THE FULL MARKOV-NEWMAN INEQUALITY FOR MÜNTZ POLYNOMIALS ON POSITIVE INTERVALS

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Abstract. For a function f defined on an interval $[a,b]$ let

$$\|f\|_{[a,b]} := \sup\{|f(x)| : x \in [a,b]\}\,.$$

The principal result of this paper is the following Markov-type inequality for Müntz polynomials.

Theorem. *Let $n \geq 1$ be an integer. Let $\lambda_0, \lambda_1, \dots, \lambda_n$ be $n+1$ distinct real numbers. Let $0 < a < b$. Then*

$$\frac{1}{3}\sum_{j=0}^{n}|\lambda_j| + \frac{1}{4\log(b/a)}(n-1)^2 \leq \sup_{0 \neq Q} \frac{\|xQ'(x)\|_{[a,b]}}{\|Q\|_{[a,b]}} \leq 11\sum_{j=0}^{n}|\lambda_j| + \frac{128}{\log(b/a)}(n+1)^2\,,$$

where the supremum is taken for all $Q \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$ (the span is the linear span over $\mathbb{R}$).

1. Introduction and Notation

Let $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ be a set of $n+1$ distinct real numbers. The span of

$$\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$$

over $\mathbb{R}$ will be denoted by

$$M(\Lambda_n) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}.$$

Elements of $M(\Lambda_n)$ are called Müntz polynomials of $n+1$ terms. For a function f defined on an interval $[a,b]$ let

$$\|f\|_{[a,b]} := \sup\{|f(x)| : x \in [a,b]\}$$

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and let

$$\|f\|_{L_p[a,b]} := \left(\int_a^b |f(x)|^p \, dx \right)^{1/p}, \qquad p > 0\,,$$

whenever the Lebesgue integral exists. Newman's beautiful inequality [4] is an essentially sharp Markov-type inequality for $M(\Lambda_n)$ on $[0,1]$ in the case when each λ_j is nonnegative.

Theorem 1.1 (Newman's Inequality). *Let $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ be a set of $n+1$ distinct nonnegative numbers. Then*

$$\frac{2}{3} \sum_{j=0}^{n} \lambda_j \leq \sup_{0 \neq Q \in M(\Lambda_n)} \frac{\|xQ'(x)\|_{[0,1]}}{\|Q\|_{[0,1]}} \leq 11 \sum_{j=0}^{n} \lambda_j\,.$$

Note that the interval $[0,1]$ plays a special role in the study of Müntz polynomials. A linear transformation $y = \alpha x + \beta$ does not preserve membership in $M(\Lambda_n)$ in general (unless $\beta = 0$), that is $Q \in M(\Lambda_n)$ does not necessarily imply that $R(x) := Q(\alpha x + \beta) \in M(\Lambda_n)$. An analogue of Newman's inequality on $[a,b]$, $a > 0$, cannot be obtained by a simple transformation. We can, however, prove the following result.

2. New Results

Theorem 2.1. *Let $n \geq 1$ be an integer. Let $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ be a set of $n+1$ distinct real numbers. Let $0 < a < b$. Then*

$$\frac{1}{3} \sum_{j=0}^{n} |\lambda_j| + \frac{1}{4\log(b/a)}(n-1)^2 \leq \sup_{0 \neq Q \in M(\Lambda_n)} \frac{\|xQ'(x)\|_{[a,b]}}{\|Q\|_{[a,b]}} \leq 11 \sum_{j=0}^{n} |\lambda_j| + \frac{128}{\log(b/a)}(n+1)^2\,.$$

Remarks 2.2. Of course, we can have $Q'(x)$ instead of $xQ'(x)$ in the above estimate; since an obvious corollary of the above theorem is

$$\frac{1}{3b} \sum_{j=0}^{n} |\lambda_j| + \frac{1}{4b\log(b/a)}(n-1)^2 \leq \sup_{0 \neq Q \in M(\Lambda_n)} \frac{\|Q'\|_{[a,b]}}{\|Q\|_{[a,b]}} \leq \frac{11}{a} \sum_{j=0}^{n} |\lambda_j| + \frac{128}{a\log(b/a)}(n+1)^2\,.$$

The reason we formulated Theorem 2.1 in the given form is that when $a \to 0$ then we obtain Theorem 1.1 (with worse absolute constants).

