SIEVE-TYPE LOWER BOUNDS FOR THE MAHLER MEASURE OF POLYNOMIALS ON SUBARCS

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Abstract. We prove sieve-type lower bounds for the Mahler measure of polynomials on subarcs of the unit circle of the complex plane. This is then applied to give an essentially sharp lower bound for the Mahler measure of the Fekete polynomials on subarcs.

1. Introduction

The large sieve of number theory [M-84] asserts that if

$$P(z) = \sum_{k=-n}^{n} a_k z^k$$

is a trigonometric polynomial of degree at most n,

$$0 \le t_1 < t_2 < \cdots < t_m \le 2\pi\,,$$

and

$$\delta := \min\left\{t_2 - t_1, t_3 - t_2, \dots, t_m - t_{m-1}, 2\pi - (t_m - t_1)\right\}\,,$$

then

$$\sum_{j=1}^{m} \left|P\left(e^{it_j}\right)\right|^2 \le \left(\frac{n}{2\pi} + \delta^{-1}\right) \int_0^{2\pi} \left|P\left(e^{it}\right)\right|^2\, dt\,.$$

There are numerous extensions of this to L_p norm (or involving $\psi\left(\left|P\left(e^{it}\right)\right|^p\right)$, where ψ is a convex function), $p > 0$, and even to subarcs. See [LMN-87] and [GLN-01]. There are versions of this that estimate Riemann sums, for example, with $t_0 := t_m - 2\pi$,

$$\sum_{j=1}^{m} \left|P\left(e^{it_j}\right)\right|^2 (t_j - t_{j-1}) \le C \int_0^{2\pi} \left|P\left(e^{it}\right)\right|^2\, dt\,,$$

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with a constant C independent of n, P, and $\{t_1, t_2, \dots, t_m\}$. These are often called forward Marcinkiewicz-Zygmund inequalities. Converse Marcinkiewicz-Zygmund inequalities provide estimates for the integrals above in terms of the sums on the left-hand side, see [L-98], [MR-99], [ZZ-95], [KL-04]. A particularly interesting case is that of the L_0 norm. A result in [EL-07] asserts that if $\{z_1, z_2, \dots, z_n\}$ are the n-th roots of unity, and P is a polynomial of degree at most n, then

$$\prod_{j=1}^{n} |P(z_j)|^{1/n} \le 2M_0(P)\,, \tag{1.1}$$

where

$$M_0(P) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|P(e^{it})|\,dt\right)$$

is the Mahler measure of P. In [EL-07] we were focusing on showing that methods of subharmonic function theory provide a simple and direct way to generalize previous results. We also extended (1.1) to points other than the roots of unity and exponentials of logarithmic potentials of the form

$$P(z) = c\,\exp\left(\int \log|z-t|\,d\nu(t)\right)\,,$$

where $c \ge 0$ and ν is a positive Borel measure of compact support with $\nu(\mathbb{C}) \ge 0$. Inequalities for exponentials of logarithmic potentials and generalized polynomials were studied by several authors, see [ELS-94], [EMN-92], [BE-95], and [EL-07], for instance.

Let $\alpha < \beta$ be real numbers. The Mahler measure $M_0(Q, [\alpha, \beta])$ is defined for bounded measurable functions Q defined on $[\alpha, \beta]$ as

$$M_0(Q, [\alpha,\beta]) := \exp\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \log|Q(e^{it})|\,dt\right)\,.$$

It is well known that

$$M_0(Q, [\alpha,\beta]) = \lim_{p\to 0+} M_p(Q, [\alpha,\beta])\,,$$

where

$$M_p(Q, [\alpha,\beta]) := \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \left|Q(e^{it})\right|^p\,dt\right)^{1/p}\,, \qquad p > 0\,.$$

It is a simple consequence of the Jensen formula that

$$M_0(Q) := M_0(Q, [0, 2\pi]) = |c| \prod_{k=1}^{n} \max\{1, |z_k|\}$$

for every polynomial of the form

$$Q(z) = c\prod_{k=1}^{n}(z - z_k)\,, \qquad c, z_k \in \mathbb{C}\,.$$

Finding polynomials with suitably restricted coefficients and maximal Mahler measure has interested many authors. The classes

$$\mathcal{L}_n := \left\{ p : p(z) = \sum_{k=0}^{n} a_k z^k \,, \quad a_k \in \{-1, 1\} \right\}$$

of Littlewood polynomials and the classes

$$\mathcal{K}_n := \left\{ p : p(z) = \sum_{k=0}^{n} a_k z^k \,, \quad a_k \in \mathbb{C}, \ |a_k| = 1 \right\}$$

of unimodular polynomials are two of the most important classes considered. Beller and Newman [BN-73] constructed unimodular polynomials of degree n whose Mahler measure is at least $\sqrt{n} - c/\log n$. For a prime number p the p-th Fekete polynomial is defined as

$$f_p(z) := \sum_{k=1}^{p-1} \left(\frac{k}{p} \right) z^k \,,$$

where

$$\left(\frac{k}{p} \right) = \begin{cases} 1, & \text{if} \quad x^2 \equiv k \pmod{p} \ \text{ has a nonzero solution,} \\ 0, & \text{if} \quad p \ \text{divides} \ k \,, \\ -1, & \text{otherwise} \end{cases}$$

is the usual Legendre symbol. Since f_p has constant coefficient 0, it is not a Littlewood polynomial, but g_p defined by $g_p(z) := f_p(z)/z$ is a Littlewood polynomial, and has the same Mahler measure as f_p. Fekete polynomials are examined in detail in [B-02]. In [EL-07] we proved the following result.

