THE NUMBER OF CERTAIN INTEGRAL POLYNOMIALS AND NONRECURSIVE SETS OF INTEGERS, PART 1

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Abstract. Given $r > 2$, we establish a good upper bound for the number of multivariate polynomials (with as many variables and with as large degree as we wish) with integer coefficients mapping the "cube" with real coordinates from $[-r, r]$ into $[-t, t]$. This directly translates to a nice statement in logic (more specifically recursion theory) with a corresponding phase transition case of 2 being open. We think this situation will be of real interest to logicians. Other related questions are also considered. In most of these problems our main idea is to write the multivariate polynomials as a linear combination of products of scaled Chebyshev polynomials of one variable.

In some private communications, Harvey Friedman raised the following problem: given $r > 2$, give an upper bound for the number of multivariate polynomials (with as many variables and with as large degree as we wish) with integer coefficients mapping the "cube" with real variables from $[-r, r]$ into $[-t, t]$. Robin Pemantle has established a rough upper bound. Here, utilizing Chebyshev polynomials, we establish a reasonably good upper bound. Namely, in this paper we prove our main result and some related ones, applications of which in recursion theory are given by Harvey Friedman in a separate article. We think that the two papers are so closely related that we decided to publish them in the same journal.

The Main Result

Theorem 1. *Let $r > 2$. The number of multivariate polynomials (with as many variables and with as large degree as we wish) with integer coefficients mapping the "cube" with real variables from $[-r, r]$ into $[-t, t]$ is at most*

$$(2t+1)^{t^2} \left(t^{(4\log^2 t)/((\log 2)(\log(r/2)))}\right)^{t^2} \leq \exp(c\, t^2 \log^3 t)\,,$$

where the constant c depends only on r.

In the above theorem, and throughout the paper, log without a specified base means the natural logarithm with the base e.

To prove the theorem we need a few lemmas.

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Lemma 1. *Let P_d be a polynomial of exactly d variables with integer coefficients (the degree is irrelevant). Then the maximum modulus of P_d on the d-cube $I_d(2) := [-2,2]^d$ is at least $(2d)^{1/2}$.*

Problem 1. *As the map $x_1 + x_2 + \cdots + x_d$ suggests the right lower bound in Lemma 1 may be $2d$ (or cd). In any case the optimal bound in Lemma 2 is somewhere between $(2d)^{1/2}$ and $2d$. Close the gap. Can the magnitude of the lower bound $(2d)^{1/2}$ in Lemma 1 be improved? Also, are there polynomials P_d of exactly d variables with integer coefficients (the degree is irrelevant) so that the maximum modulus of P_d on the d-cube $I_d(2) := [-2,2]^d$ is significantly lower than $2d$?*

Proof of Lemma 1. Let T_j be the j-th Chebyshev polynomial defined by

$$T_j(x) = \cos(jt), \qquad x = \cos t\,.$$

Let

$$Q_0(x) = 1\,, \qquad Q_j(x) = 2T_j(x/2)\,, \qquad j = 1, 2, \ldots\,.$$

The following facts are easy to check:

(i) Q_j is a polynomial of degree j with integer coefficients and with leading coefficient 1. This follows from the three-term-recursion

$$T_j(x) = 2xT_{j-1}(x) - T_{j-2}(x)\,, \qquad j = 2, 3, 4 \ldots\,,$$

$$T_0(x) = 1\,, \quad T_1(x) = x\,, \quad T_2(x) = 2x^2 - 1\,,$$

that is

$$Q_j(x) = xQ_{j-1}(x) - Q_{j-2}(x)\,, \qquad j = 3, 4, 5 \ldots\,,$$

$$Q_0(x) = 1, \quad Q_1(x) = x\,, \quad Q_2(x) = x^2 - 1\,.$$

(ii) The polynomials

$$Q_0\,, \quad 2^{-1/2}Q_j\,, \quad j = 1, 2, \ldots\,,$$

are orthonormal on $[-2,2]$ with respect to the unit measure

$$\mu(x) = \frac{dx}{\pi\sqrt{4-x^2}}\,.$$

(iii) It follows from (i) and (ii) above that every polynomial in variables $x_1, x_2, \ldots, x_d$ with integer coefficients can be written as a linear combination of the products

$$S_{n_1,n_2,\ldots,n_d}(x_1, x_2, \ldots, x_d) := Q_{n_1}(x_1)Q_{n_2}(x_2)\cdots Q_{n_d}(x_d)$$

with integer coefficients and the products $S_{n_1,n_2,\ldots,n_d}$ are orthogonal on $I_d(2) := [-2,2]^d$ with respect to the unit measure

$$\mu^d(x_1, x_2, \ldots, x_d) := \mu(x_1)\,dx_1 \times \mu(x_2)\,dx_2 \times \cdots \times \mu(x_d)\,dx_d\,.$$

(iv) We obtain by the Parseval formula that if P_d is a polynomial of exactly d variables $x_1, x_2, \dots, x_d$, then

$$\int_{I_d(2)} P_d(x_1, x_2, \dots, x_d)^2 \, d\mu^d(x_1, x_2, \dots, x_d) \geq 2^{k_1} + 2^{k_2} + \dots + 2^{k_m} \geq 2d\,,$$

where $k_1, k_2, \dots, k_m$ are positive integers with sum at least d.

