THE ASYMPTOTIC VALUE OF THE MAHLER MEASURE OF THE RUDIN-SHAPIRO POLYNOMIALS

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Abstract. In signal processing the Rudin-Shapiro polynomials have good autocorrelation properties and their values on the unit circle are small. Binary sequences with low autocorrelation coefficients are of interest in radar, sonar, and communication systems. In this paper we show that the Mahler measure of the Rudin-Shapiro polynomials of degree $n-1$ with $n=2^k$ is asymptotically $(2n/e)^{1/2}$, as it was conjectured by B. Saffari in 1985. Our approach is based heavily on the Saffari and Montgomery conjectures proved recently by B. Rodgers.

1. Introduction and Notation

Let $D := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk of the complex plane. Let $\partial D := \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle of the complex plane. The Mahler measure $M_0(f)$ is defined for bounded measurable functions f on ∂D by

$$M_0(f) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{it})|\,dt\right).$$

It is well known, see [HL-52], for instance, that

$$M_0(f) = \lim_{q\to 0+} M_q(f)\,,$$

where

$$M_q(f) := \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(e^{it})\right|^q\,dt\right)^{1/q}, \qquad q>0\,.$$

It is also well known that for a function f continuous on ∂D we have

$$M_\infty(f) := \max_{t\in[0,2\pi]} |f(e^{it})| = \lim_{q\to\infty} M_q(f)\,.$$

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It is a simple consequence of the Jensen formula that

$$M_0(f) = |c| \prod_{j=1}^{n} \max\{1, |z_j|\}$$

for every polynomial of the form

$$f(z) = c \prod_{j=1}^{n} (z - z_j)\,, \qquad c, z_j \in \mathbb{C}\,.$$

See [BE-95, p. 271] or [B-02, p. 3], for instance.

Let $\mathcal{P}_n^c$ be the set of all algebraic polynomials of degree at most n with complex coefficients. Let $\mathcal{T}_n$ be the set of all real (that is, real-valued on the real line) trigonometric polynomials of degree at most n. Finding polynomials with suitably restricted coefficients and maximal Mahler measure has interested many authors. The classes

$$\mathcal{L}_n := \left\{ f: \;\; f(z) = \sum_{j=0}^{n} a_j z^j\,, \quad a_j \in \{-1, 1\} \right\}$$

of Littlewood polynomials and the classes

$$\mathcal{K}_n := \left\{ f: \;\; f(z) = \sum_{j=0}^{n} a_j z^j\,, \quad a_j \in \mathbb{C}, \;\; |a_j| = 1 \right\}$$

of unimodular polynomials are two of the most important classes considered. Observe that $\mathcal{L}_n \subset \mathcal{K}_n$ and

$$M_0(f) \le M_2(f) = \sqrt{n+1}$$

for every $f \in \mathcal{K}_n$. Beller and Newman [BN-73] constructed unimodular polynomials $f_n \in \mathcal{K}_n$ whose Mahler measure $M_0(f_n)$ is at least $\sqrt{n} - c/\log n$.

Section 4 of [B-02] is devoted to the study of Rudin-Shapiro polynomials. Littlewood asked if there were polynomials $f_{n_k} \in \mathcal{L}_{n_k}$ satisfying

$$c_1\sqrt{n_k+1} \le |f_{n_k}(z)| \le c_2\sqrt{n_k+1}\,, \qquad z \in \partial D\,,$$

with some absolute constants $c_1 > 0$ and $c_2 > 0$, see [B-02, p. 27] for a reference to this problem of Littlewood. To satisfy just the lower bound, by itself, seems very hard, and no such sequence (f_{n_k}) of Littlewood polynomials $f_{n_k} \in \mathcal{L}_{n_k}$ is known. A sequence of Littlewood polynomials that satisfies just the upper bound is given by the Rudin-Shapiro polynomials. The Rudin-Shapiro polynomials appear in Harold Shapiro's 1951 thesis [S-51] at MIT and are sometimes called just Shapiro polynomials. They also arise independently in Golay's paper [G-51]. They are remarkably simple to construct and are a rich source of counterexamples to possible conjectures.

The Rudin-Shapiro polynomials are defined recursively as follows:

$$P_0(z) := 1\,, \qquad Q_0(z) := 1\,,$$
$$P_{k+1}(z) := P_k(z) + z^{2^k} Q_k(z)\,,$$
$$Q_{k+1}(z) := P_k(z) - z^{2^k} Q_k(z)\,,$$

for $k = 0, 1, 2, \ldots$. Note that both P_k and Q_k are polynomials of degree $n-1$ with $n := 2^k$ having each of their coefficients in $\{-1, 1\}$. In signal processing, the Rudin-Shapiro polynomials have good autocorrelation properties and their values on the unit circle are small. Binary sequences with low autocorrelation coefficients are of interest in radar, sonar, and communication systems.