Theorem 1.1. *For every $\varepsilon > 0$ there is a constant c_ε such that*

$$M_0(f_p, [0, 2\pi]) \geq \left(\frac{1}{2} - \varepsilon \right) \sqrt{p}$$

for all primes $p \geq c_\varepsilon$.

One of the key lemmas in the proof the above theorem formulates a remarkable property of the Fekete polynomials. A simple proof is given in [B-02, pp. 37-38].

Lemma 1.2 (Gauss). *We have*

$$|f_p(z_p^j)| = \sqrt{p} \,, \qquad j = 1, 2, \ldots, p-1 \,,$$

and $f_p(1) = 0$, where $z_p := \exp(2\pi i/p)$ is the first p-th root of unity.

The distribution of the zeros of Littlewood polynomials plays a key role in the study of the Mahler measure of Littlewood polynomials. There are many papers on the distribution of zeros of polynomials with constraints on their coefficients, see [ET-50], [BE-95], [BE-97], [B-97], [B-02], [BEK-99], and [E-08], for example. Results of this variety have been exploited in [EL-07] to obtain Theorem 1.1.

From Jensen's inequality,

$$M_0(f_p,[0,2\pi]) \le M_2(f_p,[0,2\pi]) = \sqrt{p-1}\,.$$

However, as it is observed in [EL-07], $1/2-\varepsilon$ in Theorem 1.1 cannot be replaced by $1-\varepsilon$. Indeed if p is prime of the form $p = 4m+1$, then the polynomial f_p is self-reciprocal, that is, $z^p f_p(1/z) = f_p(z)$, and hence

$$f_p(e^{2it}) = e^{ipt} \sum_{k=0}^{(p-3)/2} a_k \cos((2k+1)t), \qquad a_k \in \{-2,2\}\,.$$

A result of Littlewood [L-66] implies that

$$M_0(f_p,[0,2\pi]) \le \frac{1}{2\pi}\int_0^{2\pi} |f_p(e^{it})|\,dt = \frac{1}{2\pi}\int_0^{2\pi} |f_p(e^{2it})|\,dt \le (1-\varepsilon)\sqrt{p-1}\,,$$

for some absolute constant $\varepsilon > 0$. A similar argument shows that the same estimate holds when p is a prime of the form $p = 4m+3$. It is an interesting open question whether or not there is a sequence of Littlewood polynomials (f_n) such that

$$M_0(f_n,[0,2\pi]) \ge (1-\varepsilon)\sqrt{n}$$

for all $\varepsilon > 0$ and sufficiently large $n \ge N_\varepsilon$.

2. New Results

Let D denote the open unit circle of the complex plane. Let ∂D denote the unit circle. For a complex-valued function f defined on ∂D let

$$\|f\|_{\partial D} := \sup_{z\in\partial D} |f(z)|\,.$$

Theorem 2.1. *Let* $\omega_1 < \omega_2 \le \omega_1 + 2\pi$,

$$\omega_1 \le t_0 < t_1 < \cdots < t_m \le \omega_2\,,$$

$$t_{-1} := \omega_1 - (t_0 - \omega_1)\,, \qquad t_{m+1} := \omega_2 + (\omega_2 - t_m)\,,$$

$$\delta := \max\{t_0 - t_{-1}, t_1 - t_0, \dots, t_{m+1} - t_m\} \le \frac{1}{2}\sin\frac{\omega_2-\omega_1}{2}\,.$$

There is an absolute constant $c_1 > 0$ *such that*

$$\sum_{j=0}^{m} \frac{t_{j+1}-t_{j-1}}{2} \log|P(e^{it_j})| \le \int_{\omega_1}^{\omega_2} \log|P(e^{it})|\,dt + c_1 E(n,\delta,\omega_1,\omega_2)$$

for every polynomial P of the form

$$P(z) = \sum_{j=0}^{n} b_j z^j \,, \qquad b_j \in \mathbb{C}\,, \;\; b_0 b_n \neq 0\,,$$

where

$$E(n,\delta,\omega_1,\omega_2) := (\omega_2-\omega_1)n\delta + n\delta^2\log(1/\delta) + \sqrt{n\log R}\left(\delta\log(1/\delta) + \frac{\delta^2}{\omega_2-\omega_1}\right)$$

and $R := |b_0 b_n|^{-1/2}\|P\|_{\partial D}$.

Observe that R appearing in the above theorem can be easily estimated by

$$R \leq |b_0 b_n|^{-1/2}(|b_0| + |b_1| + \cdots + |b_n|)\,.$$

As a reasonably straightforward consequence of our sieve-type inequality above, the lower bound for the Mahler measure of Fekete polynomials below follows.

Theorem 2.2. *There is an absolute constant $c_2 > 0$ such that*

$$M_0(f_p,[\alpha,\beta]) \geq c_2\sqrt{p}$$

for all prime numbers p and for all $\alpha,\beta \in \mathbb{R}$ such that

$$\frac{4\pi}{p} \leq \frac{(\log p)^{3/2}}{p^{1/2}} \leq \beta - \alpha \leq 2\pi\,.$$

It looks plausible that Theorem 2.2 holds whenever $4\pi/p \leq \beta-\alpha \leq 2\pi$, but we do not seem to be able to handle the case $4\pi/p \leq \beta-\alpha \leq (\log p)^{3/2}p^{-1/2}$ in this paper.