The conclusion of the lemma now follows from (iv), since the integration takes place with respect to the unit measure μ^d on $I_d(2)$. □

Using the notation introduced in the proof of Lemma 1 we have that every polynomial $P_{d,n}$ of at most d variables $x_1, x_2, \dots, x_d$ and of degree at most n can be written as

$$(1) \qquad P_{d,n}(x_1, x_2, \dots, x_d) = \sum_{n_d=0}^{n} \cdots \sum_{n_2=0}^{n} \sum_{n_1=0}^{n} a_{n_1, n_2, \dots, n_d} Q_{n_1}(x_1) Q_{n_2}(x_2) \cdots Q_{n_d}(x_d)$$

with some integer coefficients $a_{n_1, n_2, \dots, n_d}$. (If $P_{d,n}$ is of exactly d variables this representation is unique.)

Lemma 2. *Let P_d be a polynomial of at most d variables $x_1, x_2, \dots, x_d$. Assume that P_d is a polynomial of x_k of degree $m \geq 1$. Let $r > 2$. Then the maximum modulus of P_d on the d-cube $I_d(r) := [-r, r]^d$ is at least $2(r/2)^m$.*

Proof of Lemma 2. The proof follows easily from the classical Chebyshev's inequality stating that if P is a polynomial of degree m of one variable with leading coefficient 1, then the maximum modulus of P on $[-1, 1]$ is at least 2^{1-m}. See Theorem 2.1.1 on page 30 of [BE].

The leading coefficient of P_d as a polynomial of x_k is a polynomial Q_{d-1} of at most $d - 1$ variables

$$(2) \qquad x_1, x_2, \dots, x_{k-1}, x_{k+1}, x_{k+2}, \dots, x_d\,.$$

with integer coefficients. Assume that Q_{d-1} is a polynomial of exactly ν variables out of the variables under (2). If $\nu \geq 1$, then by Lemma 1, for certain values of the variables under (2) we have that

$$|Q_{d-1}(x_1, x_2, \dots, x_{k-1}, x_{k+1}, x_{k+2}, \dots, x_d)| \geq \sqrt{2}\,,$$

while if $\nu = 0$, then Q_{d-1} is a nonzero integer constant. Now using Chebyshev's inequality transformed from $[-1, 1]$ to $[-r, r]$ to P_d as a polynomial of x_k with the special values of the variables under (2), we can couple these special choices of variables with a choice of x_k so that P_d takes a value with modulus at least $2(r/2)^m$ at the special values of $x_1, x_2, \dots, x_d$. □

Lemma 3. *Let $r > 2$. If $P_{d,n}$ is of form (1) and the maximum modulus of $P_{d,n}$ on $I_d(r)$ is at most t, then*
(a) *the coefficients satisfy*

$$|a_{n_1,n_2,\dots,n_d}| \le t\,, \qquad 0 \le n_1, n_2, \dots, n_d \le n\,,$$

(b) *all but t^2 coefficients $a_{n_1,n_2,\dots,n_d}$ are 0,*
(c) *for the nonzero coefficients $a_{n_1,n_2,\dots,n_d}$ we have*

$$|\{j = 1, 2, \dots, d : n_j > 0\}| \le \frac{2\log t}{\log 2}\,,$$

(d) *for the nonzero coefficients $a_{n_1,n_2,\dots,n_d}$ we have*

$$0 \le n_1, n_2, \dots, n_d \le \frac{\log(t/2)}{\log(r/2)}\,.$$

Proof of Lemma 3. Statements (a), (b), and (c) follow from evaluating the integral

$$\int_{I_d(2)} P_d(x_1, x_2, \dots, x_d)^2 \, d\mu^d(x_1, x_2, \dots, x_d)$$

by the Parseval formula by noting that the polynomials

$$Q_0\,, \quad 2^{-1/2} Q_j\,, \quad j = 1, 2, \dots\,,$$

are orthonormal on $[-2, 2]$ with respect to the unit measure

$$\mu(x) = \frac{dx}{\pi\sqrt{4 - x^2}}$$

(we use the notation introduced in the proof of Lemma 1). Statement (d) follows from Lemma 2. $\square$

Proof of Theorem 1. The proof is a straightforward counting with the help of Lemmas 1,2, and 3. $\square$

Remark 1. It is easy to see that the number $N_r(t)$ of multivariate polynomials (with as many variables and with as large degree as we wish) with integer coefficients mapping the "cube" with real variables from $[-r, r]$ into $[-t, t]$ is at least $\exp(ct)$ with a constant $c > 0$ depending only on r. To see this consider the different linear maps of the form $\sum_{j=1}^m \varepsilon_j x_j$, where $m = \lfloor t/(2r) \rfloor$[1] and the value of each ε_j is in $\{-1, 1\}$. This coupled with Theorem 1 yields that

$$\exp(c_1 t) \le N_r(t) \le \exp(c_2\, t^2 \log^3 t)$$

with positive constants c_1 and c_2 depending only on r.

