MARKOV-BERNSTEIN TYPE INEQUALITIES UNDER LITTLEWOOD-TYPE COEFFICIENT CONSTRAINTS

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Abstract. Let $\mathcal{F}_n$ denote the set of polynomials of degree at most n with coefficients from $\{-1,0,1\}$. Let $\mathcal{G}_n$ be the collection of polynomials p of the form

$$p(x) = \sum_{j=m}^{n} a_j x^j\,, \qquad |a_m| = 1\,, \quad |a_j| \leq 1\,,$$

where m is an unspecified nonnegative integer not greater than n.

We establish the right Markov-type inequalities for the classes $\mathcal{F}_n$ and $\mathcal{G}_n$ on $[0,1]$. Namely there are absolute constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 n \log(n+1) \leq \max_{0 \neq p \in \mathcal{F}_n} \frac{|p'(1)|}{\|p\|_{[0,1]}} \leq \max_{0 \neq p \in \mathcal{F}_n} \frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}} \leq C_2 n \log(n+1)$$

and

$$C_1 n^{3/2} \leq \max_{0 \neq p \in \mathcal{G}_n} \frac{|p'(1)|}{\|p\|_{[0,1]}} \leq \max_{0 \neq p \in \mathcal{G}_n} \frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}} \leq C_2 n^{3/2}\,.$$

It is quite remarkable that the right Markov factor for $\mathcal{G}_n$ is much larger than the right Markov factor for $\mathcal{F}_n$. We also show that there are absolute constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 n \log(n+1) \leq \max_{0 \neq p \in \mathcal{L}_n} \frac{|p'(1)|}{\|p\|_{[0,1]}} \leq \max_{0 \neq p \in \mathcal{L}_n} \frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}} \leq C_2 n \log(n+1)\,,$$

where $\mathcal{L}_n$ denotes the set of polynomials of degree at most n with coefficients from $\{-1,1\}$. For polynomials $p \in \mathcal{F} := \bigcup_{n=0}^{\infty} \mathcal{F}_n$ with $|p(0)| = 1$ and for $y \in [0,1)$ the Bernstein-type inequality

$$\frac{C_1 \log\left(\frac{2}{1-y}\right)}{1-y} \leq \max_{\substack{p \in \mathcal{F} \\ |p(0)|=1}} \frac{\|p'\|_{[0,y]}}{\|p\|_{[0,1]}} \leq \frac{C_2 \log\left(\frac{2}{1-y}\right)}{1-y}$$

is also proved with absolute constants $C_1 > 0$ and $C_2 > 0$.

This completes earlier work of the authors where the upper bound in the first inequality is obtained.

1991 *Mathematics Subject Classification.* 41A17.

Key words and phrases. Markov, Bernstein, inequality, restricted coefficients, small integer coefficients, polynomial.

Research supported in part by the NSERC of Canada (P. Borwein) and by the NSF of the USA under Grant No. DMS–9623156 (T. Erdélyi). Research conducted while an International Postdoctoral Fellow of the Danish Research Council at University of Copenhagen (T. Erdélyi).

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Littlewood had a particular fascination with the class of polynomials with coefficients restricted to being in the set $\{-1, 0, 1\}$. See in particular [22] and the many references included later in the introduction. Many of the problems he considered concerned rates of possible growth of such polynomials in different norms on the unit circle. Others concerned location of zeros of such polynomials. The best known of these is the now solved Littlewood conjecture which asserts that there exists an absolute constant $c > 0$ such that

$$\int_{-\pi}^{\pi} \left| \sum_{k=0}^{n} a_k \exp(i\lambda_k t) \right| dt \geq c \log n$$

whenever the $|a_k| \geq 1$ and the exponents λ_k are distinct integers. (Here and in what follows the expression "absolute constant" means a constant that is independent of all the variables in the inequality).

We are primarily concerned in this paper with establishing the correct Markov-type inequalities on the interval $[0, 1]$ for various classes of polynomials related to these Littlewood problems. One of the notable features is that theses bounds are quite distinct from those for unrestricted polynomials.

In this paper n always denotes a nonnegative integer. We introduce the following classes of polynomials. Let

$$\mathcal{P}_n := \left\{ p : p(x) = \sum_{j=0}^{n} a_j x^j \,, \;\; a_j \in \mathbb{R} \right\}$$

denote the set of all algebraic polynomials of degree at most n with real coefficients.