We remark that Cauchy's inequality implies

$$\begin{aligned}
M_0(f_p,[\alpha,\beta]) \leq M_1(f_p,[\alpha,\beta]) &= \frac{1}{\beta-\alpha}\int_\alpha^\beta |f_p(e^{it})|\,dt \\
&= \frac{1}{\beta-\alpha}\int_\alpha^{\alpha+2\pi} |f_p(e^{it})|\,\chi_{[\alpha,\beta]}(e^{it})\,dt \\
&\leq \frac{1}{\beta-\alpha}\left(\int_\alpha^{\alpha+2\pi}|f_p(e^{it})|^2\,dt\right)^{1/2}\left(\int_\alpha^{\alpha+2\pi}|\chi_{[\alpha,\beta]}(e^{it})|^2\,dt\right)^{1/2} \\
&\leq \frac{1}{\beta-\alpha}\sqrt{2\pi}\sqrt{p-1}\sqrt{\beta-\alpha} \\
&= \left(\frac{2\pi}{\beta-\alpha}\right)^{1/2}\sqrt{p-1}
\end{aligned}$$

whenever $0 < \beta-\alpha \leq 2\pi$. However, it seems plausible that there is a constant $C(q,\varepsilon)$ depending only on $q>0$ and $\varepsilon>0$ such that

$$M_0(f_p,[\alpha,\beta]) \leq \left(\frac{1}{\beta-\alpha}\int_I |f_p(z)|^q\,|dz|\right)^{1/q} \leq C(q,\varepsilon)\sqrt{p}\,,$$

whenever $2p^{-1/2+\varepsilon} \leq \beta-\alpha \leq 2\pi$. We expect to prove this in a forthcoming paper.

3. Lemmas

To prove the theorems we need several lemmas. Our first three lemmas look quite similar to each other. Our Lemma 3.1 deals with a subdivision of the period in which case we can exploit the formula

$$\int_0^{2\pi} \log|e^{it} - a|\, dt = \log^+ |a| := \max\{\log|a|, 0\}\,, \qquad a \in \mathbb{C}\,.$$

In Lemma 3.2 we deal with a subdivision of a subinterval $[\omega_1, \omega_2]$ of the period. In this lemma the location of the zero $a = |a|e^{i\varphi} \in \mathbb{C}$ is special. The geometric implications of the assumptions on the location of a are exploited heavily in the proof of Lemma 3.2. In our Lemmas 3.1 – 3.4 below we use the notation

$$\delta := \max\{t_1 - t_0, t_2 - t_1, \dots, t_m - t_{m-1}\}\,.$$

Lemma 3.1. *Let*

$$t_1 < t_2 < \cdots < t_m < t_1 + 2\pi\,, \qquad t_0 := t_m - 2\pi\,, \quad t_{m+1} := t_1 + 2\pi\,.$$

Then

$$\sum_{j=1}^{m} \frac{t_{j+1} - t_{j-1}}{2} \log|e^{it_j} - a| \leq \int_0^{2\pi} \log|e^{it} - a|\, dt + 5\delta = \log^+ |a| + 5\delta \tag{3.1}$$

for every $a \in \mathbb{C}$. Here $\log^+ |a| := \max\{\log|a|, 0\}$.

Lemma 3.2. *Let $\omega_1 < \omega_2 \leq \omega_1 + 2\pi$,*

$$\omega_1 =: t_{-1} = t_0 < t_1 < \cdots < t_m = t_{m+1} := \omega_2\,.$$

Let $\Delta \leq \sin((\omega_2 - \omega_1)/2)$. Then

$$\left| \sum_{j=0}^{m} \frac{t_{j+1} - t_{j-1}}{2} \log|e^{it_j} - a| - \int_{\omega_1}^{\omega_2} \log|e^{it} - a|\, dt \right| \leq 5\delta^2 \Delta^{-1} \tag{3.2}$$

for every $a = |a|e^{i\varphi} \in \mathbb{C}$ satisfying

$$|e^{i\omega_1} - a| \geq \Delta \qquad \textit{and} \qquad |e^{i\omega_2} - a| \geq \Delta \tag{3.3}$$

and

$$\omega_2 - 2\pi \leq \varphi \leq \omega_1\,. \tag{3.4}$$

Our next lemma follows immediately from Lemmas 3.1 and 3.2. (Lemma 3.2 is applied with $\delta = \Delta$.)

Lemma 3.3. *Let* $\omega_1 < \omega_2 \le \omega_1 + 2\pi$,

$$\omega_1 =: t_{-1} = t_0 < t_1 < \cdots < t_m = t_{m+1} := \omega_2 \,.$$

Let $\delta \le \sin((\omega_2 - \omega_1)/2)$. *Then*

$$\sum_{j=0}^{m} \frac{t_{j+1} - t_{j-1}}{2} \log |e^{it_j} - a| \le \int_{\omega_1}^{\omega_2} \log |e^{it} - a| \, dt + 10\delta$$

for every $a \in \mathbb{C}$ *satisfying*

$$|e^{i\omega_1} - a| \ge \delta \qquad \textit{and} \qquad |e^{i\omega_2} - a| \ge \delta \,. \tag{3.5}$$

Combining Lemmas 3.3 and 3.5 we obtain the following.