[1] Here, and in what follows, $\lfloor a \rfloor$ denotes the greatest integer not greater than a.

Problem 2. *Close the gap between* $\exp(c_1 t)$ *and* $\exp(c_2 t^2 \log^3 t)$ *in Remark 1.*

Remark 2. Note that in Theorem 1 as well as in Lemmas 1 and 2, $r = 2$ is the turning point. To see that in the case $0 < r < 2$, or even in the more general case of $[a, b]$ with $b - a < 4$, there is no upper bound for the number of variables in multivariate polynomials with integer coefficients mapping real arguments from $[-r, r]$ into $[-t, t]$ $(t \to \infty)$, one can use the following simple result on page 50 in [LGM].

Theorem A. *If* $b - a < 4$*, then there is a monic polynomial* Q *with integer coefficients satisfying* $0 \le Q(x) < 1$ *on* $[a, b]$*.*

Now take

$$Q(x_1)^n + Q(x_2)^n + \cdots + Q(x_n)^n$$

that maps the "cube" $[a, b]^n$ into $[0, 1]$ if n is sufficiently large.

Problems and Further Results

Harvey Friedman was particularly interested in the answer to the questions in Problems 3,4, and 5 below. Note that these questions are in fact the same, but we had reasons to speculate that the answers may be different depending on the magnitude of r.

Problem 3. *Let* $r > 2$*. Is it true that for every sufficiently large* n *there is a polynomial* P_n *with integer coefficients such that* $|P_n(x)| \le \log\log n$ *on* $[-r, r]$*, and the maximum of* P_n *on integer arguments is* n*?*

Problem 4. *Is it true that for every sufficiently large* n *there is a polynomial* P_n *with integer coefficients such that* $|P_n(x)| \le \log\log n$ *on* $[-2, 2]$*, and the maximum of* P_n *on integer arguments is* n*?*

Problem 5. *Let* $0 < r < 2$*. Is it true that for every sufficiently large* n *there is a polynomial* P_n *with integer coefficients such that* $|P_n(x)| \le \log\log n$ *on* $[-r, r]$*, and the maximum of* P_n *on integer arguments is* n*?*

The negative answer to Problem 3 (even to its multivariate analogue) comes from the following result which is a special case of Theorem 1.

Theorem 2. *Let* $r > 2$*. If* n *is sufficiently large, then there are at most* $n/2$ *multivariate polynomials* P_n *with integer coefficients such that* $|P_n| \le (\log n)^{1/3}$ *on the "cube" with real variables from* $[-r, r]$*.*

At the moment we do not know the answer to Problem 4. Nevertheless we can prove the following result.

Theorem 3. *For every positive integer* n *there is a polynomial* P_n *with integer coefficients such that* $|P_n(x)| \le 192 \log_6(n/7) + 49$ *on* $[-2, 2]$*, and the maximum of* P_n *on integer arguments is* n*.*