Let

$$\mathcal{P}_n^c := \left\{ p : p(x) = \sum_{j=0}^{n} a_j x^j \,, \;\; a_j \in \mathbb{C} \right\}$$

denote the set of all algebraic polynomials of degree at most n with complex coefficients.

Let

$$\mathcal{F}_n := \left\{ p : p(x) = \sum_{j=0}^{n} a_j x^j \,, \;\; a_j \in \{-1, 0, 1\} \right\}$$

denote the set of polynomials of degree at most n with coefficients from $\{-1, 0, 1\}$.

Let

$$\mathcal{L}_n := \left\{ p : p(x) = \sum_{j=0}^{n} a_j x^j \,, \;\; a_j \in \{-1, 1\} \right\}$$

denote the set of polynomials of degree at most n with coefficients from $\{-1, 1\}$. (Here we are using $\mathcal{L}_n$ in honor of Littlewood.)

Let $\mathcal{K}_n$ be the collection of polynomials $p \in \mathcal{P}_n^c$ of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j \,, \qquad |a_0| = 1 \,, \quad |a_j| \leq 1 \,.$$

Let $\mathcal{G}_n$ be the collection of polynomials $p \in \mathcal{P}_n^c$ of the form

$$p(x) = \sum_{j=m}^{n} a_j x^j \,, \qquad |a_m| = 1\,, \quad |a_j| \leq 1\,,$$

where m is an unspecified nonnegative integer not greater than n.

Obviously

$$\mathcal{L}_n \subset \mathcal{F}_n,\, \mathcal{K}_n \subset \mathcal{G}_n \subset \mathcal{P}_n^c\,.$$

The following two inequalities are well known in approximation theory. See, for example, Duffin and Schaeffer [12], Cheney [9], Lorentz [24], DeVore and Lorentz [11], Lorentz, Golitschek, and Makovoz [25], Borwein and Erdélyi [4].

Markov Inequality. *The inequality*

$$\|p'\|_{[-1,1]} \leq n^2 \|p\|_{[-1,1]}$$

holds for every $p \in \mathcal{P}_n$.

Bernstein Inequality. *The inequality*

$$|p'(y)| \leq \frac{n}{\sqrt{1-y^2}}\, \|p\|_{[-1,1]}$$

holds for every $p \in \mathcal{P}_n$ *and* $y \in (-1,1)$.

In the above two theorems and throughout the paper $\|\cdot\|_A$ denotes the supremum norm on $A \subset \mathbb{R}$.

Our intention is to establish the right Markov-type inequalities on $[0,1]$ for the classes $\mathcal{F}_n$, $\mathcal{L}_n$, $\mathcal{K}_n$, and $\mathcal{G}_n$. We also prove an essentially sharp Bernstein-type inequality on $[0,1)$ for polynomials $p \in \mathcal{F} := \bigcup_{n=0}^{\infty} \mathcal{F}_n$ with $|p(0)| = 1$.

For further motivation and introduction to the topic we refer to Borwein and Erdélyi [5]. This paper is, in part, a continuation of the work presented in [5]. The books by Lorentz, Golitschek, and Makovoz [25], and by Borwein and Erdélyi [4] also contain sections on Markov- and Bernstein-type inequalities for polynomials under various constraints.

The classes $\mathcal{F}_n$, $\mathcal{L}_n$, and other classes of polynomials with restricted coefficients have been thoroughly studied in many (mainly number theoretic) papers. See, for example, Beck [1], Bloch and Pólya [2], Bombieri and Vaaler [3], Borwein, Erdélyi, and Kós [5], Borwein and Ingalls [7], Byrnes and Newman [8], Cohen [10], Erdős [13], Erdős and Turán [14], Ferguson [15], Hua [16], Kahane [17] and [18], Konjagin [19], Körner [20], Littlewood [21] and [22], Littlewood and Offord [23], Newman and Byrnes [26], Newman and Giroux [27], Odlyzko and Poonen [28], Salem and Zygmund [29], Schur [30], and Szegő [31].

For several extremal problems the classes $\mathcal{F}_n$ tend to behave like $\mathcal{G}_n$. See, for example, Borwein, Erdélyi, and Kós [5]. It is quite remarkable that as far as the Markov-type inequality on $[0,1]$ is concerned, there is a huge difference between these classes. Compare Theorems 2.1 and 2.4.

