INEQUALITIES FOR LORENTZ POLYNOMIALS

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Abstract. We prove a few interesting inequalities for Lorentz polynomials. A highlight of this paper states that the Markov-type inequality

$$\max_{x \in [-1,1]} |f'(x)| \leq n \max_{x \in [-1,1]} |f(x)|$$

holds for all polynomials f of degree at most n with real coefficients for which f' has all its zeros outside the open unit disk. Equality holds only for $f(x) := c((1 \pm x)^n - 2^{n-1})$ with a constant $0 \neq c \in \mathbb{R}$. This should be compared with Erdős's classical result stating that

$$\max_{x \in [-1,1]} |f'(x)| \leq \frac{n}{2} \left(\frac{n}{n-1} \right)^{n-1} \max_{x \in [-1,1]} |f(x)|$$

for all polynomials f of degree at most n having all their zeros in $\mathbb{R} \setminus (-1,1)$.

1. Introduction

Let $\mathcal{P}_n$ denote the collection of all polynomials of degree at most n with *real coefficients*. Let $\mathcal{P}_n^c$ denote the collection of all polynomials of degree at most n with *complex coefficients*. Let

$$\|f\|_A := \sup_{x \in A} |f(x)|$$

denote the supremum norm of a complex-valued function f defined on a set A. The Markov inequality asserts that

$$\|f'\|_{[-1,1]} \leq n^2 \|f\|_{[-1,1]}$$

holds for all $f \in \mathcal{P}_n^c$. The inequality

$$|f'(x)| \leq \frac{n}{\sqrt{1-x^2}} \|f\|_{[-1,1]}$$

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holds for all $f \in \mathcal{P}_n^c$ and for all $x \in (-1,1)$, and is known as Bernstein inequality. For proofs of these see [2] or [5], for instance. Various analogues of the above two inequalities are known in which the underlying intervals, the maximum norms, and the family of polynomials are replaced by more general sets, norms, and families of functions, respectively. These inequalities are called Markov-type and Bernstein-type inequalities. If the norms are the same in both sides, the inequality is called "Markov-type", while "Bernstein-type inequality" usually means a pointwise estimate for the derivative. Markov- and Bernstein-type inequalitiies are known on various regions of the complex plane and the n-dimensional Euclidean space, for various norms such as weighted L_p norms, and for many classes of functions such as polynomials with various constraints, exponential sums of n terms, just to mention a few. Markov- and Bernstein-type inequalities have their own intrinsic interest. In addition, they play a fundamental role in approximation theory.

It had been observed by Bernstein that Markov's inequality for monotone polynomials is not essentially better than for arbitrary polynomials. Bernstein proved that

$$\sup_f \frac{\|f'\|_{[-1,1]}}{\|f\|_{[-1,1]}} = \begin{cases} \frac{1}{4}(n+1)^2\,, & \text{if } n \text{ is odd} \\ \frac{1}{4}n(n+2)\,, & \text{if } n \text{ is even}\,, \end{cases}$$

where the supremum is taken over all $f \in \mathcal{P}_n$ which are monotone on $[-1,1]$. See [23], for instance. This is surprising, since one would expect that if a polynomial is this far away from the "equioscillating" property of the Chebyshev polynomial T_n, then there should be a more significant improvement in the Markov inequality. In [16] Erdős gave a class of restricted polynomials for which the Markov factor n^2 improves to cn. He proved that there is an absolute constant c such that

$$|f'(x)| \leq \min\left\{\frac{c\sqrt{n}}{(1-x^2)^2}\,,\ \frac{en}{2}\right\} \|f\|_{[-1,1]}\,, \qquad x \in (-1,1)\,,$$