Proof of Theorem 3. As in the proof of Lemma 1, let

$$Q_0(x) = 1\,, \qquad Q_j(x) = 2T_j(x/2)\,, \qquad j = 1, 2, \ldots\,,$$

with
$$T_j(x) = \cos(jt), \qquad x = \cos t\,, \qquad j = 0, 1, 2, \dots .$$
We have
$$Q_n(3) = 2T_n(3/2) = \left(3/2 + \sqrt{5/4}\right)^n + \left(3/2 - \sqrt{5/4}\right)^n .$$
Let
$$a := \left(3/2 + \sqrt{5/4}\right)^2 = 6.854\dots .$$
Observe that every positive y can be written as
$$y = \sum_{j=1}^{m(y)} d_j(y) a^j + r(y)\,, \qquad m(y) = \lfloor \log_a y \rfloor\,, \quad d_j(y) \in \{0,1,2,3,4,5,6\}\,, \quad 0 \le r(y) < a\,.$$
Let
$$S_y(x) := (16 - x^2) \sum_{j=1}^{m(y)} d_j(y) Q_{2j}(x)\,.$$
Then, denoting the set of all integers by $\mathbb{Z}$, we have
$$\max_{x\in\mathbb{Z}} S_y(x) = 7 \sum_{j=1}^{m(y)} d_j(y) Q_{2j}(3) = 7 \sum_{j=1}^{m(y)} d_j(y)(a^j + a^{-j})\,,$$
so
$$7(y-7) \le \max_{x\in\mathbb{Z}} S_y(x) \le 7(y - r(y)) + 7\cdot 6 \sum_{j=1}^{m} a^{-j} \le 7y + 8\,.$$
Therefore $R_n := S_y$ with $y := n/7$ satisfies
$$\max_{x\in\mathbb{Z}} R_n(x) = n + k_n$$
with a suitable integer $-49 \le k_n \le 8$. Now let $P_n := R_n - k_n$. Then P_n is a polynomial with integer coefficients and
$$\max_{x\in\mathbb{Z}} P_n(x) = n\,.$$
Also, $y = n/7$, $m(y) = \lfloor \log_a y \rfloor$, $d_j(y) \in \{0,1,2,3,4,5,6\}$, $\max_{x\in[-2,2]} |Q_{2j}(x)| \le 2$, and $-49 \le k_n \le 8$ imply that for $x \in [-2,2]$ we have
$$\begin{aligned}|P_n(x)| = |S_y(x) - k_n| &\le 16 \left| \sum_{j=1}^{m(y)} d_j(n/7) Q_{2j}(x) \right| + |k_n| \\ &\le 16(\log_a(y))6\cdot 2 + 49 \le 192 \log_6(n/7) + 49\,,\end{aligned}$$
and the theorem is proved. $\square$

As far as Problem 5 is concerned, using Theorem A, one can easily prove the even stronger result below.

Theorem 4. *Let $0 < r < 2$. For every sufficiently large n there is a polynomial P_n with integer coefficients such that $|P_n(x)| \le c$ on $[-r,r]$ with a constant $c > 0$ independent of n, and the maximum of P_n on integer arguments is n.*

Proof of Theorem 4. Let $0 < r < 2$. By Theorem A there is a monic polynomial Q with integer coefficients so that

$$M_r(Q) := \max_{x\in[-r,r]} |Q(x)| < 1\,.$$

We choose $\alpha \in \mathbb{N}$ so that the zeros of Q are in $[-\alpha,\alpha]$, and let

$$S_k(x) := ((\alpha+2)^2 - x^2)Q(x)^{2k}\,.$$

It is easy to see that if the positive integer k is sufficiently large, then

$$M_r(S_k) := \max_{x\in[-r,r]} |S_k(x)| < \frac{1}{2}$$

and

$$m := \max_{x\in\mathbb{N}} S_k(x) \ge 2$$

is a finite integer. Now write n in the number system with base m, that is,

$$n = \sum_{j=0}^{\mu} a_j m^j\,, \qquad a_j \in \{0,1,2,\dots,m-1\}\,.$$

We define $P_n := \sum_{j=0}^{\mu} a_j S_k^j$. Then

$$\max_{x\in\mathbb{N}} P_n(x) = n$$

and

$$M_r(P_n) := \max_{x\in[-r,r]} |P_n(x)| \le (m-1)\sum_{j=0}^{\mu} M_r(S_k)^j \le \frac{m-1}{1-M_r(S_k)}\,,$$

and the theorem is proved. $\square$

Harvey Friedman raised the following questions as well.

Problem 6. *Let $r > 2$. Is it true that for every sufficiently large n there is a polynomial P_n with integer coefficients such that $|P_n(x)| \le \log\log n$ on $[-r,r]$, and the number of integer arguments where P_n takes positive values is n?*

Problem 7. *Is it true that for every sufficiently large n there is a polynomial P_n with integer coefficients such that $|P_n(x)| \le \log\log n$ on $[-2,2]$, and the number of integer arguments where P_n takes positive values is n?*

Problem 8. *Let $0 < r < 2$. Is it true that for every sufficiently large n there is a polynomial P_n with integer coefficients such that $|P_n(x)| \le \log\log n$ on $[-r,r]$, and the number of integer arguments where P_n takes positive values is n?*

The negative answer to Problem 6 (even to its multivariate analogue) follows from Theorem 2 above. As far as Problems 7 is concerned we can prove even the stronger result below.