2. New Results

Theorem 2.1. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 n \log(n+1) \leq \max_{0 \neq p \in \mathcal{F}_n} \frac{|p'(1)|}{\|p\|_{[0,1]}} \leq \max_{0 \neq p \in \mathcal{F}_n} \frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}} \leq c_2 n \log(n+1)\,.$$

Theorem 2.2. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 n \log(n+1) \leq \max_{0 \neq p \in \mathcal{L}_n} \frac{|p'(1)|}{\|p\|_{[0,1]}} \leq \max_{0 \neq p \in \mathcal{L}_n} \frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}} \leq c_2 n \log(n+1)\,.$$

Theorem 2.3. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 n \log(n+1) \leq \max_{0 \neq p \in \mathcal{K}_n} \frac{|p'(1)|}{\|p\|_{[0,1]}} \leq \max_{0 \neq p \in \mathcal{K}_n} \frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}} \leq c_2 n \log(n+1)\,.$$

Theorem 2.4. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 n^{3/2} \leq \max_{0 \neq p \in \mathcal{G}_n} \frac{|p'(1)|}{\|p\|_{[0,1]}} \leq \max_{0 \neq p \in \mathcal{G}_n} \frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}} \leq c_2 n^{3/2}\,.$$

The following theorem establishes an essentially sharp Bernstein-type inequality on $[0,1)$ for polynomials $p \in \mathcal{F} := \bigcup_{n=0}^{\infty} \mathcal{F}_n$ with $|p(0)| = 1$.

Theorem 2.5. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\frac{c_1 \log\left(\frac{2}{1-y}\right)}{1-y} \leq \max_{\substack{p \in \mathcal{F} \\ |p(0)|=1}} \frac{\|p'\|_{[0,y]}}{\|p\|_{[0,1]}} \leq \frac{c_2 \log\left(\frac{2}{1-y}\right)}{1-y}$$

for every $y \in [0,1)$.

Our next result is an essentially sharp Bernstein-type inequality on $[0,1)$ for polynomials $p \in \mathcal{L} := \bigcup_{n=0}^{\infty} \mathcal{L}_n$.

Theorem 2.6. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\frac{c_1 \log\left(\frac{2}{1-y}\right)}{1-y} \leq \max_{p \in \mathcal{L}} \frac{\|p'\|_{[0,y]}}{\|p\|_{[0,1]}} \leq \frac{c_2 \log\left(\frac{2}{1-y}\right)}{1-y}$$

for every $y \in [0,1)$.

Our final result is an essentially sharp Bernstein-type inequality on $[0,1)$ for polynomials $p \in \mathcal{K} := \bigcup_{n=0}^{\infty} \mathcal{K}_n$.

Theorem 2.7. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\frac{c_1 \log\left(\frac{2}{1-y}\right)}{1-y} \leq \max_{p \in \mathcal{K}} \frac{\|p'\|_{[0,y]}}{\|p\|_{[0,1]}} \leq \frac{c_2 \log\left(\frac{2}{1-y}\right)}{1-y}$$

for every $y \in [0,1)$.

A Bernstein-type inequality on $[0,1)$ for polynomials $p \in \mathcal{G} := \bigcup_{n=0}^{\infty} \mathcal{G}_n$ is also established in [5]. However, there is a gap between the upper bound in [5] and the lower bound we are able to prove at the moment.

3. Lemmas for Theorem 2.4

To prove Theorem 2.4 we need several lemmas. The first one is a result from [6].

Lemma 3.1. *There are absolute constants $c_3 > 0$ and $c_4 > 0$ such that*

$$\exp\left(-c_3\sqrt{n}\right) \leq \inf_{0 \neq p \in \mathcal{G}_n} \|p\|_{[0,1]} \leq \inf_{0 \neq p \in \mathcal{F}_n} \|p\|_{[0,1]} \leq \exp\left(-c_4\sqrt{n}\right) .$$

We will also need a corollary of the following well known result.

