg(P_n) + 1 = \deg(Q_n) + 1\,.$$

2. New Result

In this paper we improve the factor $\left(\frac{1}{2} - \varepsilon\right)$ in Theorem 1.2 to an absolute constant $c > 1/2$. Namely we prove the following.

Theorem 2.1. *There is an absolute constant $c > 1/2$ such that*

$$M_0(f_p) \geq c\sqrt{p}$$

for all sufficiently large primes.

3. Lemmas

To prove the theorem we need a few lemmas. In this section first we state all the lemmas we need. Lemmas 3.1, 3.4, 3.5, 3.6, and 3.8 are known results, and we give give the references to them before stating these lemmas in this section. The remaining lemmas are proved in Section 4. For a natural number p let

$$\zeta_p := \exp\left(\frac{2\pi i}{p}\right)$$

be the first p-th root of unity. Our first lemma formulates a characteristic property of the Fekete polynomials. A simple proof is given in [B-02, pp. 37-38].

Lemma 3.1 (Gauss). *We have*

$$f_p(\zeta_p^j) = \sqrt{\left(\frac{-1}{p}\right) p}\,, \qquad j = 1, 2, \ldots, p-1\,,$$

and $f_p(1) = 0$.

Our starting idea is the following observation.

Lemma 3.2. *We have*

$$\left(\prod_{j=0}^{p-1} |Q(\zeta_p^j)|\right)^{1/p} \leq 2M_0(Q)$$

for all polynomials Q of degree at most p with complex coefficients.

However, we cannot use this lemma to prove Theorem 2.1. What we will need is the following extension of Lemma 3.2. It may look somewhat more technical than Lemma 3.2, but it turns out to be a useful tool to prove our main result in this paper.

Lemma 3.3. *Let $0 < \eta \leq \pi/2$ be fixed. Suppose a polynomial Q of degree at most p with complex coefficients has at least k zeros*

$$b_j = e^{it_j}, \qquad j = 1, 2, \ldots, k,$$

such that

$$t_j \in [0, 2\pi) \setminus \bigcup_{\nu=0}^{p-1} \left(\frac{(2\nu+1)\pi}{p} - \frac{\eta}{p}, \ \frac{(2\nu+1)\pi}{p} + \frac{\eta}{p}\right).$$

We have

$$\left(\prod_{j=0}^{p-1} |Q(\zeta_p^j)|\right)^{1/p} \leq 2\left(\cos\frac{\eta}{2}\right)^{k/p} M_0(Q).$$

In the proof of Theorem 2.1 we need one of the following two results. For proofs see [B-97a] and [B97-b], respectively.

Lemma 3.4. *There is an absolute constant $c > 0$ such that every $Q \in \mathcal{K}_n$ has at most $c\sqrt{n}$ real zeros.*

Lemma 3.5. *There is an absolute constant $c > 0$ such that every $Q \in \mathcal{L}_n$ has at most $\dfrac{c\log^2 n}{\log\log n}$ zeros at 1.*

The large sieve of number theory [M-78] asserts the following.

Lemma 3.6. *If*

$$P(z) = \sum_{k=-n}^{n} a_k z^k, \qquad a_k \in \mathbb{C},$$

is a trigonometric polynomial of degree at most n,

$$0 \leq t_1 < t_2 < \cdots < t_m \leq 2\pi,$$

and

$$\delta := \min\{t_2 - t_1, t_3 - t_2, \ldots, t_m - t_{m-1}, 2\pi - (t_m - t_1)\},$$

then

$$\sum_{j=1}^{m} \left|P\left(e^{it_j}\right)\right|^2 \le \left(\frac{2n+1}{2\pi} + \delta^{-1}\right) \int_0^{2\pi} \left|P\left(e^{it}\right)\right|^2 \, dt\,.$$

It turns out to be fairly easy to show that at least half of the zeros of f_p are on the unit circle ∂D. First note that

$$F_p(z) := z^{-p/2} f_p(z) = \sum_{a=1}^{(p-1)/2} \left(\frac{a}{p}\right) \left(z^{a-p/2} + \left(\frac{-1}{p}\right) z^{p/2-a}\right).$$

Observe also that

$$F_p(e^{2i\pi t}) := \begin{cases} 2 \displaystyle\sum_{a=1}^{(p-1)/2} \left(\frac{a}{p}\right) \cos((2a-p)\pi t) & \text{if } p \equiv 1 \pmod 4 \\ 2i \displaystyle\sum_{a=1}^{(p-1)/2} \left(\frac{a}{p}\right) \sin((2a-p)\pi t) & \text{if } p \equiv 3 \pmod 4\,. \end{cases} \tag{3.1}$$

Define $H_p(t) := F_p(e^{2i\pi t})$ if $p \equiv 1 \pmod 4$, and $H_p(t) := -iF_p(e^{2i\pi t})$ if $p \equiv 3 \pmod 4$. By (3.1) we see that $H_p(t)$ is a periodic, continuous, real-valued function when t is real.