Theorem 5. *Suppose P is a polynomial of even degree with integer coefficients and with negative leading coefficient. Then $P(x)$ is negative outside the interval*

$$[-(4K_P+3), 4K_P+3]\,, \qquad \textit{with} \qquad K_P := \max_{t\in[-2,2]} |P(t)|\,. \tag{3}$$

Proof of Theorem 5. Let P be a polynomial of even degree n with integer coefficients and with negative leading coefficient. Then $P = \sum_{j=0}^{n} a_j Q_j$ with some integer coefficients a_j, where $a_n < 0$ and, as in the proof of Lemma 1,

$$Q_0(x) = 1\,, \qquad Q_j(x) = 2T_j(x/2)\,, \qquad j = 1, 2, \ldots\,,$$

with

$$T_j(x) = \cos(jt), \qquad x = \cos t\,, \qquad j = 0, 1, 2, \ldots\,.$$

Since the polynomials

$$Q_0\,, \quad 2^{-1/2}Q_j\,, \quad j = 1, 2, \ldots\,,$$

are orthonormal on $[-2,2]$ with respect to the unit measure

$$\mu(x) = \frac{dx}{\pi\sqrt{4-x^2}}\,,$$

it follows from the Parseval formula that

$$a_0^2 + \sum_{j=1}^{n} 2a_j^2 \le K_P^2 = \max_{t\in[-2,2]} |P(t)|^2\,.$$

We use the well-known formula

$$Q_j(y+y^{-1}) = 2T_j\left(\frac{y+y^{-1}}{2}\right) = y^j + y^{-j}\,. \tag{4}$$

Note that for every $x \in \mathbb{R}$ with $|x| > 4K_P + 3$ there is an $y > 4K_P + 2$ so that $x = y + y^{-1}$.

Hence $|x| > 4K_P + 3$ implies that with $x = y + y^{-1}$ we have

$$\begin{aligned}
P(x) = P(y+y^{-1}) &= a_n Q_n(y+y^{-1}) + \sum_{j=1}^{n-1} a_j Q_j(y+y^{-1}) \\
&\le -Q_n(y+y^{-1}) + \left(\sum_{j=0}^{n-1} a_j^2\right)^{1/2} \left(\sum_{j=0}^{n-1} Q_j(y+y^{-1})^2\right)^{1/2} \\
&\le -y^n - y^{-n} + K_P \left(\sum_{j=0}^{n-1} (y^{2j} + y^{-2j} + 2)\right)^{1/2} \\
&\le -y^n - y^{-n} + K_P \left(\frac{4y^{2n}}{y^2-1}\right)^{1/2} \le -y^n + K_P \left(\frac{4y^{2n}}{y^2/4}\right)^{1/2} \\
&\le |y|^{n-1}(4K_p - |y|) < 0\,,
\end{aligned}$$

and the proof is finished. □

Theorem 5 clearly implies that the answer to Problem 7 is "no" even in the multivariate analogue of Problem 7, see the result below.

Theorem 6. *There is no sequence $(P_n)_{n=m}^{\infty}$ of multivariate polynomials with integer coefficients mapping the "cube" with real arguments from $[-2,2]$ into $[-\log\log n, \log\log n]$ so that the number of points with integer coordinates where P_n takes positive values is n.*

Proof of Theorem 6. Suppose P is a polynomial of exactly d variables and with integer coefficients mapping the "cube" $I_d(2) := [-2,2]^d$ into $[-\log\log n, \log\log n]$, and the number of points with integer coordinates where P takes positive values is finite. Then, similarly to the proof of Lemma 1, we can use the Parseval formula to deduce that

$$2d \le \max_{(x_1,x_2,\dots,x_d)\in I_d(2)} |P(x_1,x_2,\dots,x_d)|^2 \le (\log\log n)^2\,.$$

Assume now that $P(x_1,x_2,\dots,x_d) > 0$. By fixing $d-1$ integer variables and using Theorem 5, we obtain that the remaining variable must be in

$$[-(4\log\log n + 3), 4\log\log n + 3]\,.$$

Therefore all the variables $x_1, x_2, \dots, x_d$ must come from the above interval. Since

$$(8\log\log n + 7)^d \le (8\log\log n + 7)^{(\log\log n)^2/2} < n\,,$$

the number of points with integer coordinates where P takes positive values is less than n, and the proof is finished. □

As far as Problem 8 is concerned, by using Theorem A, one can easily prove the even stronger result below.

Theorem 7. *Let $0 < r < 2$. Suppose (c_n) is an arbitrary sequence of positive numbers. For every sufficiently large n there is a polynomial P_n with integer coefficients such that $|P_n(x)| \le c_n$ on $[-r, r]$, the number of integer arguments where P_n takes positive values is n, and the positive values taken by P_n in integer arguments are distinct.*