Hadamard Three Circles Theorem. *Suppose f is regular in*

$$\{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}.$$

For $r \in [r_1, r_2]$, let

$$M(r) := \max_{|z|=r} |f(z)|.$$

Then

$$\log(r_2/r_1) \log M(r) \leq \log(r_2/r) \log M(r_1) + \log(r/r_1) \log M(r_2) .$$

Note that the conclusion of the Hadamard Three Circles Theorem can be rewritten as

$$\log M(r) \leq \log M(r_1) + \frac{\log(r/r_1)}{\log(r_2/r_1)} \left(\log M(r_2) - \log M(r_1)\right) .$$

Corollary 3.2. *Let $\alpha \in \mathbb{R}$. Suppose $1 \leq \alpha \leq 2n$. Suppose f is regular inside and on the ellipse $A_{n,\alpha}$ with foci at 0 and 1 and with major axis $\left[-\frac{\alpha}{n}, 1 + \frac{\alpha}{n}\right]$. Let $B_{n,\alpha}$ be the ellipse with foci at 0 and 1 and with major axis $\left[-\frac{1}{\alpha n}, 1 + \frac{1}{\alpha n}\right]$. Then there is an absolute constant $c_5 > 0$ such that*

$$\max_{z \in B_{n,\alpha}} \log|f(z)| \leq \max_{z \in [0,1]} \log|f(z)| + \frac{c_5}{\alpha}\left(\max_{z \in A_{n,\alpha}} \log|f(z)| - \max_{z \in [0,1]} \log|f(z)|\right) .$$

Proof. This follows from the Hadamard Three Circles Theorem with the substitution $w = \frac{1}{4}(z + z^{-1}) + \frac{1}{2}$. See also the remark following the Hadamard Three Circles Theorem, which is applied with the circles centered at 0 with radii $r_1 := 1$, $r := 1 + c\sqrt{1/(\alpha n)}$, and $r_2 := 1 + \sqrt{\alpha/n}$, respectively, with a suitable choice of c. □

Lemma 3.3. *Let $p \in \mathcal{G}_n$ with $\|p\|_{[0,1]} =: \exp(-\alpha)$, $\alpha \geq \log(n+1)$. Then there is an absolute constant $c_6 > 0$ such that*

$$\max_{z \in B_{n,\alpha}} |p(z)| \leq c_6 \max_{z \in [0,1]} |p(z)|\,,$$

where $B_{n,\alpha}$ is the same ellipse as in Corollary 3.2.

Proof. Note that $\|p\|_{[0,1]} \geq |p(1/4)| \geq 4^{-n}$ for every $p \in \mathcal{G}_n$. Therefore $\alpha \leq (\log 4)n$. Our assumption on $p \in \mathcal{G}_n$ can be written as

$$\max_{z \in [0,1]} \log |p(z)| = -\alpha\,.$$

Also, $p \in \mathcal{G}_n$ and $z \in A_{n,\alpha}$ imply that

$$\begin{aligned} \log |p(z)| &\leq \log \left((n+1)\big(1 + \tfrac{\alpha}{n}\big)^{n+1} \right) \\ &\leq \log(n+1) + (n+1)\tfrac{\alpha}{n} \leq \log(n+1) + 2\alpha \leq 3\alpha\,. \end{aligned}$$

Now the lemma follows from Corollary 3.2. □

Lemma 3.4. *There is an absolute constant $c_7 > 0$ such that*

$$\|p'\|_{[0,1]} \leq c_7 \alpha n\, \|p\|_{[0,1]}$$

for every $p \in \mathcal{G}_n$ with $\|p\|_{[0,1]} = \exp(-\alpha) \leq (n+1)^{-1}$.

Proof. This follows from Lemma 3.3 and the Cauchy Integral Formula. Note that for a sufficiently large absolute constant $c > 0$, the disks centered at $y \in [0,1]$ with radius $1/(c\alpha n)$ are inside the ellipse $B_{n,\alpha}$ (see the definition in Corollary 3.2). □

Lemma 3.5. *There is an absolute constant $c_8 > 0$ such that*

$$\|p'\|_{[0,1]} \leq c_8 n \log(n+1)\, \|p\|_{[0,1]}$$

for every $p \in \mathcal{G}_n$ with $\|p\|_{[0,1]} \geq (n+2)^{-1}$.