It is well known and easy to check by using the parallelogram law that

$$|P_{k+1}(z)|^2 + |Q_{k+1}(z)|^2 = 2(|P_k(z)|^2 + |Q_k(z)|^2)\,, \qquad z \in \partial D\,.$$

Hence

$$|P_k(z)|^2 + |Q_k(z)|^2 = 2^{k+1} = 2n\,, \qquad z \in \partial D\,. \tag{1.1}$$

It is also well known (see Section 4 of [B-02], for instance), that

$$|Q_k(z)| = |P_k(-z)|\,, \qquad z \in \partial D\,. \tag{1.2}$$

P. Borwein's book [B-02] presents a few more basic results on the Rudin-Shapiro polynomials. Various properties of the Rudin-Shapiro polynomials are discussed in [B-73] and [BL-76]. Obviously $M_2(P_k) = 2^{k/2}$ by the Parseval formula. In 1968 Littlewood [L-68] evaluated $M_4(P_k)$ and found that $M_4(P_k) \sim (4^{k+1}/3)^{1/4}$. The M_4 norm of Rudin-Shapiro like polynomials on ∂D are studied in [BM-00]. P. Borwein and Lockhart [BL-01] investigated the asymptotic behavior of the mean value of the M_q norms of Littlewood polynomials for arbitrary $q > 0$. They proved that

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} \frac{(M_q(f))^q}{n^{q/2}} = \Gamma\left(1 + \frac{q}{2}\right)\,.$$

In [C-15c] we proved that

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} \frac{M_q(f)}{n^{1/2}} = \left(\Gamma\left(1 + \frac{q}{2}\right)\right)^{1/q}$$

for every $q > 0$. In [CE-15c] we showed also that

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} \frac{M_0(f)}{n^{1/2}} = e^{-\gamma/2}\,,$$

where

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) = 0.577215\ldots$$

is the Euler constant and $e^{-\gamma/2} = 0.749306\ldots$. These are analogues of the results proved earlier by Choi and Mossinghoff [CM-11] for polynomials in $\mathcal{K}_n$.

Let $K := \mathbb{R} \pmod{2\pi}$. Let $m(A)$ denote the one-dimensional Lebesgue measure of $A \subset K$. In 1980 Saffari conjectured the following.

Conjecture 1.1. *Let* P_k *and* Q_k *be the Rudin-Shapiro polynomials of degree* $n-1$ *with* $n := 2^k$*. We have*

$$M_q(P_k) = M_q(Q_k) \sim \frac{2^{(k+1)/2}}{(q/2+1)^{1/q}}$$

for all real exponents $q > 0$*. Equivalently, we have*

$$\lim_{n\to\infty} m\left(\left\{t \in K : \left|\frac{P_k(e^{it})}{\sqrt{2^{k+1}}}\right|^2 \in [\alpha, \beta]\right\}\right)$$

$$= \lim_{n\to\infty} m\left(\left\{t \in K : \left|\frac{Q_k(e^{it})}{\sqrt{2^{k+1}}}\right|^2 \in [\alpha, \beta]\right\}\right) = 2\pi(\beta - \alpha)$$

whenever $0 \le \alpha < \beta \le 1$*.*

This conjecture was proved for all even values of $q \le 52$ by Doche [D-05] and Doche and Habsieger [DH-04]. Recently B. Rodgers [R-17] proved Saffari's Conjecture 1.1 for all $q > 0$. See also [EZ-17]. An extension of Saffari's conjecture is Montgomery's conjecture below.

Conjecture 1.2. *Let* P_k *and* Q_k *be the Rudin-Shapiro polynomials of degree* $n-1$ *with* $n := 2^k$*. We have*

$$\lim_{n\to\infty} m\left(\left\{t \in K : \frac{P_k(e^{it})}{\sqrt{2^{k+1}}} \in E\right\}\right)$$

$$= \lim_{n\to\infty} m\left(\left\{t \in K : \frac{Q_k(e^{it})}{\sqrt{2^{k+1}}} \in E\right\}\right) = 2m(E)$$

for any rectangle $E \subset D := \{z \in \mathbb{C} : |z| < 1\}$*.*

B. Rodgers [R-17] proved Montgomery's Conjecture 1.2 as well.

Despite the simplicity of their definitions not much is known about the Rudin-Shapiro polynomials. It has been shown in [E-16] fairly recently that the Mahler measure (M_0 norm) and the M_∞ norm of the Rudin-Shapiro polynomials P_k and Q_k of degree $n-1$ with $n := 2^k$ on the unit circle of the complex plane have the same size, that is, the Mahler measure of the Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$ is bounded from below by $cn^{1/2}$, where $c > 0$ is an absolute constant.

It is shown in this paper that the Mahler measure of the Rudin-Shapiro polynomials P_k and Q_k of degree $n-1 = 2^k - 1$ is asymptotically $(2n/e)^{1/2}$, as it was conjectured by B. Saffari in 1985. Note that $(2/e)^{1/2} = 0.85776388496\ldots$ is larger than $e^{-\gamma/2} = 0.749306\ldots$ in the average Mahler measure result for the class of Littlewood polynomials $\mathcal{L}_n$ we mentioned before.

2. NEW RESULT

Let P_k and Q_k be the Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$.