Proof of Theorem 7. Let $0 < r < 2$. By Theorem A there is a polynomial Q with integer coefficients so that

$$M_r(Q) := \max_{x \in [-r,r]} |Q(x)| < 1\,.$$

Suppose that the zeros of Q and Q' are in $[-\alpha, \alpha]$, where α is a positive integer, and denote by m the number of integer arguments from $[-\alpha, \alpha]$ where Q takes nonzero values. Now let

$$S_{k,n}(x) := -Q(x)^{2k}(x + \alpha + 1)(x - (\alpha + n - m + 1))\,.$$

It is easy to see that if the positive integer $k = k(r, n)$ is sufficiently large, then

$$M_r(S_{k,n}) := \max_{x \in [-r,r]} |S_{k,n}(x)| < c_n\,,$$

the number of integer arguments where $S_{k,n}$ takes positive values is n. To see that the positive values taken by P_n in integer arguments are distinct if the positive integer $k = k(n)$ is sufficiently large, we argue as follows. Suppose $S_{k,n}(x_1) = S_{k,n}(x_2) > 0$ for two distinct integers in $[-\alpha, \alpha + n - m]$ (outside this interval $S_{k,n}$ is nonpositive). Then

$$\left(\frac{Q(x_1)}{Q(x_2)}\right)^{2k} = \frac{(x_2 + \alpha + 1)(x_2 - (\alpha + n - m + 1))}{(x_1 + \alpha + 1)(x_1 - (\alpha + n - m + 1))} \tag{5}$$

Observe that $Q(x_1)$ and $Q(x_2)$ are positive integers not greater than $c_1 n^d$, where d is the degree of Q and c_1 is a constant depending only on Q. First assume that $Q(x_1) > Q(x_2)$ in (5). Then if the positive integer $k = k(r, n)$ is sufficiently large, then

$$\left(\frac{Q(x_1)}{Q(x_2)}\right)^{k} \ge \left(1 + \frac{1}{c_1 n^d}\right)^{k} \ge (n + 2\alpha)^2 > \frac{(x_2 + \alpha + 1)(x_2 - (\alpha + n - m + 1))}{(x_1 + \alpha + 1)(x_1 - (\alpha + n - m + 1))}\,,$$

which contradicts (5). Now assume that $Q(x_1) = Q(x_2)$ in (5). Observe that $|Q(x)|$ is increasing on $[\alpha, \infty)$, so at least one of x_1 and x_2, say x_1, must be an element of $[-\alpha, \alpha]$. Also $Q(x_1) = Q(x_2)$ together with (5) yields that $x_2 = n - m - x_1$. Since $x_1 \in [-\alpha, \alpha]$, we have

$$|Q(x_1)| \le c_2 \alpha^d\,, \tag{6}$$

where d is the degree of Q and c_2 is a constant depending only on Q. Since $|Q(x)|$ is increasing on $[\alpha, \infty)$ and takes integer values in integer arguments, we have

$$|Q(x_2)| \ge n - m - 2\alpha\,. \tag{7}$$

However, for sufficiently large n (6) and (7) contradict the assumption that $Q(x_1) = Q(x_2)$. So the positive values taken by P_n in integer arguments are distinct, indeed. $\square$

The following result seems to be useful as well.

Theorem 8. *For every positive integer n there is a polynomial P_n with integer coefficients such that $|P_n(x)| \le 1 + 2\log_2 n$ on $[-2,2]$, and $\lfloor P_n(5/2) \rfloor = n$.*

Proof of Theorem 8. With $m := \lfloor \log_2 n \rfloor$ let $P_n := \sum_{j=0}^{m} \varepsilon_{j,m} Q_j$, where, as in the proof of Lemma 1,

$$Q_0(x) = 1\,, \qquad Q_j(x) = 2T_j(x/2)\,, \qquad j = 1, 2, \dots\,,$$

with

$$T_j(x) = \cos(jt), \qquad x = \cos t\,, \qquad j = 0, 1, 2, \dots\,.$$

By considering the binary representation of n, it is easy to see that for every positive integer n there are $\varepsilon_{j,m} \in \{-1, 1\}$, $j = 0, 1, \dots, m$, so that

$$\lfloor P_n(5/2) \rfloor = \left\lfloor \varepsilon_{0,m} + \sum_{j=1}^{m} \varepsilon_{j,m}\,(2^j + 2^{-j}) \right\rfloor = \varepsilon_{0,m} + \sum_{j=1}^{m} \varepsilon_{j,m}\, 2^j = n\,.$$

Also, with these values of $\varepsilon_{j,m}$, $j = 0, 1, \dots, m$, we have

$$\max_{x \in [-2,2]} |P_n(x)| \le 1 + 2m \le 1 + 2\log_2 n\,.$$

□

Harvey Friedman raised the question whether or not $\log n$ in Theorem 8 can be replaced by $c \log\log n$. In this direction we can prove the following theorem.

Theorem 9. *Let (m_n) be an increasing sequence of positive numbers. Suppose that for every sufficiently large n there is a polynomial P_n with integer coefficients such that $|P_n(x)| \le m_n$ on $[-2,2]$ and $\lfloor P_n(5/2) \rfloor = n$. Then*

$$m_n \ge \left(\frac{c \log n}{\log\log n} \right)^{1/2}$$

for every sufficiently large n with an absolute constant $c > 0$.