Proof. Applying Corollary 3.2 with $\alpha = \log(n+2)$, we obtain that there is an absolute constant $c_9 > 0$ such that

$$\max_{z \in B_{n,\log(n+2)}} |p(z)| \leq c_9 \max_{z \in [0,1]} |p(z)|$$

for every $p \in \mathcal{G}_n$ with $\|p\|_{[0,1]} \geq (n+2)^{-1}$. To see this note that

$$\max_{z \in [0,1]} \log |p(z)| \geq -\log(n+2)$$

and

$$\max_{z \in A_{n,\alpha}} \log |p(z)| \leq \log \left(n \left(1 + \frac{\log(n+2)}{n} \right)^n \right) \leq 2 \log(n+2)\,.$$

Now the Cauchy Integral Formula yields that

$$\|p'\|_{[0,1]} \leq c_{10} n \log(n+1) \|p\|_{[0,1]}$$

with an absolute constant $c_{10} > 0$. Note that for a sufficiently large absolute constant $c > 0$, the disks centered at $y \in [0,1]$ with radius $1/(cn \log(n+2))$ are inside the ellipse $B_{n,\log(n+2)}$ (see the definition in Corollary 3.2). □

4. Proof of the Theorems

Proof of Theorem 2.1. The upper bound of the theorem was proved in [5]. It is sufficient to establish the lower bound of the theorem for degrees of the form $N = 16^n(n+1)$, $n = 1, 2, \dots$. This follows from the fact that if we have polynomials

$$P_N \in \mathcal{F}_N, \qquad N = 16^n(n+1), \qquad n = 1, 2, \dots,$$

showing the lower bound ot the theorem with a constant $c > 0$, then the polynomials

$$Q_N := P_{16^n(n+1)} \in \mathcal{F}_N, \qquad N = 1, 2, \dots,$$

show the lower bound of the theorem with the constant $c/1024 > 0$, where n is the largest integer for which $16^n(n+1) \leq N$. To show the lower bound of the theorem for the values $N = 16^n(n+1), n = 1, 2, \dots$, we proceed as follows. Let $n \geq 1$ and let T_n be the Chebyshev polynomial defined by

$$T_n(x) = \cos n\theta\,, \qquad x = \cos\theta\,, \quad \theta \in [0, \pi]\,.$$

Then $\|T_n\|_{[-1,1]} = 1$. Denote the coefficients of T_n by $a_k = a_{k,n}$, that is,

$$T_n(x) = \sum_{k=0}^{n} a_k x^k\,.$$

It is well known that the Chebyshev polynomials T_n satisfy the three-term recursion

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)\,.$$

Hence each a_k is an integer and, as a trivial bound for the coefficients of T_n, we have

$$|a_k| \leq 3^n\,, \qquad k = 0, 1, \dots, n\,. \tag{4.1}$$

Also, either $a_0 = 0$ or $a_0 = \pm 1$. Let $A := 16^n$ and let

$$P_n(x) := \sum_{k=0}^{n} \operatorname{sign}(a_k) \sum_{j=0}^{|a_k|-1} x^{Ak+j}\,.$$

We will show that P_n gives the required lower bound (with n replaced by $N := 16^n(n+1)$ in the theorem). It is straightforward from (4.1) that

$$P_n \in \mathcal{F}_N \qquad \text{with} \quad N := 16^n(n+1)\,. \tag{4.2}$$

Observe that for $x \in [0, 1]$,

$$\begin{aligned}
|P_n(x) - T_n(x^A)| &= \left| \sum_{k=0}^{n} \operatorname{sign}(a_k) \sum_{j=0}^{|a_k|-1} x^{Ak+j} - \sum_{k=0}^{n} a_k x^{Ak} \right| \\
&= \sum_{k=0}^{n} \operatorname{sign}(a_k) x^{Ak} \sum_{j=0}^{|a_k|-1} (x^j - 1) \\
&= \left| \sum_{k=1}^{n} \operatorname{sign}(a_k) x^{Ak}(1-x) \sum_{j=0}^{|a_k|-1} (1 + x + x^2 + \cdots + x^{j-1}) \right| \\
&\leq \sum_{k=1}^{n} \frac{|a_k|^2 - |a_k|}{2} \left(\max_{x \in [0,1]} x^{Ak}(1-x) \right) \leq \sum_{k=1}^{n} \frac{|a_k|^2}{Ak} \\
&\leq \frac{n9^n}{16^n} \leq 1\,.
\end{aligned}$$