Theorem 2.1. *We have*

$$\lim_{n\to\infty} \frac{M_0(P_k)}{n^{1/2}} = \lim_{n\to\infty} \frac{M_0(Q_k)}{n^{1/2}} = \left(\frac{2}{e}\right)^{1/2}.$$

3. LEMMAS

Let $D(a,r) := \{z \in \mathbb{C} : |z-a| < r\}$ denote the open disk of the complex plane centered at $a \in \mathbb{C}$ of radius $r > 0$.

To prove our theorem we need some lemmas. Our first lemma states Jensen's formula. Its proof may be found in most of the complex analysis textbooks.

Lemma 3.1. *Suppose h is a nonnegative integer and*

$$f(z) = \sum_{k=h}^{\infty} c_k (z-z_0)^k\,, \qquad c_h \neq 0\,,$$

is analytic on the closure of the disk $D(z_0,r)$. Let $a_1, a_2, \dots, a_m$ denote the zeros of f in $D(z_0,r) \setminus \{z_0\}$, where each zero is listed as many times as its multiplicity. We have

$$\log|c_h| + h\log r + \sum_{k=1}^{m} \log \frac{r}{|a_k - z_0|} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(z_0 + re^{i\theta})|\,d\theta\,.$$

Lemma 3.2. *There exists a constant c_1 depending only on $c_2 > 0$ such that every polynomial $P \in \mathcal{P}_n^c$ has at most $c_1(nr+1)$ zeros in any open disk $D(z_0,r)$ with $z_0 \in \partial D$ and*

$$|P(z_0)| \geq c_2 M_\infty(P) \tag{3.1}$$

Proof of Lemma 3.2. Without loss of generality we may assume that $z_0 := 1$ and

$$n^{-1} \leq r \leq 1\,.$$

Indeed, the case $0 < r < n^{-1}$ follows from the case $r = n^{-1}$, and the case $r > 1$ is obvious. Let $P \in \mathcal{P}_n^c$ satisfy (3.1). A well known polynomial inequality observed by Bernstein states that

$$|P(\zeta)| \leq \max\{1, |\zeta|^n\} \max_{z \in \partial D} |P(z)| \tag{3.2}$$

for any polynomials $P \in \mathcal{P}_n^c$ and for any $\zeta \in \mathbb{C}$. This is a simple consequence of the Maximum Principle, see [BE-95, p. 239], for instance. Using (3.2) we can deduce that

$$\log|P(z)| \leq \log((1+2r)^n M_\infty(P)) \leq \log M_\infty(P) + 2nr\,, \qquad |z| \leq 1+2r\,. \tag{3.3}$$

Let m denote the number of zeros of P in the open disk $D(z_0,r)$. Using Lemma 3.1 with the disk $D(z_0,2r)$ and $h=0$, then using (3.1) and (3.3), we obtain

$$\log c_2 + \log M_\infty(P) + m\log 2 \leq \log|P(z_0)| + m\log 2 \leq \frac{1}{2\pi}\,2\pi(\log M_\infty + 2nr)\,.$$

This, together with $n^{-1} \leq r \leq 1$, implies $\log c_2 + m\log 2 \leq 2nr$, and the lemma follows. □

Our next lemma is stated as Lemma 3.5 in [E-16], where its proof may also be found.

Lemma 3.3. *If P_k and Q_k are the k-th Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$, $\delta := \sin^2(\pi/8)$, and*

$$z_j := e^{it_j}\,, \quad t_j := \frac{2\pi j}{n}\,, \qquad j \in \mathbb{Z}\,,$$

then

$$\max\{|P_k(z_j)|^2, |P_k(z_{j+1})|^2\} \geq \delta 2^{k+1} = 2\delta n \qquad j \in \mathbb{Z}\,.$$

By Lemma 3.3, for every $n = 2^k$ there are

$$0 \leq \tau_1 < \tau_2 < \cdots < \tau_m < \tau_{m+1} := \tau_1 + 2\pi$$

such that

$$\tau_j - \tau_{j-1} = \frac{2\pi l}{n}\,, \qquad l \in \{1, 2\}\,,$$

and with

$$a_j := e^{i\tau_j}, \qquad j = 1, 2, \ldots, m+1, \tag{3.4}$$

we have

$$|P_k(a_j)|^2 \geq 2\delta n\,, \qquad j = 1, 2, \ldots, m+1\,. \tag{3.5}$$

(Moreover, each a_j is an n-th root of unity.) For the sake of brevity let $R_n \in \mathcal{T}_n$ be defined by

$$R_n(t) := |P_k(e^{it})|^2, \qquad n = 2^k\,.$$

Using the above notation we formulate the following observation.

Lemma 3.4. *There is an absolute constant $c_3 > 0$ such that*

$$\mu := \left|\left\{j \in \{2, 3, \ldots, m+1\} : \min_{t \in [\tau_{j-1}, \tau_j]} R_n(t) \leq \varepsilon n\right\}\right| \leq c_3 n \varepsilon^{1/2}$$

for every sufficiently large $n = 2^k \geq n_\varepsilon$, $k = 1, 2, \ldots,$ and $\varepsilon > 0$.

