FUNCTIONS WITH IDENTICAL L_p NORMS

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Abstract. Suppose $P := (p_j)_{j=1}^\infty$ is a sequence of distinct real numbers $p_j > 0$. We prove that the equalities

$$\|f\|_p = \|g\|_p \,, \qquad p \in P \,,$$

imply

$$\mu(\{x \in E : |f(x)| < \alpha\}) = \mu(\{x \in E : |g(x)| < \alpha\}) \,, \qquad \alpha \geq 0 \,,$$

whenever $0 < \mu(E) < \infty$ and $f, g \in L_\infty(E)$ if and only if $\displaystyle\sum_{j=1}^\infty \frac{p_j}{p_j^2+1} = \infty$.

1. Introduction

Associated with a measure space $(E, \mathcal{A}, \mu)$ let

$$\|f\|_p := \left(\int_E |f(x)|^p \, d\mu(x) \right)^{1/p} \,, \qquad p > 0 \,,$$

and

$$\|f\|_\infty := \inf\{\alpha \in \mathbb{R} : \mu(\{x \in E : |f(x)| > \alpha\}) = 0\} \,.$$

Using the"Full Müntz Theorem in $C[0,1]$" [1,2] G. Klun [6] proved the following result.

Theorem 1.1. *Suppose $f, g \in L_p(E)$ for all $p \geq 1$, $f, g \in L_\infty(E)$, and $P := (p_j)_{j=1}^\infty$ is a sequence of distinct real numbers $p_j \geq 1$ such that*

$$\sum_{j=1}^\infty \frac{p_j - 1}{(p_j - 1)^2 + 1} = \infty \,.$$

The equalities

$$\|f\|_p = \|g\|_p \,, \qquad p \in P \,,$$

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imply

$$\mu(\{x \in E : |f(x)| < \alpha\}) = \mu(\{x \in E : |g(x)| < \alpha\}), \qquad \alpha \geq 0.$$

It is quite remarkable that Klun does not assume $\mu(E) < \infty$ in the above theorem as he applies it elegantly when $E := \mathbb{N}$ and $\mu(A)$ is the number of elements in $A \subset \mathbb{N}$.

2. New Result

In this note we prove the following result.

Theorem 2.1. *Suppose $P := (p_j)_{j=1}^{\infty}$ is a sequence of distinct real numbers $p_j > 0$. The equalities*

$$\|f\|_p = \|g\|_p, \qquad p \in P, \tag{2.1}$$

imply

$$\mu(\{x \in E : |f(x)| < \alpha\}) = \mu(\{x \in E : |g(x)| < \alpha\}), \qquad \alpha \geq 0,$$

whenever $0 < \mu(E) < \infty$ and $f, g \in L_\infty(E)$ if and only if

$$\sum_{j=1}^{\infty} \frac{p_j}{p_j^2 + 1} = \infty. \tag{2.2}$$

Note that under the assumption $0 < \mu(E) < \infty$ the above theorem allows arbitrary sequences $P := (p_j)_{j=1}^{\infty}$ of distinct real numbers $p_j > 0$ rather than only $p_j \geq 1$, and it is an "if and only if" extension of Theorem 1.1.

Note also that a careful reading of the "only if" part of Theorem 2.1 means that if (2.2) does not hold, then there is a measure space $(E, \mathcal{A}, \mu)$ with $0 < \mu(E) < \infty$ and there are two functions $f, g \in L_\infty(E)$ such that (2.1) holds but

$$\mu(\{x \in E : |f(x)| < \alpha\}) \neq \mu(\{x \in E : |g(x)| < \alpha\})$$

for at least one value of $\alpha \geq 0$.

3. Proof

The proof of the "if" part of Theorem 2.1 is based on the following result called "Full Müntz Theorem" in $L_p(A)$ for $p \in (0, \infty)$ and for compact sets $A \subset [0,1]$ with positive lower density at 0. In Theorem 3.1 below $L_p(A)$ is considered with respect to the Lebesgue measure.

Theorem 3.1. *Let $A \subset [0,1]$ be a compact set with positive lower density at 0. Let $p \in (0, \infty)$. Suppose $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct real numbers greater than $-(1/p)$. Then $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ is dense in $L_p(A)$ if and only if*

$$\sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} = \infty.$$

Theorem 3.1 is proved in [5] by Erdélyi and Johnson, and it improves and extends earlier results of Müntz [7], Szász [10], Clarkson and Erdős [3], P. Borwein and Erdélyi [1,2], and Operstein [8]. Another proof of Theorem 3.1 is given in [4]. In fact, to prove the "if" part of Theorem 2.1 we need only the case where $p = 1$ and $A = [0,1]$ proved first in [2]. To prove the "only if" part of Theorem 2.1 we also need the following result.

Theorem 3.2 (Full Müntz Theorem in $C[0,1]$). *Suppose $(\lambda_j)_{j=1}^\infty$ is a sequence of distinct positive real numbers. Then* $\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ *is dense in* $C[0,1]$ *if and only if*

$$\sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j^2 + 1} = \infty\,.$$

Theorem 3.2 is proved by Borwein and Erdélyi [1,2].