Lemma 3.4. *Let* $\omega_1 < \omega_2 \le \omega_1 + 2\pi$,

$$\omega_1 =: t_{-1} = t_0 < t_1 < \cdots < t_m = t_{m+1} := \omega_2 \,,$$

$\delta \le 1/2$. *Then there is an absolute constant* $c_3 > 0$ *such that*

$$\sum_{j=0}^{m} \frac{t_{j+1} - t_{j-1}}{2} \log |e^{it_j} - a| \le \int_{\omega_1}^{\omega_2} \log |e^{it} - a| \, dt + c_3 \delta \log(1/\delta)$$

for every $a \in \mathbb{C}$ *such that either* $|e^{i\omega_1} - a| \le \delta$ *or* $|e^{i\omega_2} - a| \le \delta$.

Lemma 3.5. *Let* $\beta - \alpha \le \delta \le 1/2$ *and* $|a| \ge 1/2$. *There is an absolute constant* $c_4 > 0$ *such that*

$$0 \le \int_{\alpha}^{\beta} \log |e^{it} - a| \, dt + c_4 \delta \log(1/\delta) \,.$$

The following two lemmas will be needed to estimate the Mahler measure of polynomials on short intervals (of size at most δ) next to the endpoints of the interval.

Lemma 3.6. *Let* $\alpha \le \gamma \le \beta$ *and* $\beta - \alpha \le \delta \le 1/2$. *Suppose that* $|e^{i\gamma} - a| \ge (M+1)\delta$ *with some* $M > 0$. *Then*

$$(\beta - \alpha) \log |e^{i\gamma} - a| \le \int_{\alpha}^{\beta} \log |e^{it} - a| \, dt + \frac{\delta}{M} \,.$$

Lemma 3.7. *Let* $\alpha \le \gamma \le \beta$ *and* $\beta - \alpha \le \delta \le 1/2$. *Suppose that* $|e^{i\gamma} - a| \le 1$. *There is an absolute constant* $c_5 > 0$ *such that*

$$(\beta - \alpha) \log |e^{i\gamma} - a| \le \int_{\alpha}^{\beta} \log |e^{it} - a| \, dt + c_5 \delta \log(1/\delta) \,.$$

Our final lemma formulates a classical result of Erdős and Turán [ET-50]. (A recent improvement of the result below is given in [E-08].)

Lemma 3.8. *If the zeros of*

$$P(z) := \sum_{j=0}^{n} b_j z^j\,, \qquad b_j \in \mathbb{C}\,, \quad b_0 b_n \neq 0\,,$$

are denoted by

$$a_j = r_j \exp(i\varphi_j)\,, \qquad r_j > 0\,, \quad \varphi_j \in [0, 2\pi)\,, \quad j = 1, 2, \cdots, n\,,$$

then for every $\alpha < \beta \leq \alpha + 2\pi$ we have

$$\Big| \sum_{j \in I(\alpha,\beta)} 1 - \frac{\beta - \alpha}{2\pi} n \Big| < 16 \sqrt{n \log R}\,,$$

where $R := |b_0 b_n|^{-1/2} \|P\|_{\partial D}$ and $I(\alpha, \beta) := \{j : \alpha \leq \varphi_j \leq \beta\}$.

Lemma 3.9. *Let P be a polynomial of the form as in Lemma 3.8. Suppose $\alpha \leq \gamma \leq \beta$ and $\beta - \alpha \leq \delta \leq 1/2$. We have*

$$(\beta - \alpha) \log |P(e^{i\gamma})| \leq \int_\alpha^\beta \log |e^{it} - a|\, dt + c_1' \left(n\delta^2 \log(1/\delta) + \delta \log(1/\delta) \sqrt{n \log R} \right)$$

with an absolute constant $c_1' > 0$.

4. Proofs of the Lemmas

The proofs of Lemmas 3.1 and 3.2 follow the same lines but they are slightly different. To prove both of our first two lemmas our starting observations are as follows. Without loss of generality we may assume that a is a positive real number. Since then

$$\log |e^{it} - a| = \log |e^{it} - a^{-1}| + \log |a|$$

for all $t \in \mathbb{R}$, it is sufficient to prove (3.1) and (3.2) only in the case when $a \geq 1$. Note that elementary geometry shows that if f'' exists and does not change sign on $[\alpha, \beta]$, then

$$\left| \int_\alpha^\beta f(t)\, dt - \frac{\beta - \alpha}{2} (f(\alpha) + f(\beta)) \right| \leq \frac{1}{2} (\beta - \alpha)^2 |f'(\beta) - f'(\alpha)|\,. \tag{4.1}$$

This is just estimating the difference of the area of the region H below the graph of a (positive) convex or concave function on an interval $[\alpha, \beta]$ and the area of the trapezoid T with vertices $A(\alpha, 0)$, $B(\beta, 0)$, $C(\alpha, f(\alpha))$, and $D(\beta, f(\beta))$. Observe that the fact that f'' exists and does not change sign (without loss of generality we may assume that $f'' \geq 0$ on $[\alpha, \beta]$, so f is convex on $[\alpha, \beta]$) implies that the region $T \setminus H$ is contained in the triangle

CEF where the t coordinate of both E and F is β and the slope of CE is $f'(\alpha)$ and the slope of CF is $f'(\beta)$. Finally observe that the area of the triangle CEF is

$$\frac{1}{2}(\beta-\alpha)^2|f'(\beta)-f'(\alpha)|\,.$$

Also, if f' is continuous on $[\alpha,\beta]$, then

$$\left|\int_\alpha^\beta f(t)\,dt-\frac{\beta-\alpha}{2}(f(\alpha)+f(\beta))\right|\leq(\beta-\alpha)^2\max_{t\in[\alpha,\beta]}|f'(t)|\,. \tag{4.2}$$