To prove Lemma 3.4 we need a consequence of the so-called Bernstein-Szegő inequality formulated by our next lemma. For its proof see [BE-95, p. 232], for instance.

Lemma 3.5. *We have*

$$S'(t)^2 + n^2 S(t)^2 \leq n^2 \max_{\tau \in \mathbb{R}} S(\tau)^2$$

for every $S \in \mathcal{T}_n$.

Lemma 3.6. *We have*

$$|R_n'(t)| \le n^{3/2}\sqrt{2R_n(t)}\,, \qquad t \in \mathbb{R}\,.$$

Proof of Lemma 3.6. Let $S \in \mathcal{T}_n$ be defined by $S(t) := |P_k(e^{it})|^2 - n = R_n(t) - n$. Observe that (1.1) implies that

$$\max_{\tau \in \mathbb{R}} |S(\tau)| \le n\,.$$

Combining this with Lemma 3.5 implies that

$$\begin{aligned}|R_n'(t)| = |S'(t)| &\le n\sqrt{n^2 - S(t)^2} = \sqrt{n^2 - (R_n(t)-n)^2} \le n\sqrt{2nR_n(t) - R_n(t)^2)} \\ &\le n\sqrt{2nR_n(t)}\,.\end{aligned}$$

□

Now we are ready to prove Lemma 3.4.

Proof of Lemma 3.4. Let $j \in \{2, 3, \dots, m+1\}$ be such that

$$\min_{t \in [\tau_{j-1}, \tau_j]} R_n(t) \le \varepsilon n\,.$$

Using the notation of Lemma 3.3, without loss of generality we may assume that $0 < \varepsilon < \delta$. By recalling (3.4) and (3.5) there are $\tau_{j-1} \le \alpha_j < \beta_j \le \tau_j$ such that

$$R_n(\alpha_j) = \varepsilon n\,, \qquad R_n(\beta_j) = 2\varepsilon n\,,$$

and

$$R_n(t) \le 2\varepsilon n\,, \qquad t \in [\alpha_j, \beta_j]\,.$$

Then, by the Mean Value Theorem there is $\xi_j \in (\alpha_j, \beta_j)$ such that

$$\varepsilon n = R_n(\beta_j) - R_n(\alpha_j) = (\beta_j - \alpha_j) R_n'(\xi_j)\,,$$

and hence by Lemma 3.6 we obtain

$$\begin{aligned}\varepsilon n = (\beta_j - \alpha_j) R_n'(\xi_j) &\le (\beta_j - \alpha_j) n^{3/2} \sqrt{2R_n(\xi_j)} \\ &\le (\beta_j - \alpha_j) n^{3/2} \sqrt{4\varepsilon n}\,,\end{aligned}$$

that is,

$$\beta_j - \alpha_j \ge \frac{\varepsilon^{1/2}}{2n}\,.$$

Hence, on one hand,

$$m\left(\left\{t \in K : \frac{R_n(t)}{n} \in [0, 2\varepsilon]\right\}\right) = m\left(\left\{t \in K : \left|\frac{P_k(e^{it})}{\sqrt{2^{k+1}}}\right|^2 \in [0, \varepsilon]\right\}\right) \ge \frac{\mu \varepsilon^{1/2}}{2n}\,,$$

where μ is defined in the statement of the lemma. On the other hand, by Conjecture 1.1 proved by B. Rodgers there is an absolute constant $c_3/2 > 0$ such that

$$m\left(\left\{t \in K : \frac{R_n(t)}{n} \in [0, 2\varepsilon]\right\}\right) = m\left(\left\{t \in K : \left|\frac{P_k(e^{it})}{\sqrt{2^{k+1}}}\right|^2 \in [0, \varepsilon]\right\}\right) \leq (c_3/2)\varepsilon$$

for every sufficiently large $n \geq n_\varepsilon$. Combining the last two inequalities we obtain

$$\mu \leq c_3 n \varepsilon^{1/2}$$

for every sufficiently large $n \geq n_\varepsilon$. □

We introduce the notation

$$A_{n,\varepsilon} := \left\{t \in K : \frac{R_n(t)}{2n} \geq \varepsilon\right\}$$

and

$$B_{n,\varepsilon} := K \setminus A_{n,\varepsilon} = \left\{t \in K : \frac{R_n(t)}{2n} < \varepsilon\right\}.$$

Our next lemma is an immediate consequence of Conjecture 1.1 proved by B. Rodgers.

Lemma 3.7. *Let $\varepsilon \in (0, 1)$ be fixed. We have*

$$\lim_{n \to \infty} \frac{1}{2\pi} \int_{A_{n,\varepsilon}} \log \frac{R_n(t)}{2n}\, dt = \int_\varepsilon^1 \log x\, dx.$$

Proof of Lemma 3.7. Let

$$F_\varepsilon(x) := \begin{cases} \log x, & \text{if} \quad x \in [\varepsilon, 1], \\ 0, & \text{if} \quad x \in [0, \varepsilon). \end{cases}$$

By using the Weierstrass Approximation Theorem, it is easy to see that F_ε can be approximated by polynomials in $L_1[0, 1]$ norm, and hence the lemma follows from Conjecture 1.1 proved by B. Rodgers in a standard fashion. We omit the details of this routine argument. □

The above lemma will be coupled with the following inequality.

