NOTES ON INEQUALITIES WITH DOUBLING WEIGHTS

Tamás Erdélyi

Abstract. Various important weighted polynomial inequalities, such as Bernstein, Marcinkiewicz, Nikolskii, Schur, Remez, etc. inequalities, have been proved recently by Giuseppe Mastroianni and Vilmos Totik under minimal assumptions on the weights. In most of the cases this minimal assumption is the doubling condition. Sometimes however, like in the weighted Nikolskii inequality, the slightly stronger A_∞ condition is used. Throughout their paper the L_p norm is studied under the assumption $1 \leq p < \infty$. In this note we show that their proofs can be modified so that many of their inequalities hold even if $0 < p < 1$. The crucial tool is an estimate for quadrature sums for the pth power ($0 < p < \infty$ is arbitrary) of trigonometric polynomials established by Lubinsky, Máté, and Nevai. For technical reasons we discuss only the trigonometric cases.

1. The Weights

For Introduction we refer to Sections 1 and 2 of the Mastroianni-Totik paper [12] and the references therein. See [1] – [9], [11], and [13]. Here we just formulate the original and some equivalent definitions that we shall use. In Sections 2 – 7 we shall work with integrable, 2π-periodic weight functions W satisfying the so-called doubling condition:

$$W(2I) \leq LW(I) \tag{1.1}$$

for intervals $I \subset \mathbb{R}$, where L is a constant independent of I, $2I$ is the interval with length $2|I|$ ($|I|$ denotes the length of the interval I) and with midpoint at the midpoint of I, and

$$W(I) := \int_I W(u)\,du\,.$$

In other words, W has the doubling property if the measure of a twice enlarged interval is less than a constant times the measure of the original interval. An integrable, 2π periodic weight function on $\mathbb{R}$ satisfying the doubling condition will be called a doubling weight. We start with the following elementary observation.

1991 *Mathematics Subject Classification.* Primary: 41A17.

Key words and phrases. Bernstein's Inequality, Marcinkiewicz inequality, Christoffel function, Remez inequality, Nikolskii inequality, doubling weights, A_∞ weights.

Research is supported, in part, by NSF under Grant No. DMS–9623156

Lemma 1.1. *Associated with an integrable, 2π-periodic weight function W on $\mathbb{R}$, let*

$$W_n(x) := n \int_{x-1/n}^{x+1/n} W(u)\, du\,.$$

Then W is a doubling weight if and only if there are constants $s > 0$ and $K > 0$ depending only on W such that

$$W_n(y) \le K(1 + n|x-y|)^s W_n(x)\,.$$

holds for all $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$. Here, if L is the doubling constant, then $s = \log_2 L$ and $K := L$ are suitable choices.

2. The Main Theorem

The following basic theorem is stated for $1 \le p < \infty$ in [12]. Here we extend its validity to the case $0 < p \le 1$. The proof is a modification of Mastroianni's and Totik's arguments, but for the sake of completeness we present the whole proof. Also, note that the case $1 \le p < \infty$ follows immediately from the case $0 < p \le 1$. Let $\mathcal{T}_n$ denote the class of all real trigonometric polynomials of degree at most n.

Theorem 2.1. *Let W be a doubling weight, and let W_n be as in Lemma 1.1. Let $0 < p < \infty$ be arbitrary. Then there is a constant $C > 0$ depending only on p and on the doubling constant L such that for every $T_n \in \mathcal{T}_n$ we have*

$$C^{-1} \int_{-\pi}^{\pi} |T_n|^p W \le \int_{-\pi}^{\pi} |T_n|^p W_n \le C \int_{-\pi}^{\pi} |T_n|^p W\,. \tag{2.1}$$

Seemingly we have not gained too much, but, as the next lemma shows, W_n is very close to be a nonnegative trigonometric polynomial of degree at most n.

Theorem 2.2. *Suppose W satisfies the doubling condition. Let $0 < p < \infty$. Then there are constants $B_1 > 0$, $B_2 > 0$, and $B_3 > 0$ depending only on p and on the doubling constant L, and for each $n \in \mathbb{N}$ there is a nonnegative trigonometric polynomial Q_n of degree at most $N := ((\log_2 L)/p + 4)n$ so that*

$$B_1 W_n(x) \le Q_n(x)^p \le B_2 W_n(x) \tag{2.2}$$

and

$$|Q_n'(x)|^p \le B_3 n^p W_n(x) \tag{2.3}$$

uniformly in $x \in \mathbb{R}$.