Now let

$$f(t):=\log|e^{it}-a|\,.$$

Then

$$f'(t)=\frac{a\sin t}{1+a^2-2a\cos t}\quad\text{and}\quad f''(t)=\frac{-2a^2+(1+a^2)a\cos t}{(1+a^2-2a\cos t)^2}\,. \tag{4.3}$$

Also, since f'' has at most two zeros in the period, the total variation $V_{\omega_1}^{\omega_2}f'$ of f' on $[\omega_1,\omega_2]$ satisfies

$$V_{\omega_1}^{\omega_2}f'\leq 6\max_{t\in[\omega_1,\omega_2]}|f'(t)|\,. \tag{4.4}$$

Observe also that (4.1), (4.2), (4.4), and the fact that there are at most two intervals $[t_{j-1},t_j]$ on which f'' changes sign imply that

$$\begin{aligned}
&\left|\int_{\omega_1}^{\omega_2} f(t)\,dt-\sum_{j=0}^m\frac{t_{j+1}-t_{j-1}}{2}f(t_j)\right| \qquad (4.5)\\
=&\left|\sum_{j=1}^m\left(\int_{t_{j-1}}^{t_j} f(t)\,dt-\frac{t_j-t_{j-1}}{2}(f(t_j)+f(t_{j-1}))\right)\right|\\
\leq&\sum_{j=1}^m\left|\int_{t_{j-1}}^{t_j} f(t)\,dt-\frac{t_j-t_{j-1}}{2}(f(t_j)+f(t_{j-1}))\right|\\
\leq&\sum_{j=1}^m\frac{1}{2}(t_j-t_{j-1})^2|f'(t_j)-f'(t_{j-1})|+2\delta^2\max_{t\in[\omega_1,\omega_2]}|f'(t)|\\
\leq&\frac{1}{2}\delta^2(V_{\omega_1}^{\omega_2}f')+2\delta^2\max_{t\in[\omega_1,\omega_2]}|f'(t)|\leq 5\delta^2\max_{t\in[\omega_1,\omega_2]}|f'(t)|\,.
\end{aligned}$$

Proof of Lemma 3.1. In addition to $a\geq 1$, without loss of generality we may assume that $a\geq 1+\delta$, the case $1\leq a\leq 1+\delta$ follows easily from the case when $a=1+\delta$. To prove

(3.1) when $a \geq 1+\delta$ first observe that elementary calculus shows that $|f'(t)|$ achieves its maximum on the period when $\cos t := \dfrac{2a}{1+a^2}$. Then $|\sin t| = \dfrac{a^2-1}{a^2+1}$. Therefore

$$|f'(t)| \leq (a-a^{-1})^{-1} \leq \delta^{-1}\,, \qquad t \in \mathbb{R}\,, \tag{4.6}$$

where $a \geq 1+\delta$ has also been used. Hence,

$$\max_{t\in[0,2\pi]} |f'(t)| \leq \delta^{-1}\,. \tag{4.7}$$

Now (4.5) and (4.7) imply that

$$\left| \int_0^{2\pi} f(t)\,dt - \sum_{j=1}^m \frac{t_{j+1}-t_{j-1}}{2} f(t_j) \right| \leq 5\delta\,. \qquad \square$$

Proof of Lemma 3.2. As we observed it in the beginning of the section, it is sufficient to prove (3.2) only in the case when $a \geq 1$. To prove (3.2) when $a \geq 1$ first observe that

$$|f'(t)| = \frac{|a\sin t|}{1+a^2-2a\cos t} \leq \frac{|e^{it}-a|}{|e^{it}-a|^2} = \frac{1}{|e^{it}-a|} \leq \Delta^{-1}\,, \qquad t \in [\omega_1,\omega_2]\,,$$

that is,

$$\max_{t\in[\omega_1,\omega_2]} |f'(t)| \leq \Delta^{-1}\,. \tag{4.8}$$

Now (4.5) and (4.8) imply that

$$\left| \int_0^{2\pi} f(t)\,dt - \sum_{j=1}^m \frac{t_{j+1}-t_{j-1}}{2} f(t_j) \right| \leq 5\delta^2\Delta^{-1}\,. \qquad \square$$

Proof of Lemma 3.3. If (3.4) is satisfied then we get the conclusion of the lemma by using simply Lemma 3.2. If (3.4) is not satisfied, that is, if $\omega_1 < \varphi < \omega_1+2\pi$, then the argument is a bit trickier. Namely, if $\omega_1 < \varphi < \omega_1+2\pi$, then the conclusion of the lemma follows from a combination of Lemmas 3.1 and 3.2. Lemma 3.2 is applied with $[\omega_1,\omega_2]$ replaced by $[\omega_2,\omega_1+2\pi]$ and by extending the original subdivision of $[\omega_1,\omega_2]$ with norm δ to a subdivision of the period with norm δ. $\square$