Lemma 3.8. *Let $\varepsilon \in (0, 1)$ be fixed. There is an absolute constant $c_4 > 0$ such that*

$$\frac{1}{2\pi} \int_{B_{n,\varepsilon}} \log \frac{R_n(t)}{2n}\, dt \geq -c_4 \varepsilon^{1/2}$$

for every sufficiently large $n \geq n_\varepsilon$.

To prove Lemma 3.8 we need a few other lemmas.

Lemma 3.9. *Let f be a twice differentiable function on $[a,b]$. There is a $\xi \in [a,b]$ such that*

$$\int_a^b f(t)\,dt - \frac{1}{2}(f(a)+f(b))(b-a) = \frac{-(b-a)^3}{12} f''(\xi)\,.$$

This is the formula for the error term in the trapezoid rule. Its proof may be found in various calculus textbooks discussing numerical integration.

Let $w_j \in \mathbb{C}$, $j = 1, 2, \dots, n-1$, denote the zeros of P_k. So we have

$$\log \frac{R_n(t)}{2n} = \log \frac{|P_k(e^{it})|^2}{n} = \sum_{j=1}^{n-1} \log |e^{it} - w_j| - \log n\,.$$

It is a simple well known fact that $P_k \in \mathcal{L}_{n-1}$ implies that

$$1/2 \leq |w_j| \leq 2, \qquad j = 1, 2, \dots, n-1\,.$$

Associated with $w \in \mathbb{C}$ we introduce $\phi \in [0, 2\pi)$ uniquely defined by $w = |w|e^{i\phi}$. For the sake of brevity let

$$g_w(t) := \log |e^{it} - w| = \log |e^{it} - |w|e^{i\phi}|\,.$$

Simple calculations show that

$$g_w(t) = \frac{1}{2} \log(1 + |w|^2 - 2|w|\cos(t - \phi))\,,$$

$$g_w'(t) = \frac{|w|\sin(t-\phi)}{|e^{it} - w|^2}\,,$$

and

$$g_w''(t) = \frac{|w|\cos(t-\phi)}{|e^{it}-w|^4} - \frac{2|w|^2\sin^2(t-\phi)}{|e^{it}-w|^4}\,.$$

The inequality of the following lemma is immediate.

Lemma 3.10. *There is an absolute constant $c_5 > 0$ such that*

$$|g_w''(t)| \leq \frac{c_5}{|e^{it} - w|^2}$$

for every $t \in \mathbb{R}$ and $w \in \mathbb{C}$ with $|w| \leq 2$.

Combining Lemmas 3.9 and 3.10 we get the following.

Lemma 3.11. *Let $a_j = e^{i\tau_j}, j = 1, 2, \dots m+1$, be as before (defined after Lemma 3.3). There are $\xi_j \in [\tau_{j-1}, \tau_j]$ and an absolute constant $c_6 > 0$ such that*

$$\int_{\tau_{j-1}}^{\tau_j} g_w(t)\,dt - \frac{1}{2}(g_w(\tau_j) + g_w(\tau_{j-1}))(\tau_j - \tau_{j-1}) \geq \frac{-c_6}{n^3 |e^{i\xi_j} - w|^2}$$

for every $t \in \mathbb{R}$ and $w \in \mathbb{C}$ with $|w| \leq 2$.

We will also need an estimate better than the one given in Lemma 3.11 in the case when $w \in \mathbb{C}$ is close to $a_j := e^{i\tau_j}$.

Lemma 3.12. *Let $a_j = e^{i\tau_j}, j = 2, 3, \ldots, m+1$, be as before (defined after Lemma 3.3). There is an absolute constant $c_7 > 0$ such that*

$$\int_{\tau_{j-1}}^{\tau_j} g_w(t)\,dt - \frac{1}{2}(g_w(\tau_j) + g_w(\tau_{j-1}))(\tau_j - \tau_{j-1}) \geq \frac{-c_7}{n}$$

for every $t \in \mathbb{R}$ and $w \in \mathbb{C}$ such that $|w - a_j| \leq 8\pi/n$.

To prove Lemma 3.12 we need the following observation.

Lemma 3.13. *Let $a_j = e^{i\tau_j}, j = 1, 2, \ldots m+1$, be as before (defined after Lemma 3.3). There is an absolute constant $c_8 > 0$ such that $|a_j - w| \geq c_8/n$ for every $w \in \mathbb{C}$ for which $P(w) = 0$.*

Proof of Lemma 3.13. The proof is a routine combination of (1.1), Lemma 3.3, and a couple of Bernstein's inequalities. One of Bernstein's polynomial inequalities asserts that

$$\max_{z \in \partial D} |P'(z)| \leq n \max_{z \in \partial D} |P(z)| \tag{3.6}$$

for any polynomials $P \in \mathcal{P}_n^c$. See [BE-95, p. 232], for instance. Another polynomial inequality of Bernstein we need in this proof is (3.2).