for all $f \in \mathcal{P}_n$ having all their zeros in $\mathbb{R} \setminus (-1,1)$. This result motivated several people to study Markov- and Bernstein-type inequalities for polynomials with restricted zeros and under some other constraints. Generalizations of the above Markov- and Bernstein-type inequality of Erdős have been extended in various directions by several people including Lorentz [20], Scheick [24], Szabados [25], Máté [21], P. Borwein [1], Erdélyi [6,7,9,12,13], Rahman and Schmeisser [23], Kroó and Szabados [18,19], Halász [17], and the list can be even longer. A special attention is paid to the classes $\mathcal{P}_{n,k}$ and $\mathcal{P}_{n,k}^c$, where $\mathcal{P}_{n,k}$ denotes the set of all polynomials of degree at most n with *real coefficients* and with at most k $(0 \leq k \leq n)$ zeros in the open unit disk, and $\mathcal{P}_{n,k}^c$ denotes the set of all polynomials of degree at most n with *complex coefficients* and with at most k $(0 \leq k \leq n)$ zeros in the open unit disk. Associated with $0 \leq k \leq n$ and $x \in (-1,1)$, let

$$B_{n,k,x} := \sqrt{\frac{n(k+1)}{1-x^2}}\,, \qquad B_{n,k,x}^* := \max\left\{\sqrt{\frac{n(k+1)}{1-x^2}}\,,\ n\log\left(\frac{e}{1-x^2}\right)\right\}\,,$$

and

$$M_{n,k} := n(k+1)\,, \qquad M_{n,k}^* := \max\{n(k+1),\ n\log n\}\,.$$

In [10] and [11] it is shown that

$$c_1 \min\{B^*_{n,k,x}, M^*_{n,k}\} \le \sup_{f \in \mathcal{P}^c_{n,k}} \frac{|f'(x)|}{\|f\|_{[-1,1]}} \le c_2 \min\{B^*_{n,k,x}, M^*_{n,k}\}$$

for all $x \in (-1,1)$, where $c_1 > 0$ and $c_2 > 0$ are absolute constants. This result should be compared with the inequalities

$$c_1 \min\{B_{n,k,x}, M_{n,k}\} \le \sup_{f \in \mathcal{P}_{n,k}} \frac{|f'(x)|}{\|f\|_{[-1,1]}} \le c_2 \min\{B_{n,k,x}, M_{n,k}\}$$

for all $x \in (-1,1)$, where $c_1 > 0$ and $c_2 > 0$ are absolute constants. See [4] and [11]. It may be surprising that there is a significant difference between the real and complex cases as far as Markov- and Bernstein-type inequalities are concerned. In [3] essentially sharp Markov- and Bernstein-type inequalities for the classes $\mathcal{P}_{n,k}$ are proved even in L_p norms on $[-1,1]$ for all $p > 0$.

In this paper we revisit Erdős's paper [16] and make several remarks to his Markov-type inequality in it. Erdős claimed in [16] that his method gave a Markov factor slightly better than $en/2$, namely,

$$\|f'\|_{[-1,1]} \le \frac{n}{2}\left(\frac{n}{n-1}\right)^{n-1} \|f\|_{[-1,1]}$$

for all $f \in \mathcal{P}_n$ having all their zeros in $\mathbb{R} \setminus (-1,1)$. Indeed, at some points of his arguments, by replacing applications of the inequality $1 + x \le e^x$ with an application of the inequality between the geometric and arithmetic means of nonnegative numbers, we can easily see this slight improvement.

In 1963 Lorentz [20] proved that there is an absolute constant $c > 0$ such that

$$|f'(x)| \le c \min\left\{\sqrt{\frac{n}{1-x^2}}\,, n\right\} \|f\|_{[-1,1]}\,, \qquad x \in (-1,1)\,,$$

for all $f \in \mathcal{B}_n(-1,1)$, where

$$\mathcal{B}_d(a,b) := \left\{ f : f(x) = \sum_{j=0}^{d} a_j (b-x)^j (x-a)^{d-j}\,, \quad a_j \ge 0, \quad j = 0,1,\ldots,d \right\}.$$

for real numbers $a \le b$ and nonnegative integers d. He also made the observation that if $f \in \mathcal{P}_{n,0}$ then either $f \in \mathcal{B}_n(-1,1)$ or $-f \in \mathcal{B}_n(-1,1)$, where $\mathcal{P}_{n,0}$ denotes the collection of all $f \in \mathcal{P}_n$ having all their zeros outside the open unit disk. Scheick [24] has found the best possible constant c in Lorentz's Markov-type inequality for $f \in \mathcal{B}_n(-1,1)$. He showed that

$$\|f'\|_{[-1,1]} \le \frac{en}{2} \|f\|_{[-1,1]}$$

for all $f \in \mathcal{B}_n(-1,1)$.