Proof of Lemma 3.5. First assume that $|a| \geq 1$. Without loss of generality we may assume that a is real and $a \geq 1$. Then

$$\int_\alpha^\beta \log|e^{it}-a|\,dt \geq \int_\alpha^\beta \log\left|\frac{2t}{\pi}\right|\,dt = \left[t\log\left|\frac{2t}{\pi}\right| - t\right]_\alpha^\beta \geq c_6\delta\log\delta$$

with an absolute constant $c_6 > 0$, and the lemma follows. Now assume that $1/2 \le |a| < 1$. Without loss of generality we may assume that a is real and $1/2 \le a < 1$. Since

$$\log|e^{it} - a| = \log|e^{it} - a^{-1}| + \log a$$

for all $t \in \mathbb{R}$, it follows from the already proved case that

$$\begin{aligned}\int_\alpha^\beta \log|e^{it} - a|\,dt &\ge c_6\delta\log\delta + (\beta-\alpha)\log a \\ &\ge 2c_6\delta\log\delta + \delta\log(1/2) \ge (c_6+1)\delta\log\delta\,. \qquad \square\end{aligned}$$

Proof of Lemma 3.4. This is a consequence of Lemmas 3.3 and 3.5. $\square$

Proof of Lemma 3.6. Observe that $|e^{i\gamma} - a| \le |e^{i\gamma} - e^{it}| + |e^{it} - a|$, hence

$$|e^{it} - a| \ge |e^{i\gamma} - a| - |e^{i\gamma} - e^{it}| \ge |e^{i\gamma} - a| - \delta \ge \frac{M}{M+1}\,|e^{i\gamma} - a|$$

for every $t \in [\alpha, \beta]$. Hence

$$\begin{aligned}(\beta-\alpha)\log|e^{i\gamma} - a| - \int_\alpha^\beta \log|e^{it} - a|\,dt &\le \int_\alpha^\beta \left(\log|e^{i\gamma} - a| - \log|e^{it} - a|\right)dt \\ &\le \int_\alpha^\beta \log\left|\frac{e^{i\gamma} - a}{e^{it} - a}\right| dt \\ &\le (\beta-\alpha)\log\frac{M+1}{M} \\ &\le \delta/M\,. \qquad \square\end{aligned}$$

Proof of Lemma 3.7. Without loss of generality we may assume that a is a positive real number. Since then

$$\log|e^{it} - a| = \log|e^{it} - a^{-1}| + \log a$$

for all $t \in \mathbb{R}$, it is sufficient to prove the lemma only in the case when $a \ge 1$. Elementary calculus shows that

$$(\beta-\alpha)\log|e^{i\gamma} - a| \le 0\,,$$

while

$$\int_\alpha^\beta \log|e^{it} - a|\,dt \ge \int_\alpha^\beta \log\left|\frac{2t}{\pi}\right| dt = \left[t\log\left|\frac{2t}{\pi}\right| - t\right]_\alpha^\beta \ge c_5\delta\log\delta$$

with an absolute constant $c_5 > 0$. $\square$

Proof of Lemma 3.9. Every polynomial P of the form

$$P(z) = \sum_{j=0}^n b_j z^j\,, \qquad b_j \in \mathbb{C}\,, \quad b_n \ne 0\,,$$

can be factorized as

$$P(z) = b_n \prod_{k=1}^{n} (z - a_k)\,, \qquad a_k \in \mathbb{C}\,.$$

Without loss of generality we may assume that $b_n = 1$. Let

$$L - 1 := \left\lfloor \log_2 \left(\frac{\pi}{2\delta} \right) \right\rfloor \geq 1\,,$$

$$\beta_\mu := \beta + 2^\mu \delta\,, \qquad \mu = 1, 2, \ldots, L\,,$$

$$\alpha_\mu := \alpha - 2^\mu \delta\,, \qquad \mu = 1, 2, \ldots, L\,,$$

$$V_1 := \{re^{i\varphi} : \ \varphi \in [\alpha_1, \beta_1)\,, \ r > 0\}\,, \qquad V_L := \{re^{i\varphi} : \ \varphi \in [\beta_{L-1}, \alpha_{L-1} + 2\pi)\,, \ r > 0\}\,,$$

and

$$V_\mu := \{re^{i\varphi} : \ \varphi \in [\beta_{\mu-1}, \beta_\mu) \cup [\alpha_\mu, \alpha_{\mu-1})\,, \ r > 0\}\,, \qquad \mu = 2, 3, \ldots, L-1\,.$$

Note that

$$\bigcup_{\mu=1}^{L} V_\mu = \mathbb{C} \setminus \{0\}\,.$$

Let N_μ denote the number of zeros of P in V_μ. By Lemma 3.8 there is an absolute constant $c_7 > 0$ such that

$$N_\mu < c_7 \big(n 2^\mu \delta + \sqrt{n \log R} \big)\,, \qquad \mu = 1, 2, \ldots, L+1\,, \tag{4.9}$$

where $R := |b_0 b_n|^{-1/2} \|P\|_{\partial D}$. Observe also that there is an absolute constant $c_8 > 0$ such that

$$|t - a| \geq \delta + c_8 2^\mu \delta\,, \qquad t \in [\alpha, \beta]\,, \quad a \in V_\mu\,, \qquad \mu = 2, 3, \ldots, L\,. \tag{4.10}$$