Proof of Theorem 2.2. We define $2m$ as the smallest even number not less than $(\log_2 L)/p + 2$. In particular $2m \le (\log_2 L)/p + 4$. Let

$$S_n(t) = n^{-(2m-1)} \left(\frac{\sin((n+1/2)t)}{\sin(t/2)} \right)^{2m} \tag{2.4}$$

be the Jackson kernel. Then S_n is a trigonometric polynomial of degree at most $2mn \le ((\log_2 L)/p + 4)n$. It is well known that

$$2^{-2m}(\cos 1)^{2m} n^{-l} \le \int_{-\pi}^{\pi} |t|^l S_n(t)\,dt \le 4\pi^{2m} n^{-l} \tag{2.5}$$

for each $0 \le l < 2m-2$. Indeed, the inequalities

$$\begin{aligned} S_n(t) &\le 9^m n\,, & |t| &\le 1/n\,,\\ S_n(t) &\le \pi^{2m} n^{-(2m-1)} t^{-2m}\,, & 1/n \le |t| &\le \pi\,, \end{aligned}$$

are easy to establish, from where

$$\int_{-\pi}^{\pi} |t|^l S_n(t)\,dt \le \left(\frac{2\pi^{2m}}{|l-2m+1|} + 2\cdot 9^m\right) n^{-l} \le 4\pi^{2m} n^{-l}$$

is obvious for each $0 \le l < 2m-2$. On the other hand

$$S_n(t) \ge (\cos 1)^{2m} n\,, \qquad |t| \le 1/n\,,$$

from where

$$\int_{-\pi}^{\pi} |t|^l S_n(t)\,dt \ge 2^{-l}(\cos 1)^{2m} n^{-l}$$

for each $0 \le l < 2m-2$ follows. By this (2.5) is completely shown. It clearly implies that with $s = \log_2 L$ we have

$$2^{-2m}(\cos 1)^{2m} \le \int_{-\pi}^{\pi} (1+n|t|)^{s/p} S_n(t)\,dt \le 4(2\pi)^{2m}\,. \tag{2.6}$$

Now we define

$$Q_n(x) := \int_{-\pi}^{\pi} W_n(t)^{1/p} S_n(x-t)\,dt\,. \tag{2.7}$$

Then Q_n is a nonnegative trigonometric polynomial of degree $2mn$ and

$$Q_n'(x) = \int_{-\pi}^{\pi} W_n(t)^{1/p} S_n'(x-t)\,dt\,. \tag{2.8}$$

Applying Lemma 1.1 and (2.6), we obtain

$$\begin{aligned} Q_n(x) &= \int_{-\pi}^{\pi} W_n(x-t)^{1/p} S_n(t)\,dt\\ &\le \int_{-\pi}^{\pi} W_n(x)^{1/p} K^{1/p} (1+n|t|)^{s/p} S_n(t)\,dt\\ &\le L^{1/p} 4(2\pi)^{(\log_2 L)/p+4} W_n(x)^{1/p}\,. \end{aligned}$$

The opposite inequality is simpler. For $|t| \le 1/(2n)$, we have

$$W_n(x) \le L W_n(x-t)$$

and

$$S_n(t) \geq (\cos 1)^{2m} n\,,$$

therefore

$$\begin{aligned}
Q_n(x) &\geq \int_{-1/(2n)}^{1/(2n)} W_n(x-t)^{1/p} S_n(t)\,dt \\
&\geq L^{-1/p} (\cos 1)^{2m} W_n(x)^{1/p} \int_{-1/(2n)}^{1/(2n)} n\,dt \\
&\geq L^{-1/p} (\cos 1)^{(\log_2 L)/p+4} W_n(x)^{1/p}\,,
\end{aligned}$$

and the proof of (2.2) is complete. To prove (2.3), observe that

$$\begin{aligned}
|S_n'(t)| &\leq 2 \cdot 9^m m n^2\,, && |t| \leq 1/n\,, \\
|S_n'(t)| &\leq 7\pi^{2m} m n^{-(2m-2)} t^{-2m}\,, && 1/n \leq |t| \leq \pi\,,
\end{aligned}$$

