ON THE DENSENESS OF $\mathrm{span}\{x^{\lambda_j}(1-x)^{1-\lambda_j}\}$ IN $C_0([0,1])$

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Abstract. Associated with a sequence $\Lambda = (\lambda_j)_{j=0}^\infty$ of distinct exponents $\lambda_j \in [0,1]$, we define

$$H(\Lambda) := \mathrm{span}\{x^{\lambda_0}(1-x)^{1-\lambda_0}, x^{\lambda_1}(1-x)^{1-\lambda_1}, \dots\} \subset C([0,1])\,.$$

Answering a question of Giuseppe Mastroianni, we show that $H(\Lambda)$ is dense in $C_0[0,1] := \{f \in C[0,1] : f(0) = f(1) = 0\}$ in the uniform norm on $[0,1]$ if and only if

$$\sum_{j=0}^\infty (1/2 - |1/2 - \lambda_j|) = \infty\,.$$

Associated with a sequence $\Lambda = (\lambda_j)_{j=0}^\infty$ of distinct exponents $\lambda_j \in [0,1]$, we define

$$H_n(\Lambda) := \mathrm{span}\{x^{\lambda_0}(1-x)^{1-\lambda_0}, x^{\lambda_1}(1-x)^{1-\lambda_1}, \dots , x^{\lambda_n}(1-x)^{1-\lambda_n}\} \subset C([0,1])$$

and

$$H(\Lambda) := \bigcup_{n=0}^\infty H_n(\Lambda) = \mathrm{span}\{x^{\lambda_0}(1-x)^{1-\lambda_0}, x^{\lambda_1}(1-x)^{1-\lambda_1}, \dots\} \subset C([0,1])\,.$$

In June, 2005, G. Mastroianni approached me with the following question. Let $\Lambda = (\lambda_j)_{j=0}^\infty$ be an enumeration of the rational numbers in $(0,1)$. Is it true that $H(\Lambda)$ is dense in $C_0([0,1]) := \{f \in C([0,1]) : f(0) = f(1) = 0\}$ in the uniform norm on $[0,1]$? In this note we answer his question by proving the following result.

Theorem. *Let $\Lambda = (\lambda_j)_{j=0}^\infty$ be a sequence of exponents $\lambda_j \in (0,1)$. $H(\Lambda)$ is dense in $C_0([0,1])$ in the uniform norm on $[0,1]$ if and only if*

$$\sum_{j=0}^\infty (1/2 - |1/2 - \lambda_j|) = \infty\,. \tag{1}$$

Throughout the paper we adopt the notation

$$\|f\|_A := \sup_{x\in A} |f(x)|$$

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for complex-valued functions f defined on a set A.

To prove the "if part" of the theorem, we need the lemma below. It is a well known consequence of Jensen's formula.

Lemma 1. *Suppose f is a bounded analytic function on the open disk $D(a,r)$ centered at a with radius r. If $(z_j)_{j=0}^\infty$ is a sequence of distinct complex numbers such that $z_j \in D(a,r)$ and $f(z_j)=0$ for each $j=0,1,\dots,$ and*

$$\sum_{j=0}^\infty (r - |z_j - a|) = \infty\,,$$

then $f \equiv 0$ on $D(a,r)$.

Proof of the "if part" of the theorem. Suppose $H(\Lambda)$ is not dense in $C_0([0,1])$ in the uniform norm on $[0,1]$. Combining the Hahn-Banach Theorem and the Riesz Representation Theorem, we have a Borel measure with finite total variation on $[0,1]$ such that

$$\operatorname{supp}(\mu) \cap (0,1) \neq \emptyset\,,$$

and

$$\int_0^1 x^{\lambda_j}(1-x)^{1-\lambda_j}\,d\mu(x) = 0\,, \qquad j = 0,1,\dots\,.$$