Using Lemmas 3.6 and 3.7 and inequalities (4.9), (4.10), and

$$L - 1 \leq \log_2 \left(\frac{\pi}{2\delta} \right) \leq 1 + \log_2(1/\delta)\,,$$

we obtain

$$(\beta - \alpha) \log |P(e^{i\gamma})| = (\beta - \alpha) \sum_{k=1}^{n} \log |e^{i\gamma} - a_k| = (\beta - \alpha) \sum_{\mu=1}^{L} \sum_{a_k \in V_\mu} \log |e^{i\gamma} - a_k|$$

$$\leq \sum_{k=1}^{n} \int_\alpha^\beta \log |e^{it} - a_k|\, dt + N_1 c_5 \delta \log(1/\delta) + \sum_{\mu=2}^{L} \frac{N_\mu \delta}{c_8 2^\mu}$$

$$\leq \int_\alpha^\beta \log |P(e^{it})|\, dt + c_7 \big(n 2\delta + \sqrt{n \log R} \big) c_5 \delta \log(1/\delta)$$

$$+ \sum_{\mu=2}^{L} \frac{c_7 \big(n 2^\mu \delta + \sqrt{n \log R} \big) \delta}{c_8 2^\mu}\,,$$

and hence

$$
\begin{aligned}
&(\beta-\alpha)\log|P(e^{i\gamma})| - \int_\alpha^\beta \log|P(e^{it})|\,dt \\
\le & c_7\big(n2\delta+\sqrt{n\log R}\big)c_5\delta\log(1/\delta) + \sum_{\mu=2}^{L} \frac{c_7\big(n2^\mu\delta+\sqrt{n\log R}\big)\delta}{c_8 2^\mu} \\
\le & c_1'\left(n\delta^2\log(1/\delta)+\delta\log(1/\delta)\sqrt{n\log R}\right)
\end{aligned}
$$

with an absolute constant $c_1' > 0$. In the last inequality we used that

$$
\sum_{\mu=2}^{L} c_7(n2^\mu\delta)\frac{\delta}{c_8 2^\mu} = (L-1)\frac{c_7}{c_8}n\delta^2 \le \frac{c_7}{c_8}\left(1+\log_2(1/\delta)\right)n\delta^2\,. \qquad \Box
$$

5. Proofs of the Theorems

Proof of Theorem 2.1. Without loss of generality we may assume that $\omega_2-\omega_1 \le \pi/8$. First we assume that $t_0=\omega_1$ and $t_m=\omega_2$. Every polynomial P of the form

$$
P(z)=\sum_{j=0}^{n} b_j z^j\,, \qquad b_j\in\mathbb{C}\,, \quad b_n \neq 0\,,
$$

can be factorized as

$$
P(z)=b_n\prod_{k=1}^{n}(z-a_k)\,, \qquad a_k\in\mathbb{C}\,.
$$

Without loss of generality we may assume that $b_n=1$. Let

$$
U := D(e^{i\omega_1},\delta)\cup D(e^{i\omega_2},\delta)
$$

where $D(a,r)$ denotes the open disk of the complex plane centered at a with radius r. Let

$$
L-1 := \left\lfloor \log_2\left(\frac{\pi/2}{\omega_2-\omega_1}\right)\right\rfloor \ge 2\,,
$$

$$
\beta_\mu := \omega_2 + 2^\mu(\omega_2-\omega_1)\,, \qquad \mu=1,2,\dots,L-1\,,
$$

$$
\alpha_\mu := \omega_1 - 2^\mu(\omega_2-\omega_1)\,, \qquad \mu=1,2,\dots,L-1\,,
$$

$$
V_1 := \{re^{i\varphi}:\ \varphi\in[\alpha_1,\beta_1)\,,\ r>0\}\,, \qquad V_L := \{re^{i\varphi}:\ \varphi\in[\beta_{L-1},\alpha_{L-1}+2\pi)\,,\ r>0\}\,,
$$

and

$$
V_\mu := \{re^{i\varphi}:\ \varphi\in[\beta_{\mu-1},\beta_\mu)\cup[\alpha_\mu,\alpha_{\mu-1})\,,\ r>0\}\,, \qquad \mu=2,3,\dots,L-1\,.
$$

Note that

$$\bigcup_{\mu=1}^{L} V_\mu = \mathbb{C} \setminus \{0\}\,.$$

Let M denote the number of zeros of P in U. Let N_μ denote the number of zeros of P in V_μ. By Lemma 3.8 there is an absolute constant $c_7 > 0$ such that

$$M < c_7\big(n\delta + \sqrt{n \log R}\big) \tag{5.1}$$

and

$$N_\mu < c_7\big(n2^\mu(\omega_2 - \omega_1) + \sqrt{n \log R}\big)\,, \qquad \mu = 1, 2, \dots, L\,, \tag{5.2}$$

where $R := |b_0 b_n|^{-1/2} \|P\|_{\partial D}$. Observe also that there is an absolute constant $c_8 > 0$ such that

$$|t - a| \geq c_8 2^\mu (\omega_2 - \omega_1)\,, \qquad t \in [\omega_1, \omega_2]\,, \quad a \in V_\mu\,, \qquad \mu = 2, 3, \dots, L\,. \tag{5.3}$$

Using Lemmas 3.2, 3.3, and 3.4, and inequalities (5.1), (5.2), (5.3), and

$$L - 1 \leq \log_2\left(\frac{\pi/2}{\omega_2 - \omega_1}\right) \leq 1 + \log_2(1/\delta)\,,$$