Proof of Theorem 2.1. First we prove the "if" part of the theorem. Assume that (2.2) holds. Let $0 < \mu(E) < \infty$ and $f, g \in L_\infty(E)$. Multiplying by constants, without loss of generality, we may assume that $\mu(E) = 1$ and

$$\|f\|_\infty \le \|g\|_\infty = 1\,.$$

We define

$$F(t) := \mu(\{x \in E : |f(x)| < t\}) \qquad \text{and} \qquad G(t) := \mu(\{x \in E : |g(x)| < t\})\,.$$

Then $h(t) := G(t) - F(t)$ is well defined for all $t \in \mathbb{R}$, and $h(t) = 0$ for all $t \in \mathbb{R} \setminus [0,1]$, and $|h(t)| \le 1$ for all $t \in [0,1]$. Hence (2.1) implies that

$$0 = \int_E (|g(x)|^p - |f(x)|^p)\, d\mu(x) = \int_0^1 (G(y^{1/p}) - F(y^{1/p}))\, dy\,, \qquad p \in P\,.$$

Substituting $t = y^{1/p}$ we obtain

$$\text{(3.1)} \qquad 0 = p \int_0^1 (G(t) - F(t))\, t^{p-1}\, dt = p \int_0^1 h(t) t^{p-1}\, dt\,, \qquad p \in P\,,$$

where $h(t) := G(t) - F(t) \in L_\infty[0,1]$. In the light of the case $p = 1$ and $A = [0,1]$ of Theorem 3.1, (2.2) implies that $\operatorname{span}\{x^{p-1} : p \in P\}$ is dense in $L_1[0,1]$. Using (3.1) we obtain

$$\int_0^1 h(t) u(t)\, dt = 0$$

for every $u \in L_1[0,1]$, and hence, choosing $u = h \in L_\infty[0,1] \subset L_1[0,1]$, we have

$$\int_0^1 h(t)^2\, dt = 0\,,$$

and $h(t) = 0$ for almost every $t \in [0,1]$ follows. We conclude that $G(t) = F(t)$ for almost every $t \in [0,1]$. However, as both F and G are continuous from left on $\mathbb{R}$, we have $G(t) = F(t)$ for every $t \in \mathbb{R}$.

To prove the "only if" part of the theorem assume now that (2.2) does not holds. We show that there is a finite Borel measure μ on $E := [0,1]$ with $0 < \mu(E) < \infty$ and there are two functions $f, g \in L_\infty(E)$ such that (2.1) holds but

$$\mu(\{x \in E : |f(x)| < \alpha\}) \ne \mu(\{x \in E : |g(x)| < \alpha\})$$

for at least one value of $\alpha \geq 0$.

Combining Theorem 3.2, the Hahn-Banach Theorem (see [9, page 107]), and the Riesz Representation Theorem (see [9, page 40]) we can deduce that there is a finite signed Borel measure ν on $[0,1]$ and a function $h \in C[0,1]$ such that

$$\int_0^1 t^p \, d\nu(t) = 0 \,, \qquad p \in P \,, \tag{3.2}$$

and

$$\int_0^1 h(t) \, d\nu(t) = 1 \,. \tag{3.3}$$

Indeed, let M be the closure of the subspace spanned by $\{t^p : p \in P\}$ in $C[0,1]$, where $C[0,1]$ is the linear space of continuous functions on $[0,1]$ equipped with the uniform norm. If (2.2) does not hold then Theorem 3.2 implies that $M \neq C[0,1]$, and hence there is an $h \in C[0,1] \setminus M$. By the Hahn-Banach Theorem there is a bounded linear functional f on $C[0,1]$ such that $f(u) = 0$ for all $u \in M$ and $f(h) = 1$. By the Riesz representation Theorem this bounded linear functional f on $C[0,1]$ can be represented by a finite signed Borel measure ν on $[0,1]$ such that

$$f(u) = \int_0^1 u(t) \, d\nu(t) \,,$$

and the proof of (3.2) and (3.3) is finished.

Let $\{E_1, E_2\}$ be the Hahn decomposition of the measure ν on $[0,1]$, that is, E_1 and E_2 are Borel measurable, $[0,1] = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$, and $\nu(A) \geq 0$ for any Borel set $A \subset E_1$ and $\nu(A) \leq 0$ for any Borel measurable set $A \subset E_2$. We define the Borel measurable functions

$$f(t) := \begin{cases} t, & t \in E_1 \\ 0, & t \in E_2 \,, \end{cases} \qquad \text{and} \qquad g(t) := \begin{cases} t, & t \in E_2 \\ 0, & t \in E_1 \,. \end{cases}$$

We have $\|f\|_\infty \leq 1$ and $\|g\|_\infty \leq 1$. We define the finite nonnegative measure

$$|\nu|(A) := \nu(A \cap E_1) - \nu(A \cap E_2)$$

for Borel sets $A \subset [0,1]$. It follows from (3.2) and the definitions of f and g that

$$0 = \int_0^1 (|f(t)|^p - |g(t)|^p) \, d|\nu|(t) \,, \qquad p \in P \,,$$

that is,

$$\int_0^1 |f(t)|^p \, d|\nu|(t) = \int_0^1 |g(t)|^p \, d|\nu|(t) \,, \qquad p \in P \,.$$

Now we show that there is an $\alpha \geq 0$ such that

$$|\nu|(\{x \in E : |f(x)| < \alpha\}) \neq |\nu|(\{x \in E : |g(x)| < \alpha\}) \,.$$

Suppose to the contrary that

$$|\nu|(\{x \in E : |f(x)| < \alpha\}) = |\nu|(\{x \in E : |g(x)| < \alpha\})$$

for every $\alpha \geq 0$. Then

$$\int_0^1 (|f(t)|^p - |g(t)|^p)\, d|\nu|(t) = 0\,, \qquad p > 0\,,$$

and hence

$$\int_0^1 t^p \, d\nu(t) = 0\,, \qquad p > 0\,.$$

Hence it follows from the Weierstrass Theorem that

$$\int_0^1 u(t)\, d\nu(t) = 0$$

for every $u \in C[0,1]$, which contradicts (3.3). This completes the proof of the "only if" part of the theorem. $\square$

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