which follows from direct differentiation and from Bernstein's inequality

$$\max_{-\pi \leq t \leq \pi} |S_n'(t)| \leq 2mn \max_{-\pi \leq t \leq \pi} |S_n(t)| \leq 2 \cdot 9^m m n^2\,,$$

since (2.4) implies

$$\max_{-\pi \leq t \leq \pi} |S_n(t)| \leq 9^m n\,.$$

We have

$$\int_{-\pi}^{\pi} |t|^l S_n'(t)\,dt \leq \left(\frac{2 \cdot 7\pi^{2m}}{|l-2m+1|} + 4 \cdot 9^m m \right) n^{1-l} \leq 18\pi^{2m} m n^{1-l}\,.$$

It clearly implies that with $s = \log_2 L$ we have

$$\int_{-\pi}^{\pi} (1+n|t|)^{s/p} S_n'(t)\,dt \leq 18(2\pi)^{2m} mn\,. \tag{2.9}$$

Now combining (2.8) and (2.9), we obtain

$$\begin{aligned}
Q_n'(x) &= \int_{-\pi}^{\pi} W_n(x-t)^{1/p} S_n'(t)\,dt \\
&\leq \int_{-\pi}^{\pi} W_n(x)^{1/p} K^{1/p} (1+n|t|)^{s/p} S_n'(t)\,dt \\
&\leq L^{1/p} 18 (2\pi)^{(\log_2 L)/p+4} \left(\frac{\log_2 L}{p} + 4 \right) n W_n(x)^{1/p}\,.
\end{aligned}$$

By this (2.3) is proved. $\square$

Proof of Theorem 2.1. As we have already remarked the case $1 \leq p < \infty$ of the theorem follows immediately from the case $0 < p \leq 1$. To see this, if $1 \leq p < \infty$ then let m be the smallest integer not less than p. The $1 \leq p < \infty$ part of the theorem now follows by applying our theorem with n and p replaced by nm and $p/m \le 1$, respectively.

So from now on let $0 < p \le 1$. However, our next observation is valid for all $0 < p < \infty$. Namely we verify that there is a constant B_4 depending only on p and the doubling constant L such that for every $T_n \in \mathcal{T}_n$ we have

$$\int_{-\pi}^{\pi} |T_n'|^p W_n \le B_4 n^p \int_{-\pi}^{\pi} |T_n|^p W_n \,. \tag{2.10}$$

That is, Bernstein's inequality in $L_p, 0 < p < \infty$, holds for trigonometric polynomials T_n of degree at most n with the weight W_n. Indeed, by Theorem 2.2,

$$B_1 \int_{-\pi}^{\pi} |T_n'|^p W_n \le \int_{-\pi}^{\pi} |T_n' Q_n|^p \le B_2 \int_{-\pi}^{\pi} |T_n'|^p W_n \,.$$

Here

$$T_n' Q_n = (T_n Q_n)' - T_n Q_n' \,,$$

therefore

$$\begin{aligned}\int_{-\pi}^{\pi} |T_n'|^p W_n &\le 2^p \left(\int_{-\pi}^{\pi} |(T_n Q_n)'|^p + \int_{-\pi}^{\pi} |T_n Q_n'|^p \right) \\ &\le 2^p (n+N)^p \int_{-\pi}^{\pi} |T_n Q_n|^p + 2^p B_3 n^p \int_{-\pi}^{\pi} |T_n|^p W_n \\ &\le B_4 n^p \int_{-\pi}^{\pi} |T_n|^p W_n\end{aligned}$$

with a constant $B_4 > 0$ depending only on p and on the doubling constant L, where at the first inequality we used that $(A+B)^p \le 2^p(A^p + B^p)$ for arbitrary $A, B, p > 0$; at the second inequality, to estimate the first term, we used Bernstein's inequality [1] in L_p for $0 < p < \infty$ and for trigonometric polynomials of degree at most $n+N$ (N is defined in Theorem 2.2); while to estimate the second term, the bound for $|Q_n'|$ given by Theorem 2.2 has been used; in the third inequality Theorem 2.2 has been used again. Thus the proof of (2.10) is complete.