Lemma 3.7. *Let p be a prime. There are at least $(p-3)/2$ values of $k \in \{0, 1, \dots, p-1\}$ for which H_p has a zero between k/p and $(k+1)/p$.*

Our next lemma is Theorem 4 in [C-00]. For a proof of Lemma 3.8 below see Section 6 in [C-00].

Lemma 3.8. *Let p be a prime. For every fixed real number δ*

$$\left|\left\{k \in \{1, 2, \dots, p\} : H_p\left(\frac{k+1/2}{p}\right) < \delta\sqrt{p}\right\}\right| \sim c_\delta p$$

as $p \to \infty$, where

$$c_\delta = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \sin(\delta\pi x) C(x) \, \frac{dx}{x}\,, \qquad C(x) := \prod_{k=0}^{\infty} \cos^2\left(\frac{2x}{2k+1}\right).$$

Moreover $c_{-\delta} = 1 - c_\delta$ for all $\delta > 0$.

Lemma 3.9. *For every $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\left|\left\{k \in \{1, 2, \dots, p\} : \left|H_p\left(\frac{k+1/2}{p}\right)\right| \ge \delta\sqrt{p}\right\}\right| \ge (1-\varepsilon)p$$

for all sufficiently large primes $p \ge N_\varepsilon$.

Lemma 3.10. *Let $\gamma > 0$ be a real number. Let the subarcs I_k of the unit circle ∂D be defined by*

$$I_k := \left\{ e^{it} : \left| t - \frac{(2k+1)\pi}{p} \right| \leq \frac{\pi}{2p} \right\}, \qquad k = 0, 1, \ldots, p-1 .$$

We have

$$m := \left| \left\{ k \in \{0, 1, \ldots, p-1\} : \max_{z \in I_k} |f_p'(z)| \geq \gamma p^{3/2} \right\} \right| \leq \gamma^{-2} p$$

for all primes $p \geq 3$.

Lemma 3.11. *Given $\eta > 0$ let the subarcs $I_{k,\eta}$ of the unit circle ∂D be defined by*

$$I_{k,\eta} := \left\{ e^{it} : \left| t - \frac{(2k+1)\pi}{p} \right| < \frac{\eta}{p} \right\}, \qquad k = 0, 1, \ldots, p-1 .$$

For every $\varepsilon > 0$ there is an $\eta > 0$ such that

$$|\{k \in \{0, 1, \ldots, p-1\} : f_p(z) \neq 0 \ \text{ for all } \ z \in I_{k,\eta}\}| \geq (1-\varepsilon)p$$

for all sufficiently large primes $p \geq N_\varepsilon$.

4. Proofs of the Lemmas

Proof of Lemma 3.2. Let

$$Q(z) = c \prod_{j=1}^{m} (z - a_j), \qquad c, a_j \in \mathbb{C},$$

with some $m \leq p$. Without loss of generality we may assume that $c = 1$. Note that

$$|a_j^p - 1|^{1/p} \leq (2|a_j|^p)^{1/p} = 2^{1/p}|a_j|, \qquad |a_j| \geq 1,$$

while

$$|a_j^p - 1|^{1/p} \leq 2^{1/p}, \qquad |a_j| < 1 .$$

Multiplying these inequalities for $j = 1, 2, \ldots, m$, we obtain

$$\left(\prod_{j=0}^{p-1} |Q(\zeta_p^j)| \right)^{1/p} = \left(\prod_{j=0}^{m} |a_j^p - 1| \right)^{1/p} \leq 2^{m/p} \prod_{j=0}^{m} \max\{|a_j|, 1\} \leq 2M_0(Q) .$$