Then the function

$$f(z) := \int_0^1 t^z(1-t)^{1-z}\,d\mu(t)$$

is a bounded analytic function on the strip

$$S := \{z \in \mathbb{C} : 0 < \operatorname{Re}(z) < 1\}\,,$$

hence it is a bounded analytic function on the open disk $D(1/2,1/2)$ centered at $1/2$ with radius $1/2$. Since $f(\lambda_j)=0$ for each $j=0,1,\dots,$ and

$$\sum_{j=0}^\infty (1/2 - |1/2 - \lambda_j|) = \infty\,,$$

Lemma 1 implies that $f \equiv 0$ on $D(1/2,1/2)$, hence by the Unicity Theorem $f \equiv 0$ on the strip S as well. Hence

$$f(1/2+iy) = \int_0^1 t^{1/2+iy}(1-t)^{1/2-iy}\,d\mu(t) = 0$$

for every $y \in \mathbb{R}$. Therefore, for every $y \in \mathbb{R}$,

$$\begin{aligned} 0 = f(1/2+iy) &= \int_0^1 \sqrt{t(1-t)}\left(\frac{t}{1-t}\right)^{iy} d\mu(t) \\ &= \int_{-\infty}^\infty e^{ixy}\,d\nu(x)\,, \end{aligned}$$

with a non-zero Borel measure ν with finite total variation on $\mathbb{R}$. However, this is a contradiction, see the exercise at the end of Section 2.1 of Chapter VII in [7], for instance. $\square$

Lemma 2. *We have*

$$|y(1-y)Q'(y)| \le \left(1 + 9\sum_{j=0}^{n} \lambda_j\right) \|Q\|_{[0,1]} .$$

for every $Q \in H_n(\Lambda)$ and $y \in (0,1)$.

Associated with a sequence $\Lambda = (\lambda_j)_{j=0}^{\infty}$ of distinct nonnegative exponents λ_j, we define

$$M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots x^{\lambda_n}\} .$$

To prove Lemma 2, we need D.J. Newman's inequality [2, Theorem 6.1.1 on page 276].

Lemma 3. *Let $b > 0$. The inequality*

$$|yP'(y)| \le 9\left(\sum_{j=0}^{n} \lambda_j\right) \|P\|_{[0,b]}$$

holds for every $P \in M_n(\Lambda)$ and $y \in (0,b]$.

Note that D.J. Newman [5] proves the above inequality with the constant 11 rather than 9.

Proof of Lemma 2. Note that every $Q \in H_n(\Lambda)$ is of the form

$$Q(x) = (1-x)P\left(\frac{x}{1-x}\right) \tag{2}$$

with some $P \in M_n(\Lambda)$. Let $y \in (0,1)$. Using Lemma 3, we obtain

$$\begin{aligned}
|y(1-y)Q'(y)| &= \left| y(1-y)(1-y)P'\left(\frac{y}{1-y}\right)\frac{1}{(1-y)^2} - y(1-y)P\left(\frac{y}{1-y}\right)\right| \\
&\le \left| yP'\left(\frac{y}{1-y}\right)\right| + \left|(1-y)P\left(\frac{y}{1-y}\right)\right| \\
&\le y\frac{1-y}{y}9\left(\sum_{j=0}^{n}\lambda_j\right)\|P\|_{[0,\frac{y}{1-y}]} + \left|(1-y)P\left(\frac{y}{1-y}\right)\right| \\
&\le 9\left(\sum_{j=0}^{n}\lambda_j\right)(1-y)\left\|P\left(\frac{u}{1-u}\right)\right\|_{[0,y]} + |Q(y)| \\
&\le 9\left(\sum_{j=0}^{n}\lambda_j\right)\left\|(1-u)P\left(\frac{u}{1-u}\right)\right\|_{[0,y]} + |Q(y)| \\
&\le \left(1 + 9\sum_{j=0}^{n}\lambda_j\right)\|Q\|_{[0,y]} \le \left(1 + 9\sum_{j=0}^{n}\lambda_j\right)\|Q\|_{[0,1]} .
\end{aligned}$$

□

We need the following version of a simple lemma from [4], the short proof of which is presented in this note as well.