Theorem 2.1 was proved by P. Borwein and T. Erdélyi under the additonal assumptions that $\lambda_j \geq \delta j$ for each j with a constant $\delta > 0$ and with constants depending on a, b and δ instead of the absolute constants (see [1] or [2], for instance).

The novelty of Theorem 2.1 is the fact that $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ is an arbitrary set of $n+1$ distinct real numbers, not even the nonnegativity of the exponents λ_j is needed.

In the $L_p[a,b]$ norm ($p \geq 1$) we can establish the following.

Theorem 2.3. *Let $n \geq 1$ be an integer. Let $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ be a set of $n+1$ distinct real numbers. Let $0 < a < b$ and $1 \leq p < \infty$. Then there is a positive constant $c_1(a,b)$ depending only on a and b such that*

$$\sup_{0 \neq P \in M(\Lambda_n)} \frac{\|P'\|_{L_p[a,b]}}{\|P\|_{L_p[a,b]}} \leq c_1(a,b) \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right) .$$

Theorem 2.3 was proved by T. Erdélyi under the additonal assumptions that $\lambda_j \geq \delta j$ for each j with a constant $\delta > 0$ and with $c_1(a,b)$ replaced by $c_1(a,b,\delta)$, see [3]. The novelty of Theorem 2.3 is the fact again that $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ is an arbitrary set of $n+1$ distinct real numbers, not even the nonnegativity of the exponents λ_j is needed.

3. Lemmas

The following comparison theorem for Müntz polynomials is proved in [1, E.4 f] of Section 3.3].

Lemma 3.1 (A Comparison Theorem). *Suppose*

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\} \qquad \text{and} \qquad \Gamma_n := \{\gamma_0 < \gamma_1 < \cdots < \gamma_n\} ,$$

$\lambda_n \geq 0$, and $\lambda_j \leq \gamma_j$ for each $j = 0, 1, \dots, n$. Let $0 < a < b$. Then

$$\max_{0 \neq Q \in M(\Lambda_n)} \frac{|Q'(b)|}{\|Q\|_{[a,b]}} \leq \max_{0 \neq Q \in M(\Gamma_n)} \frac{|Q'(b)|}{\|Q\|_{[a,b]}} .$$

The following result is essentially proved by Saff and Varga [5]. They assume that $\Lambda := (\lambda_j)_{j=0}^{\infty}$ is an increasing sequence of nonnegative integers and $\delta = 1$ in the next lemma, however, this assumption can be easily dropped from their theorem, see [1, E.9 of Section 6.1]. In fact, their proof remains valid almost word for word, the modifications are straightforward.

Lemma 3.2 (The Interval Where the Norm of a Müntz Polynomial Lives). *Let*

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\} \qquad \text{and} \qquad \lambda_0 \geq 0 .$$

Let $0 \neq P \in M(\Lambda_n)$ and $Q(x) := x^{k\delta} P(x)$, where k is a nonnegative integer and δ is a positive real number. Let $\xi \in [0,1]$ be a point so that $|Q(\xi)| = \|Q\|_{[0,1]}$. Suppose $\lambda_j \geq \delta j$ for each j. Then

$$\left(\frac{k}{k+n} \right)^{2/\delta} \leq \xi.$$

4. Proofs

Proof of Theorem 2.1. First we prove the upper bound. Let $P \in M(\Lambda_n)$. We want to show that

$$y|P'(y)| \le \left(11 \sum_{j=0}^{n} |\lambda_j| + \frac{128(n+1)^2}{\log(b/a)} \right) \|P\|_{[a,b]}$$

for every $y \in [a,b]$. To this end we distinguish two cases. Without loss of generality we may assume that $\lambda_k = 0$ for some k, otherwise we add the 0 exponent by changing n for $(n+1)$.