□

Proof of Lemma 3.3. Let

$$Q(z) = c \prod_{j=1}^{m} (z - a_j), \qquad c, a_j \in \mathbb{C},$$

with some $m \le p$, where $a_j = b_j$, $j = 1, 2, \dots, k$. Without loss of generality we may assume that $c = 1$. Note that

$$|a_j^p - 1|^{1/p} \le 2^{1/p} \max\{|a_j|, 1\}\,, \qquad j = k+1, k+2, \dots, m\,,$$

and

$$|a_j^p - 1|^{1/p} \le \left(2\cos\frac{\eta}{2}\right)^{1/p} = \left(2\cos\frac{\eta}{2}\right)^{1/p} |a_j|\,, \quad j = 1, 2, \dots, k\,,$$

Multiplying these inequalities for $j = 1, 2, \dots, m$, we obtain

$$\begin{aligned}\left(\prod_{j=0}^{p-1} |Q(\zeta_p^j)|\right)^{1/p} = \left(\prod_{j=0}^{m} |a_j^p - 1|\right)^{1/p} &\le 2^{m/p}\left(\cos\frac{\eta}{2}\right)^{k/p} \prod_{j=0}^{m} \max\{|a_j|, 1\} \\ &\le 2\left(\cos\frac{\eta}{2}\right)^{k/p} M_0(Q)\,.\end{aligned}$$

□

Proof of Lemma 3.7. By Lemma 3.1 If $\zeta_p = e^{2i\pi/p}$ then, for all k not divisible by p we have $|f_p(\zeta_p^k)| = \sqrt{p}$, and hence $|F_p(\zeta_p^k)| = \sqrt{p}$. Moreover

$$F_p(\zeta_p^k) = (\zeta_p^k)^{-p/2} \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \zeta_p^{ak} = (-1)^k \left(\frac{k}{p}\right) \sum_{a=1}^{p-1} \left(\frac{ak}{p}\right) \zeta_p^{ak} = (-1)^k \left(\frac{k}{p}\right) F_p(\zeta_p)\,.$$

Therefore if $\left(\frac{k}{p}\right) = \left(\frac{k+1}{p}\right)$, then $H_p\left(\frac{k}{p}\right)$ and $H_p\left(\frac{k+1}{p}\right)$ have different signs. Since $H_p(t)$ is real-valued and continuous on the real line, it must have a zero between k/p and $(k+1)/p$ by the Intermediate Value Theorem. However, by Lemma 2 in [C-00] we have

$$\left|\left\{k \in \{1, 2, \dots, p-2\} : \ \left(\frac{k}{p}\right) = \left(\frac{k+1}{p}\right)\right\}\right| = \frac{p-3}{2}\,,$$

and hence the values of $k \in \{0, 1, \dots, p-1\}$ for which H_p has a zero between k/p and $(k+1)/p$ is at least $(p-3)/2$. □

Proof of Lemma 3.9. Note that

$$I_\delta := \int_0^\infty \sin(\delta\pi x) C(x)\, \frac{dx}{x}$$

converges for every fixed $\delta > 0$, and

$$\lim_{\delta \to 0+} I_\delta = 0\,.$$

Indeed, there is an absolute constant $c_1 > 0$ such that

$$C(x) \le c_1 2^{-3x/\pi}\,, \qquad x \ge 1\,,$$

as

$$\left|\cos\left(\frac{2x}{2k+1}\right)\right| < \frac{1}{2}\,, \qquad \frac{3x}{\pi} < 2k+1 < \frac{6x}{\pi}\,.$$

Also,

$$\left|\frac{\sin(\delta\pi x)}{x}\right| \le \delta\pi\,, \qquad x > 0\,.$$

Therefore

$$I_\delta \le \int_0^\infty \left|\frac{\sin(\delta\pi x)}{x}\right| |C(x)|\,dx \le A_\delta + B_\delta\,,$$

where

$$A_\delta := \int_0^{\delta^{-1/2}} \left|\frac{\sin(\delta\pi x)}{x}\right| |C(x)|\,dx \le \delta^{-1/2}\delta\pi \le \delta^{1/2}\pi\,,$$

and

$$B_\delta := \int_{\delta^{-1/2}}^\infty \frac{|C(x)|}{x}\,dx \le \delta^{1/2}\int_{\delta^{-1/2}}^\infty c_1 2^{-3x/\pi}\,dx \le \delta^{1/2}\frac{c_1\pi}{3\log 2}\,.$$