Hence, for $x \in [0,1]$,

$$|P_n(x)| \leq |T_n(x^A)| + |P_n(x) - T_n(x^A)| \leq 1 + 1 = 2\,.$$

We conclude that

$$\|P_n\|_{[0,1]} \leq 2\,. \tag{4.3}$$

Let $Q_{n,A}(x) := T_n(x^A)$. Then

$$Q'_{n,A}(1) = AT'_n(1) = An^2\,.$$

Now

$$\begin{aligned} P'_n(1) &= \sum_{k=1}^{n} \operatorname{sign}(a_k) \sum_{j=0}^{|a_k|-1} (Ak+j) \\ &= Q'_{n,A}(1) + \sum_{k=1}^{n} \operatorname{sign}(a_k) \sum_{j=0}^{|a_k|-1} j \\ &\geq An^2 - n|a_k|^2 \geq 16^n n^2 - 9^n n \geq \frac{9}{16} 16^n n^2\,. \end{aligned} \tag{4.4}$$

Now (4.2), (4.3), and (4.4) give the lower bound of the theorem.

Proof of Theorem 2.2. The upper bound of the theorem follows from Theorem 2.1. To show the lower bound we modify the construction in Theorem 2.1 by replacing the 0 coefficients in P_n by ± 1 coefficients with alternating signs. We use the notation of the proof of Theorem 2.1. Let $R_n \in \mathcal{L}_N$ be the polynomial arising from P_n in this way. Recall that $N := 16^n(n+1)$. Then, using the fact that $R_n - P_n$ is of the form

$$R_n(x) - P_n(x) = \pm \sum_{j=1}^{m} (-1)^j x^{k_j}\,, \qquad 0 \leq k_1 < k_2 < \cdots < k_m \leq N\,,$$

we have $\|R_n - P_n\|_{[0,1]} \leq 1$. Combining this with (4.2), we obtain

$$\|R_n\|_{[0,1]} \leq \|P_n\|_{[0,1]} + \|R_n - P_n\|_{[0,1]} \leq 2 + 1 = 3\,. \tag{4.5}$$

On the other hand, using the form of $R_n - P_n$ given above, we have $|(R_n - P_n)'(1)| \leq N$. This, together with (4.4) gives

$$R'_n(1) \geq P'_n(1) - |(R_n - P_n)'(1)| \geq \frac{9}{16} 16^n n^2 - 16^n(n+1) \geq \frac{1}{4} 16^n n^2 \tag{4.6}$$

for $n \geq 4$. Now (4.5) and (4.6) together with $R_n \in \mathcal{L}_N$ and $N := 16^n(n+1)$, give the lower bound of the theorem.

Proof of Theorem 2.3. Since $\|p\|_{[0,1]} \geq |p(0)| = 1$, the upper bound of the theorem follows as a special case of Lemma 3.5. The lower bound of the theorem follows from the lower bound in Theorem 2.2.

Proof of Theorem 2.4. To see the upper bound, note that Lemma 3.1 implies $\alpha \leq c_3 n^{1/2}$ in Lemma 3.4. So the upper bound of the theorem follows from Lemmas 3.4 and 3.5. Now we prove the lower bound of the theorem. Let k be a natural number and let

$$Q_n(x) := x^{n+1} T_k(x^k) = \sum_{j=n+1}^{n+1+k^2} b_j x^j \,,$$

where T_k is, once again, the Chebyshev polynomial defined by

$$T_k(x) = \cos n\theta \,, \qquad x = \cos\theta \,, \quad \theta \in [0,\pi] \,.$$

Then $\|Q_n\|_{[0,1]} = 1$ and

$$Q_n'(1) = k^3 + n + 1 \,, \tag{4.7}$$

while, the coefficients b_j of Q_n satisfy

$$|b_j| \leq 3^k \,, \qquad j = 0, 1, \dots, n+1+k^2 \tag{4.8}$$

(see the argument in the proof of Theorem 2.1). By Lemma 3.1 there is a polynomial $R_n \in \mathcal{F}_n$ such that

$$\|R_n\|_{[0,1]} \leq \exp(-c_4\sqrt{n}) \,.$$

Let

$$P_n := R_n + \exp(-c_4\sqrt{n}) Q_n =: \sum_{j=0}^{n+1+k^2} a_j x^j \,.$$

Then

$$\|P_n\|_{[0,1]} \leq 2\exp(-c_4\sqrt{n}) \,. \tag{4.9}$$

It follows from (4.8) that $|a_j| \leq \exp(-c_4\sqrt{n})3^k$ for each $j \geq n+1$. From now on let