Now let M be a large positive integer to be chosen later, and set

$$I_k := \left[\frac{2k\pi}{Mn}, \frac{2(k+1)\pi}{Mn} \right], \qquad k = 0, 1, \dots, Mn-1 \,.$$

Let $\zeta_k \in I_k$ be the place where $|T_n|$ attains its maximum on I_k, and let $\theta_k \in I_k$ be a place where W_n attains its maximum on I_k (note that W_n is positive continuous). Finally we define

$$R_n := \sum |T_n(\zeta_k)|^p W_n(\theta_k) \,,$$

where, and in what follows, the summation is taken for $k = 0, 1, \dots, Mn-1$. Let $\xi_k \in I_k$ be arbitrary. Using $0 < p \le 1$, we have

$$\begin{aligned}R_n - \sum |T_n(\xi_k)|^p W_n(\theta_k) &= \sum \left(|T_n(\zeta_k)|^p - |T_n(\xi_k)|^p \right) W_n(\theta_k) \\ &\le \sum \left(|T_n(\zeta_k)| - |T_n(\xi_k)| \right)^p W_n(\theta_k) \le \sum |T_n(\zeta_k) - T_n(\xi_k)|^p \, W_n(\theta_k) \\ &\le \sum |T_n'(\tau_k)(\zeta_k - \xi_k)|^p \, W_n(\theta_k) \le (Mn)^{-p} \sum |T_n'(\tau_k)|^p W_n(\theta_k)\end{aligned}$$

with appropriate $\tau_k \in I_k$. Using the fact that for $u, v \in I_k$ we have

$$L^{-1}W_n(u) \le W_n(v) \le LW_n(u)$$

uniformly, then applying Theorem 2.2, we can continue

$$\le (Mn)^{-p} L \sum |T_n'(\tau_k)|^p W_n(\tau_k) \le (Mn)^{-p} L B_1^{-1} \sum |T_n'(\tau_k)|^p \, |Q_n(\tau_k)|^p \,.$$

Now using Theorem 2 of Lubinsky, Máté, and Nevai from [10] (see Lemma 2.3 after the proof), then applying Theorem 2.2, and (2.10), we can continue

$$\begin{aligned}
&\le (Mn)^{-p} L B_1^{-1} \frac{9}{2} Mn \int_0^{2\pi} |T_n' Q_n|^p \\
&\le (Mn)^{-p} L B_1^{-1} \frac{9}{2} B_2 Mn \int_0^{2\pi} |T_n'|^p W_n \\
&\le (Mn)^{-p} L B_1^{-1} \frac{9}{2} B_2 B_4 Mn n^p \int_0^{2\pi} |T_n|^p W_n \\
&= L B_1^{-1} \frac{9}{2} B_2 B_4 M^{1-p} n \int_0^{2\pi} |T_n|^p W_n \,,
\end{aligned}$$

where we assume that $M \ge (\log_2 L)/p + 5$, that is $Mn \ge ((\log_2 L)/p + 5)n \ge N + n$, ($N$ is defined in Theorem 2.2), and where B_1 and B_2 are the same as in Theorem 2.2, while B_4 is defined earlier in this proof.

Now it is clear that

$$\begin{aligned}
\int_0^{2\pi} |T_n|^p W_n &= \sum \int_{I_k} |T_n|^p W_n \\
&\le \sum |I_k| |T_n(\zeta_k)|^p W_n(\theta_k) = \frac{2\pi}{Mn} R_n \,.
\end{aligned}$$

So we have proven

$$R_n - \sum |T_n(\xi_k)|^p W_n(\theta_k) \le L B_1^{-1} \frac{9}{2} B_2 B_4 \, \frac{2\pi}{M^p} \, R_n$$

from which it follows that

$$R_n - \sum |T_n(\xi_k)|^p W_n(\theta_k) \le \frac{1}{2} R_n$$

provided

$$M \ge \left(4\pi L B_1^{-1} \frac{9}{2} B_2 B_4 \right)^{1/p} + \frac{\log_2 L}{p} + 5 \,. \tag{2.11}$$