An elementary, but very useful tool for proving inequalities for polynomials with restricted zeros is the *Bernstein* or *Lorentz representation* of polynomials. Namely, as Lorentz observed it, if $f \in \mathcal{P}_{n,0}$ is positive on $(-1,1)$ then it is of the form

$$f(x) = \sum_{j=0}^{d} a_j (1-x)^j (x+1)^{d-j}\,, \qquad a_j \geq 0\,, \quad j = 0, 1, \ldots, d\,, \tag{1.1}$$

with $d = n$. This is formulated as Lemma 3.2 in this paper and its simple proof is reproduced. Moreover, if a polynomial $f \in \mathcal{P}_n$ is positive on $(-1,1)$ and has no zeros in the ellipse L_ε with large axis $[-1,1]$ and small axis $[-\varepsilon i, \varepsilon i]$ ($\varepsilon \in [-1,1]$) then it has a Lorentz representation (1.1) with $d \leq 3n\varepsilon^{-2}$. See [14]. Combining this with Lorentz's Markov- and Bernstein-type inequality gives that there is an absolute constant $c > 0$ such that

$$f'(x)| \leq c \min\left\{ \frac{\sqrt{n}}{\varepsilon\sqrt{1-x^2}}\,,\ \frac{n}{\varepsilon^2} \right\} \|f\|_{[-1,1]}\,, \qquad x \in (-1,1)\,,$$

for all $f \in \mathcal{P}_n$ having no zeros in L_ε.

The minimal value of $d \in \mathbb{N}$ for which a polynomial f has a representation (1.1) is called the *Lorentz degree* of the polynomial and it is denoted by $d(f)$. It follows from the already mentioned result in [14] that $d(f) < \infty$ if and only if f has no zeros in $(-1,1)$. This is a theorem ascribed to Hausdorff. In addition, it has been proved in [8] that if

$$f(x) = ((x-a)^2 + \varepsilon^2(1-a^2))^n, \qquad 0 < \varepsilon \leq 1, \quad -1 < a < 1\,,$$

then

$$c_1 n\varepsilon^{-2} \leq d(f) \leq c_2 n\varepsilon^{-2}$$

with absolute constants $c_1 > 0$ and $c_2 > 0$. Lorentz degree of trigonometric polynomials on an interval $(-\omega, \omega)$ shorter than the period is studied in [15].

The well known results of Nikolskii assert that the essentially sharp inequality

$$\|f\|_{L_q[-1,1]} \leq c(p,q) n^{2/p-2/q} \|f\|_{L_p[-1,1]}$$

holds for all algebraic polynomials $f \in \mathcal{P}_n^c$ and for all $0 < p < q \leq \infty$, while the essentially sharp inequality

$$\|f\|_{L_q[-\pi,\pi]} \leq c(p,q) n^{1/p-1/q} \|f\|_{L_p[-\pi,\pi]}$$

holds for all trigonometric polynomials f of degree at most n with complex coefficients and for all $0 < p < q \leq \infty$. The subject started with [22] and [26]. There are quite a few related papers in the literature. In this paper we establish the right Nikolskii-type inequalities for the classes $\mathcal{B}_d(-1,1)$ and $\mathcal{P}_{n,0}$.