$$k := \left\lfloor \frac{c_4\sqrt{n}}{\log 3} \right\rfloor \,, \tag{4.10}$$

which implies that $|a_j| \leq 1$ for each $j \geq n+1$, while $a_j \in \{-1, 0, 1\}$ for each $j \leq n$. Hence

$$P_n \in \mathcal{G}_{n+1+k^2} \,, \qquad n + 1 + k^2 \leq c_5 n \,, \tag{4.11}$$

with an absolute constant c_5. Note that if n is large enough, then $R_n'(1) = 0$. Otherwise, as an integer, $|R_n'(1)|$ would be at least 1 and Markov's inequality would imply that

$$1 \leq |R_n'(1)| \leq 2n^2 \|R_n\|_{[0,1]} \leq 2n^2 \exp(-c_4\sqrt{n}) \,,$$

which is impossible for a large enough n, say for $n \geq c_6$. Hence, combining (4.9), (4.10), and (4.7), we conclude

$$\begin{aligned} P_n'(1) &= R_n'(1) + \exp(-c_4\sqrt{n}) Q_n'(1) = \exp(-c_4\sqrt{n})(k^3 + n + 1) \\ &\geq \exp(-c_4\sqrt{n}) c_7 n^{3/2} \geq \frac{c_7}{2} n^{3/2} \|P_n\|_{[0,1]} \,. \end{aligned}$$

for every $n \geq c_6$ with an absolute constant $c_7 > 0$. This, together with (4.11) finishes the proof of the lower bound of the theorem.

Proof of Theorem 2.5. The upper bound of the theorem was proved in [5]. To show the lower bound we proceed as follows. Let P_n be the same as in the proof of Theorem 2.1. Throughout this proof we will use the notation introduced in the proof of Theorem 2.1. We will show that P_n gives the required lower bound with a suitably chosen even n depending on y. If $n := 2m$ is even, then

$$|P_n(0)| = 1, \qquad \text{and} \qquad P_n \in \mathcal{F}_N \qquad \text{with} \quad N := 16^n(n+1)\,. \tag{4.12}$$

Recall that by (4.3) we have $\|P_n\|_{[0,1]} \leq 2$. As in the proof of Theorem 2.1, let $A := 16^n$ and $Q_{n,A}(x) := T_n(x^A)$. For $n \geq n_0$, the Chebyshev polynomial T_n has a zero δ in $[e^{-2}, e^{-1}]$. In particular, $T_n'(\delta) \geq n$. Let $n \in \mathbb{N}$ be the largest even integer for which

$$\delta^{1/A} = \delta^{16^{-n}} \leq e^{-16^{-n}} \leq y\,. \tag{4.13}$$

Without loss of generality we may assume that $n \geq n_0$, otherwise $p \in \mathcal{F}$ defined by $p(x) := 1 + x$ shows the lower bound of the theorem. Note that there are absolute constants $c_3 > 0$ and $c_4 > 0$ such that $16^n \geq c_3(1-y)^{-1}$ and hence

$$16^n n \geq \frac{c_4 \log\left(\frac{2}{1-y}\right)}{1-y} \tag{4.14}$$

Therefore

$$Q_{n,A}'(\delta^{1/A}) = A\delta^{(A-1)/A}T_n'(\delta) \geq An\delta = 16^n n\delta \geq e^{-2}16^n n\,.$$

Observe that $\delta \in [e^{-2}, e^{-1}]$ and $0 \leq j \leq |a_k| - 1 \leq A$ (see the notation introduced in the proof of Theorem 2.1) imply

$$(Ak+j)\delta^{(Ak+j-1)/A} - Ak\delta^{(Ak-1)/A} \leq j$$

and

$$\begin{aligned}(Ak+j)\delta^{(Ak+j-1)/A} - Ak\delta^{(Ak-1)/A} &\geq Ak\delta^{k-(1/A)}(\delta^{j/A} - 1)\\ &\geq Ak\delta^{k-(1/A)}\frac{-j}{A}\log\frac{1}{\delta}\\ &\geq -2j\,.\end{aligned}$$

We conclude that

$$\left|(Ak+j)\delta^{(Ak+j-1)/A} - Ak\delta^{(Ak-1)/A}\right| \leq 2j \tag{4.15}$$