Using also that

$$L^{-1} W_n(\theta_k) \le W_n(\eta_k) \le L W_n(\theta_k)$$

uniformly whenever $\eta_k \in I_k$, we obtain that for any $\xi_k, \eta_k \in I_k$ we have

$$\sum |T_n(\xi_k)|^p W_n(\eta_k) \geq \frac{1}{2L} R_n$$

provided (2.11). In particular, this is true for the points ξ_k and η_k where $|T_n|$ and W_n, respectively, attain their minimum on I_k, from which we obtain that all possible sums

$$\sum |T_k(u_k)|^p W_n(v_k)\,, \qquad u_k,\, v_k \in I_k$$

are uniformly of the same size (they are between $(2L)^{-1}R_n$ and R_n). If we also observe that $v_k \in I_k$ implies

$$n \int_{I_k} W \leq W_n(v_k) \leq L^{(\log_2 M)+1} n \int_{I_k} W\,,$$

it follows that

$$\frac{n}{2L} \sum \int_{I_k} \left(\max_{v \in I_k} |T_n(v)|^p \right) W(u)\,du \leq \sum |T_n(u_k)|^p W_n(v_k)$$

$$\leq 2L L^{(\log_2 M)+1} n \sum \int_{I_k} \left(\min_{v \in I_k} T_n(v)|^p \right) W(u)\,du$$

whenever $u_k, v_k \in I_k$. Setting $u_k = v_k = 2k\pi/(Mn) + t$ and integrating this with respect to $t \in [0, 1/(Mn)]$, it follows that

$$\frac{1}{2ML} \sum \int_{I_k} \left(\max_{v \in I_k} |T_n(v)|^p \right) W(u)\,du \leq \sum \int_{I_k} |T_n(t)|^p W_n(t)\,dt$$

$$\leq 2L \frac{L^{(\log_2 M)+1}}{M} \sum \int_{I_k} \left(\min_{v \in I_k} T_n(v)|^p \right) W(u)\,du\,.$$

We now conclude that

$$\frac{1}{2ML} \sum \int_{I_k} |T_n(t)|^p W(t)\,dt \leq \sum \int_{I_k} |T_n(t)|^p W_n(t)\,dt$$

$$\leq 2L \frac{L^{(\log_2 M)+1}}{M} \sum \int_{I_k} |T_n(t)|^p W(t)\,dt\,,$$

which we wanted to prove. $\square$

We remark that the crucial step of the proof of Theorem 2.1 is two applications of Theorem 2 of Lubinsky, Máté, and Nevai from [10]. This can be formulated as follows.

Lemma 2.3. *Let $0 < p < \infty$. Let ψ be a convex, nonnegative, and nondecreasing function on $[0, \infty)$. Let*

$$\delta := \min\{\tau_2 - \tau_1, \tau_3 - \tau_2, \dots, \tau_m - \tau_{m-1}, 2\pi - (\tau_m - \tau_1)\} > 0\,.$$

Then

$$\sum_{j=1}^{m} \psi(|S_n(\tau_j)|^p) \le (2n+\delta^{-1})(2\pi)^{-1} \int_0^{2\pi} \psi(|S_n(u)|^p(p+1)e/2)\, du$$

for every trigonometric polynomial S_n of degree at most n.

To demonstrate the power of Theorem 2.1 (together with Theorem 2.2) we prove Bernstein's Inequality in $L_p, 0<p<\infty$, with doubling weighs. This has been done in the case $1 \le p < \infty$ in the Mastroianni-Totik paper [12]. We state the extended versions of most of the remaining results of [12]. The proofs are left to the reader who needs to observe only that in the appropriate places of the proofs in [12], one needs to apply our Theorems 2.1 and 2.2 rather than their Theorems 3.1 and 3.2.

3. Bernstein's Inequality in $L_p, 0<p<\infty$, with Doubling Weights

Bernstein inequality plays a basic role in proving inverse theorems of approximation. The next result is a Bernstein-type inequality in $L_p, 0<p<\infty$, with respect to doubling weights.

Theorem 3.1. *Let W be a doubling weight, and let $0<p<\infty$ arbitrary. Then there is a constant $C>0$ depending only on p and on the doubling constant L so that*

$$\int_{-\pi}^{\pi} |T_n'|^p W \le Cn^p \int_{-\pi}^{\pi} |T_n|^p W$$

holds for every $T_n \in \mathcal{T}_n$.