Proof of Theorem 9. First let $n := 2^k$ with a positive integer k. Since (m_n) is increasing $|P_\ell(x)| \le m_n$ on $[-2,2]$ for every $\ell = \frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n$ for every sufficiently large n. As we have seen in the proof of Lemma 1, P_ℓ can be written as

$$P_\ell = \sum_{j=0}^{\mu_\ell} a_{j,\ell}\, Q_j \tag{8}$$

with integer coefficients $a_{j,\ell}$, where, as in the proof of Lemma 1,

$$Q_0(x) = 1\,, \qquad Q_j(x) = 2T_j(x/2)\,, \qquad j = 1, 2, \dots\,,$$

with

$$T_j(x) = \cos(jt), \qquad x = \cos t\,, \qquad j = 0, 1, 2, \dots\,.$$

Using observation (ii) in the proof of Lemma 1, the Parseval formula yields that

$$|a_{j,\ell}| \le m_n\,, \quad j = 0, 1, \dots, \mu_\ell, \quad \ell = \tfrac{n}{2}+1, \tfrac{n}{2}+2, \dots, n\,,$$

and the number of nonzero coefficients $a_{j,\ell}$ in (8) is at most m_n^2. Also

$$\lfloor P_\ell(5/2) \rfloor = \left\lfloor \sum_{j=0}^{\mu_\ell} a_{j,\ell}\,(2^j + 2^{-j}) \right\rfloor = \sum_{j=0}^{\mu_\ell} a_{j,\ell}\, 2^j + r_\ell \tag{9}$$

with an integer $r_\ell \in [-2m_n, 2m_n]$. Let K_n denote the cardinality of the set

$$\left\{ \lfloor P_\ell(5/2) \rfloor \pmod{2^k} :\ \ell = \tfrac{n}{2}+1, \tfrac{n}{2}+2, \dots, n \right\}\,,$$

It is easy to see that $\lfloor P_\ell(5/2) \rfloor = \ell$ implies that

$$K_n \ge \frac{n}{2} \tag{10}$$

for all sufficiently large $n = 2^k$. On the other hand, using (9) and the information on $a_{j,\ell}$ and r_ℓ, we can deduce that

$$K_n \le k^{m_n^2} (2m_n+1)^{m_n^2} (4m_n+1)\,. \tag{11}$$

Combining (10) and (11), we obtain that

$$k^{m_n^2} (2m_n+1)^{m_n^2} (4m_n+1) \ge K_n \ge \frac{n}{2} = 2^{k-1}$$

for every sufficiently large $n = 2^k$. Hence

$$m_n^2 \log k + m_n^2 \log(2m_n+1) \log(4m_n+1) \ge k-1\,,$$

which imolies

$$m_n \ge \left(\frac{ck}{\log k} \right)^{1/2}$$

for every sufficiently large $n = 2^k$ with an absolute constant $c > 0$. The theorem is now proved for $n = 2^k$, from which the general case follows by the monotonicity of the sequence (m_n). $\square$

The next result is closely related to Problem 4.

Theorem 10. *Let (m_n) be an increasing sequence of positive numbers. Suppose that for every sufficiently large n there is a polynomial P_n with integer coefficients such that $|P_n(x)| \le m_n$ on $[-2,2]$ and*

$$\max_{u \in \mathbb{Z} \setminus \{0\}} \lfloor P_n(u + u^{-1}) \rfloor = n\,.$$

Then

$$m_n \geq \left(\frac{c \log n}{\log \log n}\right)^{1/2}$$

for every sufficiently large n with an absolute constant $c > 0$.

Proof of Theorem 10. First let $n := 2^k$ with a positive integer k. Since (m_n) is increasing $|P_\ell(x)| \leq m_n$ on $[-2, 2]$ for every $\ell = \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n$ for every sufficiently large n. Then Theorem 5 implies that P_ℓ is negative outside the interval $[-(4m_n + 3), 4m_n + 3]$. As we have seen in the proof of Lemma 1, P_ℓ can be written as

$$P_\ell = \sum_{j=0}^{\mu_\ell} a_{j,\ell}\, Q_j \tag{12}$$

with integer coefficients $a_{j,\ell}$, where, as in the proof of Lemma 1 and Theorem 9,

$$Q_0(x) = 1\,, \qquad Q_j(x) = 2T_j(x/2)\,, \qquad j = 1, 2, \ldots\,,$$

with

$$T_j(x) = \cos(jt), \qquad x = \cos t\,, \qquad j = 0, 1, 2, \ldots\,.$$

Using observation (ii) in the proof of Lemma 1, the Parseval formula yields that

$$|a_{j,\ell}| \leq m_n\,, \quad j = 0, 1, \ldots, \mu_\ell, \quad \ell = \tfrac{n}{2} + 1, \tfrac{n}{2} + 2, \ldots, n\,,$$

and the number of nonzero coefficients $a_{j,\ell}$ in (12) is at most m_n^2. For fixed positive integers n and u let $K(n, u)$ denote the cardinality of the set

$$\left\{\lfloor P_\ell(u + u^{-1})\rfloor : \ \ell = \tfrac{n}{2} + 1, \tfrac{n}{2} + 2, \ldots, n\right\}$$