Suppose $w \in \mathbb{C}$, $|w - a_j| \leq c/n$, and $P(w) = 0$. Let Γ be the line segment connecting $a_j = e^{i\tau_j}$ and w. We have

$$\begin{aligned} (2\delta)^{1/2} n^{1/2} \leq & |P_k(a_j)| = |P_k(a_j) - P_k(w)| = \left| \int_\Gamma P_k'(z)\,dz \right| \\ \leq & \int_\Gamma |P_k'(z)|\,|dz|\,. \end{aligned}$$

Hence there is a $\zeta \in \Gamma$ such that

$$|P_k'(\zeta)| \cdot |a_j - w| \geq (2\delta)^{1/2} n^{1/2}\,.$$

Combining this with (3.6) and $|\zeta - a_j| \leq |w - a_j| \leq c/n$, we obtain

$$|P_k'(\zeta)| \geq \frac{(2\delta)^{1/2} n^{1/2}}{c/n} = \frac{(2\delta)^{1/2}}{c} n^{3/2}\,. \tag{3.7}$$

On the other hand, combining (3.2), (3.6), and (1.1), we obtain

$$\begin{aligned} |P_k'(\zeta)| &\leq \left(\max\{1, |\zeta|^{n-1}\}\right) \left(\max_{z \in \partial D} |P_k'(z)|\right) \\ &\leq \left(\max\{1, |\zeta|^{n-1}\}\right) \left(n \max_{z \in \partial D} |P_k(z)|\right) \leq \left(1 + \frac{c}{n}\right)^n n(2n)^{1/2} \\ &\leq e^c \sqrt{2} n^{3/2} \end{aligned}$$

Combining this with (3.7), we get $\delta^{1/2} \le ce^c$. $\square$

Proof of Lemma 3.12. Observe that Lemma 3.13 implies that there is an absolute constant $c_8 > 0$ such that

$$\frac{1}{2}(\log|e^{i\tau_j} - w| + \log|e^{i\tau_{j-1}} - w|) \le \log(c_8/n) = \log c_8 - \log n\,. \tag{3.8}$$

Now we show that there is an absolute constant $c_9 > 0$ such that

$$\int_{\tau_{j-1}}^{\tau_j} \log|e^{it} - w|\,dt \ge (\tau_j - \tau_{j-1})(-c_9 - \log n)\,. \tag{3.9}$$

To see this let $w = |w|e^{i\phi}$. We have

$$|e^{it} - w| = |e^{it} - |w|e^{i\phi}| \ge |e^{it} - e^{i\phi}| = 2\sin\left|\frac{t-\phi}{2}\right| \ge 2\,\frac{2}{\pi}\,\frac{|t-\phi|}{2} = \frac{2}{\pi}\,|t-\phi|$$

whenever $|t - \phi| \le \pi$. Hence, if

$$\phi \in \left[\tau_{j-1} + \frac{c_8}{2n}, \tau_j - \frac{c_8}{2n}\right]\,,$$

then

$$\begin{aligned}\int_{\tau_{j-1}}^{\tau_j} \log|e^{it} - w|\,dt &\ge \int_{\tau_{j-1}}^{\tau_j} \log\left(\frac{2}{\pi}\,|t-\phi|\right)\,dt \\ &= \int_{\tau_{j-1}}^{\phi} \log\left(\frac{2}{\pi}\,(\phi - t)\right)\,dt + \int_{\phi}^{\tau_j} \log\left(\frac{2}{\pi}\,(t-\phi)\right)\,dt \\ &\ge (\tau_j - \tau_{j-1})(-c_9 - \log n)\end{aligned}$$

with an absolute constant $c_9 > 0$, and (3.9) follows. While, if

$$\phi \notin \left[\tau_{j-1} + \frac{c_8}{2n}, \tau_j - \frac{c_8}{2n}\right]\,,$$

then Lemma 3.13 implies that there is an absolute constant $c_{10} > 0$ such that

$$\min_{t\in[\tau_{j-1},\tau_j]} |e^{it} - w| \ge c_{10}/n\,,$$

and hence

$$\begin{aligned}\int_{\tau_{j-1}}^{\tau_j} \log|e^{it} - w|\,dt &\ge \int_{\tau_{j-1}}^{\tau_j} \log(c_{10}/n)\,dt \\ &\ge (\tau_j - \tau_{j-1})(-c_{11} - \log n)\end{aligned}$$

with an absolute constant $c_{11} > 0$, and (3.9) follows again. Combining (3.8) and (3.9) and recalling that $g_w(t) := \log|e^{it} - w|$ and $\tau_j - \tau_{j-1} \le 4\pi/n$, we obtain the inequality of the lemma. $\square$

Lemma 3.14. *There is an absolute constant* $c_{12} > 0$ *such that*

$$\int_{\tau_{j-1}}^{\tau_j} \log\left(\frac{R_n(t)}{n}\right) \geq \frac{-c_{12}}{n}$$

for every $j \in \{2, 3, \ldots, m+1\}$.