Proof of Theorem 3.1. With the help of Theorem 2.1 and with a piece of its proof, the proof of the theorem is a triviality now. We have already proven the theorem with W replaced by W_n, see (2.9). What remains to observe is that Theorem 2.1 allows us to replace W_n by W. $\square$

4. The Christoffel Function for $0<p<\infty$ with Doubling Weights

For $0<p<\infty$ and $x \in \mathbb{R}$, we define

$$\lambda_n(x) = \lambda_n(W,p,x) := \inf_{|T_n(x)|=1} \int_{-\pi}^{\pi} |T_n|^p W\,,$$

where the infimum is taken for all $T_n \in \mathcal{T}_n$ for which $|T_n(x)|=1$. Estimates for the Christoffel functions are useful in comparing different norms of trigonometric polynomials, and (in the algebraic case) their magnitude plays an important role in the study of orthogonal polynomials (mostly in the classical $p=2$ setting). The size of $\lambda_n(W,p,x)$, where W is a doubling weight and $0<p<\infty$ is arbitrary is given by the next theorem.

Theorem 4.1. *Let W be a doubling weight, and let $0 < p < \infty$ be arbitrary. Then there is a constant $C > 0$ depending only on p and on the doubling constant L so that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have*

$$\frac{C^{-1}}{n} W_n(x) \leq \lambda_n(W, p, x) \leq \frac{C}{n} W_n(x) \,.$$

5. The Marcinkiewicz Inequalities for $0 < p < \infty$ with Doubling Weights

Marcinkiewicz-type inequalities offer a basic tool by which the (weighted) L_p norm of a trigonometric polynomial can be replaced by a finite sum. The next theorem offers such inequalities involving doubling weights.

Theorem 5.1. *Let W be a doubling weight, and let $0 < p < \infty$ be arbitrary. Then there are two constants $M > 0$ and $C > 0$ depending only on p and on the doubling constant L such that*

$$\int_{-\pi}^{\pi} |T_n|^p W \leq \frac{C}{n} \sum_{j=0}^{S} T_n(\xi_j)|^p W_n(\xi_j)$$

for every $T_n \in \mathcal{T}_n$ provided the points $\xi_0 < \xi_1 < \cdots < \xi_S$ satisfy $\xi_{j+1} - \xi_j \leq 1/(Mn)$ and $\xi_S \geq \xi_0 + 2\pi$. Furthermore, for every $M > 0$ there is a constant $C > 0$ depending only on p, M and on the doubling constant L such that

$$\frac{1}{n} \sum_{j=0}^{S} |T_n(\xi_j)|^p W_n(\xi_j) \leq C \int_{-\pi}^{\pi} |T_n|^p W$$

for every $T_n \in \mathcal{T}_n$ provided the points $\xi_0 < \xi_1 < \cdots < \xi_S$ satisfy $\xi_{j+1} - \xi_j \geq 1/(Mn)$ and $\xi_S \leq \xi_0 + 2\pi$.

6. Schur Inequality for $0 < p < \infty$ with Doubling Weights

Sometimes we need to get rid of a factor in an algebraic polynomial or trigonometric polynomial and one needs an estimate to see how the norm changes under such a transformation. Schur-type inequalities are used in such a situation. The next theorem offers a Schur-type inequality involving doubling weights and generalized Jacobi weights.

Theorem 6.1. *Let W be a doubling weight, and let $0 < p < \infty$ be arbitrary. Let H be a generalized Jacobi weight of the form*

$$H(t) = h(t) \prod_{j=1}^{k} |t - x_j|^{\gamma_j} \,, \qquad x_j, t \in [-\pi, \pi) \,, \quad \gamma_j > 0 \,,$$

where h is a positive measurable function bounded away from 0 and ∞. Then

$$\int_{-\pi}^{\pi} |T_n|^p W \leq C n^{\Gamma} \int_{-\pi}^{\pi} |T_n|^p W H \,,$$

for every $T_n \in \mathcal{T}_n$, where $\Gamma := \max\{\gamma_j : j = 1, 2, \ldots, k\}$ and C is a constant independent of n.

7. Remez Inequality for $0 < p < \infty$ with A_∞ Weights

The periodic weight W on $\mathbb{R}$ is called an A_∞ weight (or is said to satisfy the A_∞ condition) if for every $\alpha > 0$ there is a $\beta > 0$ such that

$$W(E) \geq \beta W(I)$$

for any interval $I \subset \mathbb{R}$ and any measurable set $E \subset I$ with $|E| \geq \alpha |I|$. Similarly to doubling weights, many equivalent definitions are known, see [12], for instance. A_∞ weights are obviously doubling weights; the A_∞ condition is slightly stronger than the doubling condition. The following full analogue of the trigonometric Remez inequality [6] (see also [12]) holds with A_∞ weights.