Lemma 4. *Let $\Gamma := (\gamma_j)_{j=0}^\infty$ be a sequence of distinct nonnegative real numbers with*

$$\eta := \sum_{j=0}^\infty \gamma_j < \infty\,.$$

Then

$$|P(z)| \leq \exp\Big(9\eta \|\log z\|_K\Big)\|P\|_{[0,1]}\,, \qquad z \in K\,,$$

for every $P \in M(\Gamma) := \operatorname{span}\{x^{\gamma_0}, x^{\gamma_1}, \dots\}$ and for every compact $K \subset \mathbb{C} \setminus \{0\}$.

In fact, in the "only if part" of the proof of the theorem, the following consequence of Lemma 4 is needed.

Lemma 5. *Let $\Gamma := (\gamma_j)_{j=0}^\infty$ be a sequence of distinct nonnegative real numbers with*

$$\eta := \sum_{j=0}^\infty \gamma_j \leq \frac{1}{20}\,.$$

Then

$$\|Q\|_{[3/4,1]} < \|Q\|_{[0,1/2]}$$

for every $Q \in H(\Gamma) = \operatorname{span}\{x^{\gamma_0}(1-x)^{1-\gamma_0}, x^{\gamma_1}(1-x)^{1-\gamma_1}, \dots\}$.

Proof of Lemma 4. The lemma is a consequence of D.J. Newman's Markov-type inequality. Repeated applications of Lemma 3 with $b := 1$ and the substitution $x = e^{-t}$ imply that

$$\|(P(e^{-t}))^{(m)}\|_{[0,\infty)} \leq (9\eta)^m \|P(e^{-t})\|_{[0,\infty)}\,, \qquad m = 1, 2, \dots\,,$$

and, in particular

$$|(P(e^{-t}))^{(m)}(0)| \leq (9\eta)^m \|P(e^{-t})\|_{[0,\infty)}\,, \qquad m = 1, 2, \dots\,,$$

for every $P \in M(\Gamma)$. By using the Taylor series expansion of $P(e^{-t})$ around 0, we obtain that

$$|P(z)| \leq c_1(K, \eta)\|P\|_{[0,1]}\,, \qquad z \in K\,,$$

for every $P \in M(\Gamma) = \operatorname{span}\{x^{\gamma_0}, x^{\gamma_1}, \dots\}$ and for every compact $K \subset \mathbb{C} \setminus \{0\}$, where

$$c_1(K, \eta) := \sum_{m=0}^\infty \frac{(9\eta)^m \|\log z\|_K^m}{m!} = \exp(9\eta \|\log z\|_K)\,,$$

and the result of the lemma follows. $\Box$

Proof of Lemma 5. This follows from Lemma 4 and the fact that every

$$Q \in H_n(\Gamma) = \operatorname{span}\{x^{\gamma_0}(1-x)^{1-\gamma_0}, x^{\gamma_1}(1-x)^{1-\gamma_1}, \dots, x^{\gamma_n}(1-x)^{1-\gamma_n}\} \tag{3}$$

is of the form (2) with a

$$P \in M_n(\Gamma) = \operatorname{span}\{x^{\gamma_0}, x^{\gamma_1}, \dots, x^{\gamma_n}\}\,. \tag{4}$$

□

Let $\Gamma := (\gamma_j)_{j=0}^{\infty}$ be a sequence of distinct nonnegative real numbers with $\gamma_0 := 0$. One of the most basic properties of a Müntz space $M_n(\Gamma)$ defined in (4) is the fact that it is a Chebyshev space of dimension $n+1$ on every $A \subset [0, \infty)$ containing at least $n+1$ points. That is, $M_n(\Gamma) \subset C(A)$ and every $P \in M_n(\Gamma)$ having at least $n+1$ (distinct) zeros in A is identically 0. Since any $Q \in H_n(\Gamma)$ defined in (3) is of the form (2) with a $P \in M_n(\Gamma)$ defined in (4), the space $H_n(\Gamma)$ is also a Chebyshev space of dimension $n+1$ on every $A \subset [0,1)$ containing at least $n+1$ points. The following properties of the space $H_n(\Gamma)$, as a Chebyshev space of dimension $n+1$ on every $A \subset [0,1)$ containing at least $n+1$ points, are well known (see, for example, [2, 3, 6]).