Proof of Lemma 3.14. Let, as before, $w_\nu, \nu = 1, 2, \ldots, n-1$, denote the zeros of P_k. Recall that $|w_\nu| \leq 2$ for each $\nu = 1, 2, \ldots, n-1$. We define the annuli

$$E_{j,q} := D(a_j, 2^{q+3}\pi/n) \setminus D(a_j, 2^{q+2}\pi/n)\,, \qquad q = 1, 2, \ldots\,,$$

and the disk

$$E_{j,0} := D(a_j, 8\pi/n)\,.$$

Observe that the sets $E_{j,q}$ are pairwise disjoint and

$$\mathbb{C} = \bigcup_{j=0}^{\infty} E_{j,q}\,.$$

By Lemmas 3.2 and 3.3 there is an absolute constant $c_1 > 0$ (depending only on the explicitly given value of δ) such that $E_{j,q}$ contains at most $c_1(n2^{q+3}\pi/n + 1)$ zeros of P_k and $E_{j,0}$ contains at most $c_1(8\pi + 1)$ zeros of P_k. Hence Lemmas 3.11 and Lemma 3.12 give that

$$\begin{aligned}
&\int_{\tau_{j-1}}^{\tau_j} \log(R_n(t))\,dt - \frac{1}{2}(\log(R_n(\tau_j)) + \log(R_n(\tau_{j-1})))(\tau_j - \tau_{j-1}) \\
=&\sum_{\nu=1}^{n-1}\left(\int_{\tau_{j-1}}^{\tau_j} g_{w_\nu}(t)\,dt - \frac{1}{2}(g_{w_\nu}(\tau_j) + g_{w_\nu}(\tau_{j-1}))(\tau_j - \tau_{j-1})\right) \\
=&\sum_{q=0}^{\infty}\sum_{w_\nu \in E_q}\left(\int_{\tau_{j-1}}^{\tau_j} g_{w_\nu}(t)\,dt - \frac{1}{2}(g_{w_\nu}(\tau_j) + g_{w_\nu}(\tau_{j-1}))(\tau_j - \tau_{j-1})\right) \\
=&\sum_{q=0}^{0} + \sum_{q=1}^{\infty} \geq c_1(8\pi+1)\frac{-c_7}{n} + \sum_{q=1}^{\infty} c_1(n(2^{q+3}\pi/n) + 1)\frac{-c_6}{n^3(2^{q+2}/n)^2} \\
\geq& c_1(8\pi+1)\frac{-c_7}{n} - \sum_{q=1}^{\infty}\frac{c_1 c_6}{2^q n} \\
\geq& -c_{13}/n
\end{aligned}$$

with an absolute constant $c_{13} > 0$. Now recall that $R_n(\tau_{j-1}) \geq 2\delta n$ and $R_n(\tau_j) \geq 2\delta n$, and the result follows. □

Now we are ready to prove Lemma 3.8.

Proof of Lemma 3.8. Given $\varepsilon \in (0,1)$, let

$$I_{n,\varepsilon} := \left\{ j \in \{2,3,\dots,m+1\} : \min_{t \in [\tau_{j-1},\tau_j]} R_n(t) < \varepsilon \right\},$$

and let

$$J_{n,\varepsilon} := \bigcup_{j \in I_{n,\varepsilon}} [\tau_{j-1},\tau_j]\,.$$

Using that $0 \le R_n(t) \le 2n$ for every $t \in K$, and then using Lemmas 3.4 and 3.14, we get

$$\int_{B_{n,\varepsilon}} \log \frac{R_n(t)}{2n}\,dt \ge \int_{J_{n,\varepsilon}} \log \frac{R_n(t)}{2n}\,dt = \sum_{j \in I_{n,\varepsilon}} \int_{\tau_{j-1}}^{\tau_j} \log \frac{R_n(t)}{2n}\,dt$$
$$\ge c_3 n \varepsilon^{1/2}(-c_{12}/n) \ge -c_4\varepsilon^{1/2}$$

for every sufficiently large $n \ge n_\varepsilon$, where $c_4 = c_3c_{12} > 0$, and the lemma is proved. $\square$

4. PROOF OF THE THEOREM

Proof of Theorem 2.1. It follows from (1.2) immediately that

$$\lim_{n\to\infty} \frac{M_0(P_k)}{n^{1/2}} = \lim_{n\to\infty} \frac{M_0(Q_n)}{n^{1/2}}\,,$$

so it is sufficient to prove the asymptotic formula only for $M_0(P_k)$.