Now, using the definition of P_n (see the proof of Theorem 2.1), (4.13), (4.15), and (4.14), we obtain

$$\begin{aligned}
\|P_n'\|_{[0,y]} \geq |P_n'(\delta^{1/A})| &= \Big|\sum_{k=1}^{n} \operatorname{sign}(a_k) \sum_{j=0}^{|a_k|-1} (Ak+j)\delta^{(Ak+j-1)/A}\Big| \\
&= \Big|Q_{n,A}'(\delta^{1/A}) + \sum_{k=1}^{n} \operatorname{sign}(a_k) \sum_{j=0}^{|a_k|-1} \big((Ak+j)\delta^{Ak+j-1)/A} - Ak\delta^{Ak-1)/A}\big)\Big| \\
&\geq \big|Q_{n,A}'(\delta^{1/A})\big| - \sum_{k=1}^{n} \sum_{j=0}^{|a_k|-1} 2j \\
&\geq e^{-2}16^n n - 2n|a_k|^2 \geq e^{-2}16^n n - 2n9^n \geq \tfrac{1}{16}16^n n \\
&\geq \frac{c_1 \log\left(\frac{2}{1-y}\right)}{1-y}
\end{aligned}$$

for every $n \geq 2$ with an absolute constant $c_1 > 0$. This, together with (4.3) and (4.12) gives the lower bound of the theorem. □

Proof of Theorem 2.6. The upper bound of the theorem follows from Theorem 2.5. To show the lower bound we modify the construction in Theorem 2.5 by replacing the 0 coefficients in P_n by ± 1 coefficients with alternating signs. We use the notation of the proof of Theorems 2.1 and 2.5. Let $R_n \in \mathcal{L}_N$ be the polynomial arising from P_n in this way. As in the proof of Theorem 2.2, we have $\|R_n - P_n\|_{[0,1]} \leq 1$, and combining this with (4.3), we obtain

$$\|R_n\|_{[0,1]} \leq \|P_n\|_{[0,1]} + \|R_n - P_n\|_{[0,1]} \leq 2 + 1 = 3\,. \tag{4.16}$$

On the other hand, for $a \in [0,1)$, we have

$$|(R_n - P_n)'(a)| \leq \operatorname{Var}_1^\infty f_a(x)\,, \tag{4.17}$$

where $f_a(x) := xa^{x-1}$. Now it is elementary calculus to show that the graph of f_a on $[1,\infty)$ contains two monotone pieces,

$$\max_{x\in[1,\infty)} |f_a(x)| \leq \frac{c_3}{1-a} \qquad \text{and} \qquad \lim_{x\to\infty} f_a(x) = 0\,.$$

Hence

$$\operatorname{Var}_1^\infty f_a(x) \leq \frac{c_4}{1-a}$$

with an absolute constant $c_4 > 0$. This, together with (4.17) yields

$$|(R_n - P_n)'(a)| \leq \frac{c_4}{1-a}\,. \tag{4.18}$$

Now let $y \in [0,1)$. By the proof of Theorem 2.5, there exists an $a \in [0,y]$ such that

$$|P_n'(a)| \geq \frac{c_1 \log\left(\frac{2}{1-y}\right)}{1-y}\,.$$

Combining this with (4.18), we deduce

$$\begin{aligned}\|R_n'\|_{[0,y]} &\geq |R_n'(a)| \geq |P_n'(a)| - |(R_n - P_n)'(a)| \\ &\geq \frac{c_1 \log\left(\frac{2}{1-y}\right)}{1-y} - \frac{c_4}{1-y} \geq \frac{c_5 \log\left(\frac{2}{1-y}\right)}{1-y}\end{aligned}$$

for $y \in [y_0, 1]$, where $c_5 > 0$ and $y_0 \in [0, 1)$ are absolute constants. This, together with (4.16) and $R_n \in \mathcal{L}_N$ gives the lower bound of the theorem for $y \in [y_0, 1]$. If $y \in [0, y_0)$, then the trivial example $p(x) := 1 + x$ shows the lower bound of the theorem. □

Proof of Theorem 2.7. The upper bound of the theorem was proved in [5]. The lower bound of the theorem follows from the lower bound in either Theorem 2.5 or Theorem 2.6. □

5. Acknowledgment.

We thank Fedor Nazarov for suggesting the the crucial idea for the proof of the lower bounds in Theorems 2.1 and 2.2.

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