Theorem 7.1. *Let W be an A_∞ weight, and let $0 < p < \infty$ be arbitrary. Then there is a constant $C > 0$ depending only on p and on the weight W so that if $T_n \in \mathcal{T}_n$ and E is a measurable subset of $[0, 2\pi]$ of measure at most $s \in (0, 1]$, then*

$$\int_{-\pi}^{\pi} |T_n|^p W \leq C^{1+ns} \int_{[0,2\pi]\setminus E} |T_n|^p W\,.$$

The same inequality with doubling weights holds provided that the exceptional set E is not too complicated. We have

Theorem 7.2. *Let W be a doubling weight, and let $0 < p < \infty$ be arbitrary. Then there is a constant $C > 0$ depending only on p and on the doubling constant L so that if $T_n \in \mathcal{T}_n$ and E is a measurable subset of $[0, 2\pi]$ of measure at most $s \in (0, 1]$ that is a union of intervals of length at least c/n, then*

$$\int_{-\pi}^{\pi} |T_n|^p W \leq \left(\frac{C}{c}\right)^{1+ns} \int_{[0,2\pi]\setminus E} |T_n|^p W\,.$$

8. Nikolskii Inequality for $0 < p < q < \infty$ with A_∞ Weights

Sometimes we would like to compare the L_p and L_q norms of trigonometric polynomials. The following theorem offers such Nikolskii-type inequalities with respect to A_∞ weights.

Theorem 8.1. *Let W be an A_∞ weight and let $0 < p < q < \infty$ be arbitrary. Then there is a constant $C > 0$ depending only on p and q and on the weight W so that*

$$\left(\int_{-\pi}^{\pi} |T_n| W\right)^{1/q} \leq C n^{1/p-1/q} \left(\int_{-\pi}^{\pi} |T_n|^p W^{p/q}\right)^{1/p}$$

for all $T_n \in \mathcal{T}_n$.

References

1. V.V. Arestov, *On inequalities for trigonometric polynomials and their derivatives*, Izv. Akad. Nauk SSSR, Ser. Mat. **45** (1981), 3–22.

2. V.M. Badkov, *Asymptotic and extremal properties of orthogonal polynomials in the presence of singularities in the weight*, Proc. Steklov Math. Inst. (1994), 37–82.

3. T. Erdélyi, *Bernstein and Markov type inequalities for generalized nonnegative polynomials*, Canadian J. Math. **43(3)** (1991), 495–505.

4. T. Erdélyi, *Remez-type inequalities and their applications*, J. Comput. & Appl. Math. **47** (1993), 167–209.

5. T. Erdélyi, *Nikolskii-type inequalities for generalized polynomials and zeros of orthogonal polynomials*, J. Approx. Theory **67** (1991), 80–92.

6. T. Erdélyi, *Remez-type inequalities on the size of generalized polynomials*, J. London Math. Soc. **45** (1992), 255–264.

7. C. Fefferman and B. Muckenhoupt, *Two nonequivalent conditions for weight functions*, Proc. Amer. Math. Soc. **45** (1994), 99–104.

8. K.G. Ivanov and V. Totik, *Fast decreasing polynomials*, Constr. Approx. **6** (1990), 1–20.

9. N.X. Ky, *On approximation by trigonometric polynomials in L^p_μ-spaces*, Studia Sci. Math. Hungar. **28** (1993), 183–188.

10. D. Lubinsky, A. Máté, and P. Nevai, *Quadrature sums involving pth powers of polynomials*, SIAM J. Math. Anal. **18** (1987), 531–544.

11. G. Mastroianni, *Some weighted polynomial inequalities*, J. Comput. & Appl. Math. **65** (1995), 279–292.

12. G. Mastroianni and V. Totik, *Weighted polynomial inequalities with doubling and A_∞ weights*, Constr. Approx. (to appear).

13. E.M. Stein, *Harmonic Analysis*, Princeton University Press, Princeton, New Jersey, 1993.

Department of Mathematics, Texas A&M University, College Station, Texas 77843 USA (T. Erdélyi)