Case 1. Let $y \in \left[(ab)^{1/2}, b\right]$. First we examine the subcase when $\lambda_0 := 0$. That is, we have $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n$. Let

$$0 < \delta := \min \left\{ 1\,, \min_{1 \le j \le n} \frac{\lambda_j}{j} \right\} \le 1\,,$$

Observe that the inequalities

$$\lambda_j \ge \delta j\,, \qquad j = 1, 2, \dots, n\,,$$

are satisfied. We define $Q(x) := x^{mn\delta} P(x)$, where with the choice $m := \lfloor \frac{8 \log 2}{\delta \log(b/a)} \rfloor$, using the inequality $2^{-2u} \le 1 - u$ $(0 \le u \le 1/2)$ we have

$$a = \sqrt{ab}\sqrt{\frac{a}{b}} \le \sqrt{ab}\, 2^{-\frac{4}{\delta(m+1)}} \le \sqrt{ab}\left(1 - \frac{1}{m+1}\right)^{2/\delta} = \sqrt{ab}\left(\frac{m}{m+1}\right)^{2/\delta}.$$

Scaling Newman's Inequality from $[0,1]$ to $[0,y]$, then using Lemma 3.2, we obtain

$$\begin{aligned} y|Q'(y)| &\le 11 \sum_{j=0}^{n} (\lambda_j + mn\delta) \|Q\|_{[0,y]} \\ &= 11 \left(\sum_{j=0}^{n} \lambda_j + mn(n+1)\delta \right) \|Q\|_{\left[y\left(\frac{m}{m+1}\right)^{2/\delta}, y\right]} \\ &\le 11 \left(\sum_{j=0}^{n} \lambda_j + mn(n+1)\delta \right) \|Q\|_{[a,y]}. \end{aligned}$$

Hence

$$\begin{aligned}
y\,|P'(y)| &\le |Q'(y)|y^{1-mn\delta} + mn\delta|P(y)| \\
&\le y^{-mn\delta}11\left(\sum_{j=0}^{n}\lambda_j + mn(n+1)\delta\right)\|Q\|_{[a,y]} + mn\delta\|P\|_{[a,y]} \\
&\le \left(11\sum_{j=0}^{n}\lambda_j + mn(11n+12)\delta\right)\|P\|_{[a,y]} \\
&\le \left(11\sum_{j=0}^{n}\lambda_j + \frac{128n^2}{\log(b/a)}\right)\|P\|_{[a,b]}.
\end{aligned}$$

This finishes the proof in Case 1 under the additional assumption $\lambda_0 := 0$. Now we drop this additional assumption. Suppose

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$$

and $\lambda_k = 0$ for some $0 \le k \le n$. For a fixed $\varepsilon > 0$ let

$$\Gamma_{n,\varepsilon} := \{\gamma_{0,\varepsilon} < \gamma_{1,\varepsilon} < \cdots < \gamma_{n,\varepsilon}\}$$

with

$$\gamma_{j,\varepsilon} := (j-k)\varepsilon\,, \qquad j = 0, 1, 2, \ldots k\,,$$

and

$$\gamma_{j,\varepsilon} := \lambda_j\,, \qquad j = k+1, k+2, \ldots, n\,.$$

If $\varepsilon > 0$ is sufficiently small, then by Lemma 3.1 we have

(4.1) $$\max_{0 \ne Q \in M(\Lambda_n)} \frac{y|Q'(y)|}{\|Q\|_{[a,y]}} \le \max_{0 \ne Q_\varepsilon \in M(\Gamma_{n,\varepsilon})} \frac{y|Q_\varepsilon'(y)|}{\|Q_\varepsilon\|_{[a,y]}}\,.$$

Let $Q_\varepsilon \in M(\Gamma_{n,\varepsilon})$. Then Q_ε is of the form

$$Q_\varepsilon(x) = x^{-k\varepsilon}R_\varepsilon(x)\,, \qquad R_\varepsilon \in \operatorname{span}\{x^{\gamma_0+k\varepsilon}, x^{\gamma_1+k\varepsilon}, \ldots, x^{\gamma_n+k\varepsilon}\},$$

where each $\gamma_j + k\varepsilon$ is nonnegative. Hence, using the upper bound of the theorem in the already proved case

$$\lambda_0 := 0\,, \;\; y \in \left[(ab)^{1/2}, b\right]$$

we obtain

$$y\,|R_\varepsilon'(y)| \le \left(11\sum_{j=0}^{n}(\gamma_{j,\varepsilon} + k\varepsilon) + \frac{128n^2}{\log(b/a)}\right)\|R_\varepsilon\|_{[a,y]}\,.$$