2. New Results

For $p > 0$ let

$$\|f\|_p := \left(\int_{-1}^{1} |f(x)|\, dx \right)^{1/p}, \qquad \|f\|_\infty := \max_{x \in [-1,1]} |f(x)|\,.$$

As in Section 1 we will use the following notation. Let $\mathcal{P}_n$ denote the collection of all polynomials of degree at most n with *real coefficients*. For real numbers $a \le b$ and $d \in \mathbb{N}$ let

$$\mathcal{B}_d(a,b) := \left\{ f : f(x) = \sum_{j=0}^{d} a_j (b-x)^j (x-a)^{d-j}\,, \quad a_j \ge 0, \quad j = 0,1,\dots,d \right\}.$$

Let $\mathcal{P}_{n,0}$ denote the collection of all $f \in \mathcal{P}_n$ having all their zeros outside the open unit disk. Our first two results are the right Nikolskii-type inequalities for the classes $\mathcal{B}_d(-1,1)$ and $\mathcal{P}_{n,0}$.

Theorem 2.1. *We have*

$$\|f\|_p \le \left(\frac{qd+1}{2}\right)^{1/q-1/p} \|f\|_q$$

for all $f \in \mathcal{B}_d(-1,1)$ and for all $0 < q < p \le \infty$. If $d > 0$ equality holds only for $p = \infty$ and $f(x) := c(1 \pm x)^d$ with a constant $c \ge 0$.

Theorem 2.2. *We have*

$$\|f\|_p \le \left(\frac{qn+1}{2}\right)^{1/q-1/p} \|f\|_q$$

for all $f \in \mathcal{P}_{n,0}$ and for all $0 < q < p \le \infty$. If $n > 0$ equality holds only for $p = \infty$ and $f(x) := c(1 \pm x)^n$ with a constant $0 \ne c \in \mathbb{R}$.

An application of Theorem 2.1 with $q = 1$ and $p = \infty$ allows us to prove the following sharp Markov-type inequality for all $f \in \mathcal{P}_d$ such that $f' \in \mathcal{B}_{d-1}(-1,1)$.

Theorem 2.3. *We have*

$$\|f'\|_\infty \le d\,\|f\|_\infty$$

for all $f \in \mathcal{P}_d$ with $f' \in \mathcal{B}_{d-1}(-1,1)$. Equality holds only for $f(x) := \sigma c((1+\sigma x)^d - 2^{d-1})$ with a constant $c \ge 0$ and $\sigma \in \{-1,1\}$.

Combining Theorem 2.3 with Lemma 3.2 gives the following.

Theorem 2.4. *We have*

$$\|f'\|_\infty \le n\,\|f\|_\infty$$

for all $f \in \mathcal{P}_n$ for which f' has all its zeros outside the open unit disk. Equality holds only for $f(x) := c((1 \pm x)^n - 2^{n-1})$ with a constant $0 \ne c \in \mathbb{R}$.

Our final result is a sharp Markov-type inequality for all $f \in \mathcal{P}_n$ which are monotone on $[-1,1]$ and have all their zeros in $\mathbb{R} \setminus (-1,1)$. Erdős claimed this in [16] but he did not give a hint how to prove this. Experts seem to be puzzled by this observation of Erdős even today.

Theorem 2.5. *We have*

$$\|f'\|_\infty \le \frac{n}{2}\,\|f\|_\infty$$

for all $f \in \mathcal{P}_n$ which is monotone on $[-1,1]$ and has all its zeros in $\mathbb{R} \setminus (-1,1)$. Equality holds only for $f(x) := c(1 \pm x)^n$, $f(x) := c(x+3)(x-1)$, and $f(x) := c(x-3)(x+1)$ with a constant $0 \ne c \in \mathbb{R}$.

We note that there is a hint to Part c] of Exercise 10 on page 432 of the book [2] suggesting that Theorem 2.5 holds. However, it was discovered by M. Boedihardjo that the hint to part c] of E.10 on page 432 of the book [2] does not work out. Here we claim a proof of Theorem 2.5 as a consequence of Theorem 2.1. A direct elementary proof of Theorem 2.5 by using undergraduate calculus would be desirable.