Note that

$$\lfloor P_\ell(u + u^{-1})\rfloor = \left\lfloor \sum_{j=0}^{\mu_\ell} a_{j,\ell}\,(u^j + u^{-j}) \right\rfloor = \sum_{j=0}^{\mu_\ell} a_{j,\ell}\, u^j + r_\ell \tag{13}$$

with an integer $r_\ell \in [-2m_n, 2m_n]$. Let $K^*(n, u)$ denote the cardinality of the set

$$\left\{\lfloor P_\ell(u + u^{-1})\rfloor \pmod{u^k} : \ \ell = \tfrac{n}{2} + 1, \tfrac{n}{2} + 2, \ldots, n\right\}\,.$$

Let $K^*(n, 0) := 0$,

$$K_n^* := \sum_{u=-(4m_n+3)}^{4m_n+3} K^*(n, u) \qquad \text{and} \qquad S_n^* := K^*(n, -1) + K^*(n, 1)\,.$$

Obviously $S_n^* \le 4m_n + 2$. Since there are integers v_ℓ from $[-(4m_n+3), 4m_n+3]$ such that

$$P_\ell(v_\ell) := \max_{u \in \mathbb{Z} \setminus \{0\}} \lfloor P_\ell(u + u^{-1}) \rfloor = \ell\,, \quad \ell = \tfrac{n}{2} + 1, \tfrac{n}{2} + 2, \dots, n\,,$$

whenever $n = 2^k$ is sufficiently large, it is easy to argue that

$$K_n^* \ge \frac{n}{2} - (4m_n + 2) \tag{14}$$

for all sufficiently large $n = 2^k$. On the other hand, using (13) and the information on $a_{j,\ell}$ and r_ℓ, we can deduce that

$$K_n^* \le (8m_n + 7)k^{m_n^2}(2m_n + 1)^{m_n^2}(4m_n + 1)\,. \tag{15}$$

for all sufficiently large $n = 2^k$. Combining (14) and (15), we obtain that

$$(4m_n + 2) + (8m_n + 7)k^{m_n^2}(2m_n + 1)^{m_n^2}(4m_n + 1) \ge \frac{n}{2} = 2^{k-1}$$

for all sufficiently large $n = 2^k$. Hence

$$\log(16m_n + 14) + m_n^2 \log k + m_n^2 \log(2m_n + 1) + \log(4m_n + 1) \ge k - 1\,,$$

which implies

$$m_n \ge \left(\frac{ck}{\log k}\right)^{1/2}$$

for every sufficiently large $n = 2^k$ with an absolute constant $c > 0$. The theorem is now proved for all sufficiently large $n = 2^k$, from which the general case follows by the monotonicity of the sequence (m_n). $\square$

Finally we extend the result of Theorem 6 by the following theorem.

Theorem 11. *Let (m_n) be an increasing sequence of positive numbers. Suppose that for every sufficiently large n there is a multivariate polynomial P_n with integer coefficients mapping the "cube" with real arguments from $[-2,2]$ into $[-m_n, m_n]$ so that the number of points with integer coordinates where P_n takes positive values is n. Then*

$$m_n \ge \left(\frac{c \log n}{\log \log n}\right)^{1/2}$$

for every sufficiently large n with an absolute constant $c > 0$.

Proof of Theorem 11. Suppose P_n is a polynomial with exactly d variables and with integer coefficients mapping the "cube" $I_d(2) := [-2,2]^d$ into $[-m_n, m_n]$, and the number of points with integer coordinates where P_n takes positive values is n. Then, by observation (iv) in the proof of Lemma 1, we have

$$2d \le \max_{(x_1, x_2, \dots, x_d) \in I_d(2)} |P_n(x_1, x_2, \dots, x_d)|^2 \le m_n^2\,.$$

Assume now that $P_n(x_1, x_2, \dots, x_d) > 0$. By fixing $d-1$ integer variables and using Theorem 4, we obtain that the remaining variable must be in

$$[-(4m_n+3), 4m_n+3]\,.$$

Therefore all the variables $x_1, x_2, \dots, x_d$ must come from the above interval. Hence,

$$(8m_n+7)^{m_n^2/2} \geq (8m_n+7)^d \geq n\,,$$

that is,

$$\frac{1}{2}m_n^2 \log(8m_n+7) \geq \log n\,,$$

and

$$m_n \geq \left(\frac{c\log n}{\log\log n}\right)^{1/2}$$

follows for every sufficiently large n with an absolute constant $c > 0$. $\square$

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