Recalling (4.1), and taking the limit when $\varepsilon > 0$ tends to 0, we obtain

$$\begin{aligned}
\max_{0 \neq Q \in M(\Lambda_n)} \frac{y|Q'(y)|}{\|Q\|_{[a,y]}} &\leq \lim_{\varepsilon \to 0+} \max_{0 \neq Q_\varepsilon \in M(\Gamma_{n,\varepsilon})} \frac{y|Q'_\varepsilon(y)|}{\|Q_\varepsilon\|_{[a,y]}} \\
&\leq \lim_{\varepsilon \to 0+} 11 \sum_{j=0}^{n} (\gamma_{j,\varepsilon} + k\varepsilon) + \frac{128n^2}{\log(b/a)} \\
&= 11 \sum_{j=k+1}^{n} \lambda_j + \frac{128n^2}{\log(b/a)} \\
&\leq 11 \sum_{j=0}^{n} |\lambda_j| + \frac{128n^2}{\log(b/a)}
\end{aligned}$$

The proof of the upper bound of the theorem is now finished in Case 1.

Case 2. Let $y \in \left[a, (ab)^{1/2}\right]$. Suppose again that

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$$

and $\lambda_k = 0$ for some $0 \leq k \leq n$. Associated with $P \in M(\Lambda_n)$ let $\widetilde{P} \in M(\widetilde{\Lambda}_n)$ be defined by

$$\widetilde{P}(x) := P(ab/x)\,,$$

$$\widetilde{\Lambda_n} := \{\widetilde{\lambda}_0 < \widetilde{\lambda}_1 < \cdots < \widetilde{\lambda}_n\} := \{-\lambda_n < -\lambda_{n-1} < \cdots < -\lambda_0\}\,.$$

Using the upper bound of the theorem in the already proved Case 1 with $\widetilde{P} \in M(\widetilde{\Lambda}_n)$ and $\widetilde{y} = ab/y \in \left[(ab)^{1/2}, b\right]$, we obtain

$$\begin{aligned}
y|P'(y)| = |\widetilde{P}'(\widetilde{y})|(ab/y) &\leq \left(11 \sum_{j=0}^{n} |\widetilde{\lambda}_j| + \frac{128(n+1)^2}{\log(b/a)} \right) \|\widetilde{P}\|_{[a,b]} \\
&= \left(11 \sum_{j=0}^{n} |\lambda_j| + \frac{128(n+1)^2}{\log(b/a)} \right) \|P\|_{[a,b]}\,,
\end{aligned}$$

and the proof is finished in Case 2 as well.

Now we show the lower bound of the theorem. Suppose

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}\,,$$

and $0 \leq k \leq n$ is chosen so that $\lambda_j < 0$ for all $j = 1, 2, \dots, k$ and $\lambda_j \geq 0$ for all $j = k+1, k+2, \dots, n$. Let

$$\Lambda_n^- := \{-\lambda_k < -\lambda_{k-1} < \cdots < -\lambda_0\}$$

and

$$\Lambda_n^+ := \{\lambda_{k+1} < \lambda_{k+2} < \cdots < \lambda_n\}$$

The lower bound of Theorem 1.1 (combined with a linear scaling, if necessary) shows the existence of a $Q \in M(\Lambda_n^+)$ for which

$$|Q'(1)| \geq \frac{2}{3}\left(\sum_{j=k+1}^{n} \lambda_j\right) \|Q\|_{[0,1]} .$$

Then $R(x) := Q(x/b) \in M(\Lambda_n)$ satisfies

$$\begin{aligned}\|xR'(x)\|_{[a,b]} \geq b|R'(b)| = |Q'(1)| &\geq \frac{2}{3}\left(\sum_{j=k+1}^{n} |\lambda_j|\right) \|Q\|_{[0,1]} \\ &\geq \frac{2}{3}\left(\sum_{j=k+1}^{n} |\lambda_j|\right) \|R\|_{[a,b]}.\end{aligned}$$