3. Lemmas

Lemma 3.1. *Let $a \le a_1 \le b_1 \le b$ be real numbers, and let d be a nonnegative integer. Then $\mathcal{B}_d(a,b) \subset \mathcal{B}_d(a_1,b_1)$.*

Proof of Lemma 3.1. This follows from the identities

$$x - a = \frac{b_1 - a}{b_1 - a_1}\,(x - a_1) + \frac{a_1 - a}{b_1 - a_1}\,(b_1 - x)$$

and

$$b - x = \frac{b - b_1}{b_1 - a_1}\,(x - a_1) + \frac{b - a_1}{b_1 - a_1}\,(b_1 - x)$$

valid for all $x \in \mathbb{C}$. $\square$

Lemma 3.2. *Suppose $f \in \mathcal{P}_n$ has all its zeros outside the open unit disk. Then either $f \in \mathcal{B}_n(-1,1)$ or $-f \in \mathcal{B}_n(-1,1)$.*

Proof of Lemma 3.2. This follows from the identities

$$x - \alpha = \tfrac{1}{2}(1-\alpha)(x+1) + \tfrac{1}{2}(-1-\alpha)(1-x)$$

and

$$(x-\alpha)(x-\overline{\alpha}) = \tfrac{1}{4}|1+\alpha|^2(1-x)^2 + \tfrac{1}{2}(|\alpha|^2-1)(1-x)(1+x) + \tfrac{1}{4}|1-\alpha|^2(x+1)^2$$

valid for all $x \in \mathbb{C}$ and $\alpha \in \mathbb{C}$. Observe that

$$(1-\alpha)(-1-\alpha) = \alpha^2 - 1 \ge 0\,, \qquad \alpha \in \mathbb{R} \setminus (-1,1)\,,$$

and

$$|\alpha|^2 - 1 \ge 0\,, \qquad \alpha \in \mathbb{C}\,,\ \ |\alpha| \ge 1\,.$$

$\square$

Lemma 3.3. *We have*

$$(\max\{f(a), f(b)\})^q \le \frac{qd+1}{b-a} \int_a^b f(x)^q \, dx$$

for all $f \in \mathcal{B}_d(a,b)$ and for all $q > 0$. Equality holds only for $f(x) = c(x-a)^d$ and $f(x) = c(b-x)^d$ with a constant $c \ge 0$.

Proof of Lemma 3.3. Let $f \in \mathcal{B}_d(a,b)$ be of the form

$$f(x) = \sum_{j=0}^{d} a_j (b-x)^j (x-a)^{d-j} \,, \qquad a_j \ge 0, \quad j = 0, 1, \ldots, d \,.$$

Then

$$\begin{aligned} f(b)^q = (a_0(b-a)^d)^q &= \frac{qd+1}{b-a} \int_a^b (a_0(x-a)^d)^q \, dx \\ &\le \frac{qd+1}{b-a} \int_a^b \left(\sum_{j=0}^{d} a_j (b-x)^j (x-a)^{d-j} \right)^q dx \\ &= \frac{qd+1}{b-a} \int_a^b f(x)^q \, dx \,. \end{aligned}$$

Similarly,

$$\begin{aligned} f(a)^q = (a_d(b-a)^d)^q &= \frac{qd+1}{b-a} \int_a^b (a_d(b-x)^d)^q \, dx \\ &\le \frac{qd+1}{b-a} \int_a^b \left(\sum_{j=0}^{d} a_j (b-x)^j (x-a)^{d-j} \right)^q dx \\ &= \frac{qd+1}{b-a} \int_a^b f(x)^q \, dx \,. \end{aligned}$$

$\square$

Lemma 3.4. *We have*

$$\|f\|_\infty^q \le \frac{qd+1}{2} \, \|f\|_q^q \,.$$

for all $f \in \mathcal{B}_d(-1,1)$ and for all $q > 0$. Equality holds only for $f(x) = c(1 \pm x)^d$ with a constant $c \ge 0$.