Let $\varepsilon \in (0,1)$ be fixed. By Lemma 3.7 we have

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{A_{n,\varepsilon}} \log \frac{R_n(t)}{2n}\,dt = \int_\varepsilon^1 \log x\,dx\,.$$

while it follows from Lemma 3.8 and the inequalities $0 \le R_n(t) \le 2n$ that there is an absolute constant $c_4 > 0$ such that

$$-c_4\varepsilon^{1/2} \le \frac{1}{2\pi} \int_{B_{n,\varepsilon}} \log \frac{R_n(t)}{2n}\,dt \le 0$$

for every sufficiently large $n \ge n_\varepsilon$. As K is the disjoint union of $A_{n,\varepsilon}$ and $B_{n,\varepsilon}$, we have

$$\limsup_{n\to\infty} \frac{1}{2\pi} \int_K \log \frac{R_n(t)}{2n}\,dt \le \int_\varepsilon^1 \log x\,dx \tag{4.1}$$

and

$$\liminf_{n\to\infty} \frac{1}{2\pi} \int_K \log \frac{R_n(t)}{2n}\,dt \ge \int_\varepsilon^1 \log x\,dx - c_4\varepsilon^{1/2}\,. \tag{4.2}$$

As (4.1) and (4.2) hold for an arbitrary $\varepsilon \in (0,1)$, it follows that

$$\int_0^1 \log x\, dx \le \liminf_{n\to\infty} \frac{1}{2\pi} \int_K \log \frac{R_n(t)}{2n}\, dt \le \limsup_{n\to\infty} \frac{1}{2\pi} \int_K \log \frac{R_n(t)}{2n}\, dt \le \int_0^1 \log x\, dx\,,$$

and hence

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_K \log \frac{R_n(t)}{2n}\, dt = -1\,.$$

Hence, recalling that

$$R_n(t) := |P_k(e^{it})|^2, \qquad t \in K\,,$$

we obtain

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{2\pi} \int_K \log \frac{|P_k(e^{it})|}{(2n)^{1/2}}\, dt &= \lim_{n\to\infty} \frac{1}{2\pi} \int_K \log \left(\frac{R_n(t)}{2n}\right)^{1/2} dt \\ &= \lim_{n\to\infty} \frac{1}{2\pi} \frac{1}{2} \int_K \log \frac{R_n(t)}{2n}\, dt = -1/2\,.\end{aligned}$$

Hence

$$\begin{aligned}\lim_{n\to\infty} \frac{M_0(P_k)}{(2n)^{1/2}} &= \lim_{n\to\infty} \exp\left(\frac{1}{2\pi} \int_K \log \frac{|P_k(e^{it})|}{(2n)^{1/2}}\, dt\right) \\ &= \exp\left(\lim_{n\to\infty} \frac{1}{2\pi} \int_K \log \frac{|P_k(e^{it})|}{(2n)^{1/2}}\, dt\right) = \exp(-1/2)\,,\end{aligned}$$

which is the asymptotic formula for $M_0(P_k)$ stated in the theorem. □

5. THE MAHLER MEASURE OF THE FEKETE POLYNOMIALS

For a prime p the p-th Fekete polynomial is defined as

$$f_p(z) := \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) z^k\,,$$

where

$$\left(\frac{k}{p}\right) = \begin{cases} 1, & \text{if } \ x^2 \equiv k \pmod p \ \text{ for an } \ x \not\equiv 0 \pmod p\,, \\ 0, & \text{if } \ p \text{ divides } k\,, \\ -1, & \text{otherwise} \end{cases}$$

is the usual Legendre symbol. Since f_p has constant coefficient 0, it is not a Littlewood polynomial, but g_p defined by $g_p(z) := f_p(z)/z$ is a Littlewood polynomial of degree $p-2$, and has the same Mahler measure as f_p. Fekete polynomials are examined in detail in [B-02], [CG-00], [E-11], [E-12], [EL-07], and [M-80]. In [CE-15a] and [CE-15b] the authors examined the maximal size of the Mahler measure and the L_q norms of sums of n monomials on the unit circle as well as on subarcs of the unit circles. In the constructions appearing in [CE-15a] properties of the Fekete polynomials f_p turned out to be quite useful. Montgomery [M-80] proved the following fundamental result.

Theorem 5.1. *There are absolute constants $c_{14} > 0$ and $c_{15} > 0$ such that*

$$c_{14}\sqrt{p}\log\log p \leq \max_{z \in \partial D} |f_p(z)| \leq c_{15}\sqrt{p}\log p\,.$$

In [E-07] we proved the following result.

Theorem 5.2. *For every $\varepsilon > 0$ there is a constant c_ε such that*

$$M_0(f_p) \geq \left(\frac{1}{2} - \varepsilon\right)\sqrt{p}$$

for all primes $p \geq c_\varepsilon$.

In [E-18] the factor $\left(\frac{1}{2} - \varepsilon\right)$ in Theorem 1.2 has been improved to to an absolute constant $c > 1/2$. Namely we prove the following.

Theorem 5.3. *There is an absolute constant $c > 1/2$ such that*

$$M_0(f_p) \geq c\sqrt{p}$$

for all sufficiently large primes.

The determine the asymptotic size of the Mahler measure $M_0(f_p)$ of the Fekete polynomials f_p seems to be beyond reach at the moment. Not even a (published or unpublished) conjecture seems to be known.

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