So by choosing $\delta > 0$ so that

$$I_\delta \le A_\delta + B_\delta \le \delta^{1/2}\pi + \delta^{1/2}\frac{c_1\pi}{3\log 2} \le \frac{\pi\varepsilon}{2}\,,$$

the lemma follows from Lemma 3.8. □

Proof of Lemma 3.10. Suppose there are $0 \le k_1 < k_2 < \cdots < k_m \le p-1$ such that

$$t_j \in I_{k_j}\,, \qquad |f_p'(t_j)| \ge \gamma p^{3/2}\,, \qquad j = 1,2,\ldots,m\,.$$

Then

$$0 < t_1 < t_2 < \cdots < t_m < 2\pi\,,$$

and

$$\delta := \min\{t_2 - t_1, t_3 - t_2, \ldots, t_m - t_{m-1}, 2\pi - (t_m - t_1)\} \ge \frac{\pi}{p}\,.$$

Hence by the large sieve inequality formulated in Lemma 3.6 and the Parseval formula applied to $g_p(z) := z^{(3-p)/2} f_p'(z)$ we get

$$\begin{aligned} m\gamma^2 p^3 &\le \sum_{j=1}^m \left|f_p'\left(e^{it_j}\right)\right|^2 = \sum_{j=1}^m \left|g_p\left(e^{it_j}\right)\right|^2 \\ &\le \left(\frac{2(p-1)/2+1}{2\pi} + \delta^{-1}\right)\int_0^{2\pi} |g_p(e^{it})|^2\,dt \\ &= \left(\frac{2(p-1)/2+1}{2\pi} + \delta^{-1}\right)\int_0^{2\pi} |f_p'(e^{it})|^2\,dt \\ &\le \left(\frac{p}{2\pi} + \frac{p}{\pi}\right) 2\pi\,\frac{(p-1)p(2p-1)}{6} = 3p\,\frac{(p-1)p(2p-1)}{6} \\ &\le p^4\,. \end{aligned}$$

□

Proof of Lemma 3.11. Let $\varepsilon > 0$. By Lemma 3.9 there is a $\delta > 0$ depending only on $\varepsilon > 0$ such that

$$\left|\left\{k \in \{1,2,\dots,p\} : |f_p(e^{i(2k+1)\pi/p})| > \delta\sqrt{p}\right\}\right| \geq (1-\varepsilon/2)p \tag{4.1}$$

for all sufficiently large primes $p \geq N_\varepsilon$. Let $\gamma := (\varepsilon/2)^{-1/2}$. By Lemma 3.10 we have

$$\left|\left\{k \in \{0,1,\dots,p-1\} : \max_{z\in I_{k,\pi/2}} |f_p'(z)| \leq \gamma p^{3/2}\right\}\right| \geq p - \gamma^{-2}p = (1-\varepsilon/2)p\,. \tag{4.2}$$

Now let

$$A_{p,\delta,\gamma} := \left\{k \in \{1,2,\dots,p\} : |f_p(e^{i(2k+1)\pi/p})| > \delta\sqrt{p}\,, \quad \max_{z\in I_{k,\pi/2}} |f_p'(z)| \leq \gamma p^{3/2}\right\}\,.$$

By (4.1) and (4.2) we obtain

$$|A_{p,\delta,\gamma}| \geq (1-\varepsilon)p\,. \tag{4.3}$$

Let $0 < \eta < \min\{\delta/\gamma, \pi/2\}$. Observe that $k \in A_{p,\delta,\gamma}$ implies that f_p does not vanish in $I_{k,\eta}$. Indeed, $z := e^{it} \in I_{k,\eta}$ implies

$$\left|t - \frac{(2k+1)\pi}{p}\right| < \frac{\eta}{p}\,,$$

and hence

$$\begin{aligned}
|f_p(z)| &\geq |f_p(e^{i(2k+1)\pi/p})| - |f_p(z) - f_p(e^{i(2k+1)\pi/p})| \\
&> \delta\sqrt{p} - \left|\int_{(2k+1)\pi/p}^{t} f_p'(e^{i\tau})e^{i\tau}\,d\tau\right| \\
&\geq \delta\sqrt{p} - \int_{(2k+1)\pi/p}^{t} |f_p'(e^{i\tau})||e^{i\tau}|\,d\tau \geq \delta\sqrt{p} - \frac{\eta}{p}\gamma p^{3/2} \\
&> \delta\sqrt{p} - \delta\sqrt{p} = 0
\end{aligned}$$

for all sufficiently large primes $p \geq N_\varepsilon$, and the lemma follows from (4.3). □

Proof of Theorem 2.1

Now we are ready to prove the theorem.