we obtain

$$\begin{aligned}
&\sum_{j=0}^{m} \frac{t_{j+1} - t_{j-1}}{2} \log|P(e^{it_j})| = \sum_{j=0}^{m}\sum_{k=1}^{n} \frac{t_{j+1} - t_{j-1}}{2} \log|e^{it_j} - a_k| \\
&= \sum_{k=1}^{n}\sum_{j=0}^{m} \frac{t_{j+1} - t_{j-1}}{2} \log|e^{it_j} - a_k| = \sum_{\mu=1}^{L}\sum_{a_k \in V_\mu}\sum_{j=0}^{m} \frac{t_{j+1} - t_{j-1}}{2} \log|e^{it_j} - a_k| \\
&\leq \sum_{k=1}^{n}\left(\int_{\omega_1}^{\omega_2} \log|e^{it} - a_k|\,dt\right) + N_1(10\delta) + M(c_3\delta\log(1/\delta)) + \sum_{\mu=2}^{L} \frac{N_\mu(5\delta^2)}{c_8 2^\mu(\omega_2 - \omega_1)}\,,
\end{aligned}$$

and hence

$$\begin{aligned}
&\sum_{j=0}^{m} \frac{t_{j+1} - t_{j-1}}{2} \log|P(e^{it_j})| - \int_{\omega_1}^{\omega_2} \log|P(e^{it})|\,dt \\
&\leq c_7\big(2n(\omega_2 - \omega_1) + \sqrt{n\log R}\big)(10\delta) + c_7\big(n\delta + \sqrt{n\log R}\big)(c_3\delta\log(1/\delta)) \\
&+ \sum_{\mu=2}^{L} c_7\big(n2^\mu(\omega_2 - \omega_1) + \sqrt{n\log R}\big)\frac{5\delta^2}{c_8 2^\mu(\omega_2 - \omega_1)} \\
&\leq c_9'\left((\omega_2 - \omega_1)n\delta + n\delta^2\log(1/\delta) + \sqrt{n\log R}\left(\delta\log(1/\delta) + \frac{\delta^2}{\omega_2 - \omega_1}\right)\right) \\
&\leq c_9' E(n, \delta, \omega_1, \omega_2)
\end{aligned}$$

with an absolute constant $c_9' > 0$. In the last inequality we used that

$$\sum_{\mu=2}^{L} c_7(n2^\mu(\omega_2-\omega_1))\frac{\delta^2}{c_8 2^\mu(\omega_2-\omega_1)} = (L-1)\frac{c_7}{c_8}\,n\delta^2 \le \frac{c_7}{c_8}\,n\delta^2\left(1+\log_2(1/\delta)\right).$$

Hence the theorem is proved in the case when $t_0=\omega_1$ and $t_m=\omega_2$.

Now we eliminate the extra assumptions $t_0=\omega_1$ and $t_m=\omega_2$ from the proof. Applying the already proved case of the theorem with $\omega_1=t_0$ and $\omega_2=t_m$, we have

$$\sum_{j=1}^{m-1}\frac{t_{j+1}-t_{j-1}}{2}\log|P(e^{it_j})| \le \int_{t_0}^{t_m}\log|P(e^{it})|\,dt + c_1'E(n,\delta,\omega_1,\omega_2)\,. \tag{5.4}$$

It follows from Lemma 3.9 that

$$\frac{t_0-t_{-1}}{2}\log|P(e^{it_0})| \le \int_{\omega_1}^{t_0}\log|P(e^{it})|\,dt + c_1'E(n,\delta,\omega_1,\omega_2) \tag{5.5}$$

and

$$\frac{t_{m+1}-t_m}{2}\log|P(e^{it_m})| \le \int_{t_m}^{\omega_2}\log|P(e^{it})|\,dt + c_1'E(n,\delta,\omega_1,\omega_2)\,. \tag{5.6}$$

Now (5.4), (5.5), and (5.6) imply the theorem. $\square$

Proof of Theorem 2.2. The theorem follows from Theorem 2.1 and Lemma 1.2 in a straightforward fashion. Let $g_p(z) := f_p(z)/z$ and let

$$\omega_1 := \alpha \le t_0 < t_1 < t_2 < \cdots < t_m \le \beta =: \omega_2 \tag{5.7}$$

be chosen so that e^{it_j}, $j=0,1,\dots,m$, are exactly the primitive p-th roots of unity lying on the arc connecting $e^{i\alpha}$ and $e^{i\beta}$ on the unit circle counterclockwise. The assumption on p guarantees that the value of δ defined in Theorem 2.1 is at most $4\pi/p$. Observe also that $R \le p-2 < p$. By Lemma 1.2 we have

$$|g_p(e^{it_j})| = \sqrt{p}\,, \qquad j=0,1,\dots,m\,.$$

Applying Theorem 2.1 with $P := g_p$, $n=p-2$, and (5.7), we obtain

$$\sum_{j=0}^{m}\frac{t_{j+1}-t_{j-1}}{2}\log|g_p(e^{it_j})| \le \int_{\alpha}^{\beta}\log|g_p(e^{it})|\,dt + c_1E(p-2,4\pi/p,\alpha,\beta)\,,$$

where the assumption

$$\frac{(\log p)^{3/2}}{p^{1/2}} \le \beta-\alpha \le 2\pi$$

implies that

$$\begin{aligned} E(p-2,4\pi/p,\alpha,\beta) &\le c_{10}'\left(\frac{(\beta-\alpha)p}{p} + \frac{\log p}{p} + \sqrt{p\log p}\left(\frac{\log p}{p} + \frac{1}{p^2(\beta-\alpha)}\right)\right) \\ &\le c_{10}(\beta-\alpha) \end{aligned}$$

with absolute constants $c_{10}' > 0$ and $c_{10} > 0$. Hence

$$M_0(f_p,[\alpha,\beta]) = M_0(g_p,[\alpha,\beta]) \ge \exp(-c_1c_{10})\,\sqrt{p}\,,$$

and the theorem follows. $\square$

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