Similarly, the lower bound of Theorem 1.1 (combined with a linear scaling if necessary) shows the existence of a $Q \in M(\Lambda_n^-)$ for which

$$|Q'(1)| \geq \frac{2}{3}\left(\sum_{j=0}^{k} (-\lambda_j)\right) \|Q\|_{[0,1]} .$$

Then $R(x) := Q(a/x) \in M(\Lambda_n)$ satisfies

$$\begin{aligned}\|xR'(x)\|_{[a,b]} \geq a|R'(a)| = |Q'(1)| &\geq \frac{2}{3}\left(\sum_{j=0}^{k} |\lambda_j|\right) \|Q\|_{[0,1]} \\ &\geq \frac{2}{3}\left(\sum_{j=0}^{k} |\lambda_j|\right) \|R\|_{[a,b]}.\end{aligned}$$

The two observations above already give

$$\frac{1}{3}\sum_{j=0}^{n} |\lambda_j| \leq \sup_{0 \neq Q \in M(\Lambda_n)} \frac{\|xQ'(x)\|_{[a,b]}}{\|Q\|_{[a,b]}}.$$

To prove that

$$\frac{(n-1)^2}{4\log(b/a)} \leq \sup_{0 \neq \in M(\Lambda_n)} \frac{\|xQ'(x)\|_{[a,b]}}{\|Q\|_{[a,b]}} \tag{4.2}$$

we argue as follows. Let

$$Q_{m,\varepsilon}(x) := T_m \left(\frac{2x^\varepsilon}{b^\varepsilon - a^\varepsilon} - \frac{b^\varepsilon + a^\varepsilon}{b^\varepsilon - a^\varepsilon} \right) \in \operatorname{span}\{1, x^\varepsilon, x^{2\varepsilon}, \dots, x^{m\varepsilon}\},$$

where $T_m(x) = \cos(m \arccos x)$, $x \in [-1,1]$, is the Chebyshev polynomial of degree m. Then

$$\begin{aligned} \frac{b|Q'_{m,\varepsilon}(b)|}{\|Q_{m,\varepsilon}\|_{[a,b]}} &= |T'_m(1)| \frac{2}{b^\varepsilon - a^\varepsilon} \varepsilon b^\varepsilon \\ &= \frac{2m^2}{\varepsilon^{-1}(b^\varepsilon - 1) - \varepsilon^{-1}(a^\varepsilon - 1)} b^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} \frac{2m^2}{\log b - \log a}. \end{aligned} \tag{4.3}$$

Now suppose, as before,

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\},$$

and $0 \leq k \leq n$ is chosen so that $\lambda_j < 0$ for all $j = 1, 2, \dots, k$ and $\lambda_j \geq 0$ for all $j = k+1, k+2, \dots, n$. Using Lemma 3.1 and (4.3), we obtain that for $k \leq n-1$ there is a

$$Q \in \operatorname{span}\{x^{\lambda_{k+1}}, x^{\lambda_{k+2}}, \dots, x^{\lambda_n}\}$$

such that

$$\frac{2(n-k-1)^2}{\log b - \log a} \leq \frac{b|Q'(b)|}{\|Q\|_{[a,b]}}, \tag{4.4}$$

Similarly, using Lemma 3.1 and (4.4), we obtain for $k \geq 0$ that there is an

$$R \in \operatorname{span}\{x^{-\lambda_0}, x^{-\lambda_1}, \dots, x^{-\lambda_k}\}$$

such that

$$\frac{2k^2}{\log b - \log a} \leq \frac{b|R'(b)|}{\|R\|_{[a,b]}},$$

and hence for

$$Q \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_k}\}$$

defined by $Q(x) := R(ab/x)$ we have

$$\frac{2k^2}{\log b - \log a} = \frac{2k^2}{\log b - \log a} \leq \frac{a|Q'(a)|}{\|Q\|_{[a,b]}}. \tag{4.5}$$

Now (4.2) follows from (4.4) and (4.5), and the proof of the lower bound of the theorem is finished. □

Proof of Theorem 2.2. One can copy the proof in [3] by putting the upper bound of Theorem 1.1 in the appropriate place in the arguments. We omit the details. □

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