Lemma 6 (Existence of Chebyshev Polynomials). *Let A be a compact subset of $[0,1)$ containing at least $n+1$ points. Then there exists a unique (extended) Chebyshev polynomial*

$$T_n := T_n\{\gamma_0, \gamma_1, \dots, \gamma_n; A\}$$

for $H_n(\Gamma)$ on A defined by

$$T_n(x) = c\left(x^{\gamma_n}(1-x)^{1-\gamma_n} - \sum_{j=0}^{n-1} a_j x^{\gamma_j}(1-x)^{1-\gamma_j}\right),$$

where the numbers $a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$ are chosen to minimize

$$\left\| x^{\gamma_n}(1-x)^{1-\gamma_n} - \sum_{j=0}^{n-1} a_j x^{\gamma_j}(1-x)^{1-\gamma_j} \right\|_A$$

and where $c \in \mathbb{R}$ is a normalization constant chosen so that

$$\|T_n\|_A = 1$$

and the sign of c is determined by

$$T_n(\max A) > 0.$$

Lemma 7 (Alternation Characterization). *The Chebyshev polynomial*

$$T_n := T_n\{\gamma_0, \gamma_1, \dots, \gamma_n; A\} \in H_n(\Gamma)$$

is uniquely characterized by the existence of an alternation set

$$\{x_0 < x_1 < \cdots < x_n\} \subset A$$

for which

$$T_n(x_j) = (-1)^{n-j} = (-1)^{n-j}\|T_n\|_A, \qquad j = 0, 1, \dots, n.$$

Note that the existence of a unique (extended) Chebyshev polynomial with the above alternation characterization can be guaranteed even if we allow that $A = [0,1]$ rather than $A \subset [0,1)$. This can be seen by a standard limiting argument. Namely we take the Chebyshev polynomials $T_{n,\delta}$ on the interval $[0, 1-\delta]$ first and then we let $\delta > 0$ tend to 0.

Now we are ready to prove the "only if part" of the theorem. It turns out that the approach used in the corresponding part of the proof in [1] can be followed here, but, while the proof is still rather short, the details are slightly more subtle.

Proof of the "only if part" of the theorem. Suppose now that Λ is a sequence of distinct exponents $\lambda_j \in (0,1)$ such that (1) does not hold, that is,

$$\sum_{j=0}^{\infty} (1/2 - |1/2 - \lambda_j|) < \infty \,.$$

Then there are sequences $\Gamma := (\gamma_j)_{j=0}^{\infty}$ of distinct numbers $\gamma_j \in [0, 1/4)$ and $\Delta := (\delta_j)_{j=0}^{\infty}$ of distinct numbers $\delta_j \in [0, 1/4)$, and a finite set $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ of distinct numbers $\alpha_j \in [1/4, 3/4]$ such that

$$\sum_{j=0}^{\infty} \gamma_j < \frac{1}{20}\,, \qquad \sum_{j=0}^{\infty} \delta_j < \frac{1}{20}\,,$$

and

$$\{\lambda_j : j = 0, 1, \dots\} \subset$$
$$\{\gamma_j : j = 0, 1, \dots\} \cup \{1 - \delta_j : j = 0, 1, \dots\} \cup \{\alpha_j : j = 1, 2, \dots m\}\,.$$