Proof of Lemma 3.4. Let $y \in [-1,1]$ be such that $f(y) = \|f\|_\infty$. By Lemma 3.1 we have

$$\mathcal{B}_d(-1,1) \subset \mathcal{B}_d(-1,y) \cap \mathcal{B}_d(y,1) \,.$$

Hence Lemma 3.3 yields

$$(y+1)f(y)^q \le (qd+1)\int_{-1}^{y} f(x)^q\,dx$$

and

$$(1-y)f(y)^q \le (qd+1)\int_{y}^{1} f(x)^q\,dx$$

Adding the above two inequalities, we conclude

$$\|f\|_\infty^q = f(y)^q \le \frac{qd+1}{2}\int_{-1}^{1} f(x)^q\,dx = \frac{qd+1}{2}\,\|f\|_q^q\,.$$

□

4. Proof of the Theorems

Proof of Theorem 2.1. When $p=\infty$ the Theorem follows from Lemma 3.4. Now let $f \in \mathcal{B}_d(-1,1)$ and $0<q<p<\infty$. Using Lemma 3.4 we obtain

$$\begin{aligned}\|f\|_p^p &= \int_{-1}^{1} f(x)^p\,dx \le \left(\int_{-1}^{1} f(x)^q\,dx\right)\|f\|_\infty^{p-q} \le \|f\|_q^q\left(\frac{qd+1}{2}\right)^{(p-q)/q}\|f\|_q^{p-q}\\ &= \left(\frac{qd+1}{2}\right)^{(p-q)/q}\|f\|_q^p\,,\end{aligned}$$

hence

$$\|f\|_p \le \left(\frac{qd+1}{2}\right)^{1/q-1/p}\|f\|_q\,.$$

□

Proof of Theorem 2.2. . Combining Theorem 2.1 and Lemma 3.2 gives the result. □

Proof of Theorem 2.3. Applying Theorem 2.1 with f replaced by $f' \in \mathcal{B}_{d-1}$, $p:=\infty$, and $q:=1$, we obtain

$$\|f'\|_\infty \le \frac{d}{2}\int_{-1}^{1} f'(x)\,dx = \frac{d}{2}\left(f(1)-f(-1)\right) \le d\,\|f\|_\infty\,.$$

□

Proof of Theorem 2.4. Assume that $f' \in \mathcal{P}_{n-1}$ has no zeros in the open unit disk. Then, by Lemma 3.2 either $f' \in \mathcal{B}_{n-1}(-1,1)$ or $-f' \in \mathcal{B}_{n-1}(-1,1)$. Without loss of generality we may assume that $f' \in \mathcal{B}_{n-1}(-1,1)$, and Theorem 2.3 gives the result. □

Proof of Theorem 2.5. Assume that $f \in \mathcal{P}_n$ is monotone on $[-1,1]$ and has all its zeros in $\mathbb{R}\setminus(-1,1)$. As f has only real zeros, repeated applications of Rolle's Theorem imply that $f' \in \mathcal{P}_{n-1}$ has only real zeros and at most one simple zero in $(-1,1)$. However, as f is (strictly) monotone on $(-1,1)$, f' cannot have a simple zero in $(-1,1)$, so f' has all its zeros in $\mathbb{R} \setminus (-1,1)$. Hence Lemma 3.2 implies that either $f' \in \mathcal{B}_{n-1}(-1,1)$ or $-f' \in \mathcal{B}_{n-1}(-1,1)$. Without loss of generality we may assume that $f' \in \mathcal{B}_{n-1}(-1,1)$. Applying Theorem 2.1 with f replaced by $f' \in \mathcal{B}_{d-1}$, $p := \infty$, and $q := 1$, we obtain

$$\|f'\|_\infty \leq \frac{n}{2} \int_{-1}^{1} f'(x)\, dx = \frac{n}{2} \left(f(1) - f(-1)\right) \leq \frac{n}{2} \|f\|_{[-1,1]} \,,$$

where in the last step we used that $f(1)f(-1) \geq 0$ since $f \in \mathcal{P}_n$ has all its zeros in $\mathbb{R} \setminus (-1,1)$. $\square$

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