Proof of Theorem 2.1. As in Lemma 3.11 let the subarcs $I_{k,\eta}$ of the unit circle ∂D be defined by

$$I_{k,\eta} := \left\{e^{it} : \left|t - \frac{(2k+1)\pi}{p}\right| < \frac{\eta}{p}\right\}\,, \qquad k = 0,1,\dots,p-1\,.$$

It follows from Lemma 3.11 that for $\varepsilon := 1/8$ there is an $\eta > 0$ such that

$$|\{k \in \{0,1,\dots,p-1\} : f_p(z) \neq 0 \text{ for all } z \in I_{k,\eta}\}| \geq \frac{7p}{8}$$

for all sufficiently large primes p. Combining this with Lemma 3.7 we have that

$$|\{k \in \{0,1,\dots,p-1\} : f_p \text{ has a zero on } I_{k,\pi} \text{ and } f_p(z) \neq 0 \text{ for all } z \in I_{k,\eta}\}| \geq p/4$$

for all sufficiently large primes p. Hence the assumptions of Lemma 3.3 are satisfied with $Q := f_p$ and $k \geq p/4$ for all sufficiently large primes. Suppose that 1 is a zero of f_p with multiplicity $m = m(p)$. By either Lemma 3.4 or Lemma 3.5 we have $m = O(p^{1/2})$. Let $g_p(z) := f_p(z)/h_m(z)$ with $h_m(z) := (z-1)^m$. Note that $|g_p(1)|$ is a nonzero integer, hence $|g_p(1)| \geq 1$. Also, h_m is monic and has all its zeros on the unit circle, hence $M_0(h_m) = 1$. Combining these with the multiplicative property of the Mahler measure and Lemma 3.3 applied to g_p with $k \geq p/4$, we obtain

$$\begin{aligned} M_0(f_p) &= M_0(g_p)M_0(h_m) = M_0(g_p) \\ &\geq \left(2\left(\cos\frac{\eta}{2}\right)^{k/p}\right)^{-1} \left(|g_p(1)| \prod_{k=1}^{p-1} |g_p(\zeta_p^k)|\right)^{1/p} \\ &\geq \left(2\left(\cos\frac{\eta}{2}\right)^{1/4}\right)^{-1} \left(|g_p(1)| \prod_{k=1}^{p-1} |g_p(\zeta_p^k)|\right)^{1/p}. \end{aligned} \tag{4.4}$$

Now observe that Lemma 3.1 implies

$$|g_p(\zeta_p^k)| = |f_p(\zeta_p^k)| = p^{1/2}\,, \qquad k = 1,2,\dots,p-1\,, \tag{4.5}$$

while

$$\left|\prod_{k=1}^{p-1} (\zeta_p^k - 1)\right| = \left|\sum_{k=0}^{p-1} 1^k\right| = p\,. \tag{4.6}$$

Also, $m = O(p^{1/2})$ implies that

$$\lim_{p\to\infty} p^{-(1/2+m)/p} = 1\,. \tag{4.7}$$

Using (4.4)–(4.7) and the observation $|g_p(1)| \geq 1$, we conclude that there are absolute constants

$$c_1 := \left(2\left(\cos\frac{\eta}{2}\right)^{1/4}\right)^{-1} > c_2 > \frac{1}{2}$$

such that

$$
\begin{aligned}
M_0(f_p) &\geq \left(2\left(\cos\frac{\eta}{2}\right)^{1/4}\right)^{-1}\left(|g_p(1)|\prod_{k=1}^{p-1}|g_p(\zeta_p^k)|\right)^{1/p} \\
&\geq c_1\left(|g_p(1)|\prod_{k=1}^{p-1}\left|\frac{f_p(\zeta_p^k)}{(\zeta_p^k-1)^m}\right|\right)^{1/p} \geq c_1\left(\prod_{k=1}^{p-1}\left|\frac{f_p(\zeta_p^k)}{(\zeta_p^k-1)^m}\right|\right)^{1/p} \\
&= c_1\frac{(p^{1/2})^{(p-1)/p}}{p^{m/p}} = c_1 p^{(p-1)/(2p)-m/p} = c_1 p^{1/2}p^{-(1/2+m)/p} \\
&\geq c_2\sqrt{p}
\end{aligned}
$$

for all sufficiently large primes p. □

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