Without loss of generality we may assume that $\gamma_0 := 0$ and $\delta_0 := 0$. For notational convenience, let

$$\begin{aligned}
T_{n,\gamma} &:= T_n\{\gamma_0, \gamma_1, \dots, \gamma_n; [0,1]\} \\
T_{n,\delta} &:= T_n\{1-\delta_0, 1-\delta_1, \dots, 1-\delta_n; [0,1]\} \\
T_{2n+m+1,\gamma,\delta,\alpha} &:= T_{2n+m+1}\{\gamma_0, \dots, \gamma_n, 1-\delta_0, \dots, 1-\delta_n, \alpha_1, \dots, \alpha_m; [0,1]\}\,.
\end{aligned}$$

It follows from Lemmas 2, 5, and 7, and the Mean Value Theorem that for every $\varepsilon > 0$ there exists a $k_1(\varepsilon) \in \mathbb{N}$ depending only on $(\gamma_j)_{j=0}^{\infty}$ and ε (and not on n) so that $T_{n,\gamma}$ has at most $k_1(\varepsilon)$ zeros in $[\varepsilon, 1]$ and at least $n - k_1(\varepsilon)$ zeros in $(0, \varepsilon)$. Similarly, applying Lemmas 2, 5, and 7, and the Mean Value Theorem to $S_{n,\delta}(x) := T_{n,\delta}(1-x)$ gives that for every $\varepsilon > 0$ there exists a $k_2(\varepsilon) \in \mathbb{N}$ depending only on $(\delta_j)_{j=0}^{\infty}$ and ε (and not on n) so that $T_{n,\delta}$ has at most $k_2(\varepsilon)$ zeros in $[0, 1-\varepsilon]$ and at least $n - k_2(\varepsilon)$ zeros $(1-\varepsilon, 1)$.

Now, counting the zeros of $T_{n,\gamma} - T_{2n+m+1,\gamma,\delta,\alpha}$ and $T_{n,\delta} - T_{2n+m+1,\gamma,\delta,\alpha}$, we can deduce that $T_{2n+m+1,\gamma,\delta,\alpha}$ has at least $n - k_1(\varepsilon) - 3$ zeros in $(0, \varepsilon)$ and it has at least $n - k_2(\varepsilon) - 3$ zeros in $(1-\varepsilon, 1)$ (we count every zero without sign change twice). Hence, for every $\epsilon > 0$ there exists a $k(\epsilon) \in \mathbb{N}$ depending only on $(\lambda_j)_{j=0}^{\infty}$ and ε (and not on n) so that $T_{2n+m+1,\gamma,\delta,\alpha}$ has at most $k(\epsilon)$ zeros in $[\epsilon, 1-\epsilon]$.

Let $\varepsilon := 1/4$ and $k := k(1/4)$. Pick $k+m+5$ points

$$\frac{1}{4} < \eta_0 < \eta_1 < \cdots < \eta_{k+m+4} < \frac{3}{4}$$

and a function $f \in C_0([0,1])$ so that $f(x) = 0$ for all $x \in [0,1/4] \cup [3/4,1]$, while

$$f(\eta_j) := 2\,(-1)^j, \qquad j = 0,1,\dots,k+m+4\,.$$

Assume that there exists a $Q \in H(\Lambda)$ so that

$$\|f - Q\|_{[0,1]} < 1\,.$$

Then $Q - T_{2n+m+1,\gamma,\delta,\alpha}$ has at least $2n+m+2$ zeros in $(0,1)$. However, for sufficiently large n, $Q - T_{2n+m+1,\gamma,\delta,\alpha}$ is in the linear span of the $2n+m+2$ functions

$$x^{\gamma_j}(1-x)^{1-\gamma_j}\,, \qquad j = 0,1,\dots,n\,,$$

$$x^{1-\delta_j}(1-x)^{\delta_j}\,, \qquad j = 0,1,\dots,n\,,$$

and

$$x^{\alpha_j}(1-x)^{1-\alpha_j}\,, \qquad j = 1,2\dots,m\,,$$

so it can have at most $2n+m+1$ zeros in $(0,1)$. This contradiction shows that $H(\Lambda)$ is not dense in $C_0([0,1])$. $\square$

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