NEWMAN'S INEQUALITY FOR MÜNTZ POLYNOMIALS ON POSITIVE INTERVALS

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Abstract. The principal result of this paper is the following Markov-type inequality for Müntz polynomials.

Theorem (Newman's Inequality on $[a,b] \subset (0,\infty)$). *Let $\Lambda := (\lambda_j)_{j=0}^\infty$ be an increasing sequence of nonnegative real numbers. Suppose $\lambda_0 = 0$ and there exists a $\delta > 0$ so that $\lambda_j \geq \delta j$ for each j. Suppose $0 < a < b$. Then there exists a constant $c(a,b,\delta)$ depending only on a, b, and δ so that*

$$\|P'\|_{[a,b]} \leq c(a,b,\delta) \left(\sum_{j=0}^n \lambda_j \right) \|P\|_{[a,b]}$$

for every $P \in M_n(\Lambda)$, where $M_n(\Lambda)$ denotes the linear span of $\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$ over $\mathbb{R}$.

When $[a,b] = [0,1]$ and with $\|P'\|_{[a,b]}$ replaced with $\|xP'(x)\|_{[a,b]}$ this was proved by Newman. Note that the interval $[0,1]$ plays a special role in the study of Müntz spaces $M_n(\Lambda)$. A linear transformation $y = \alpha x + \beta$ does not preserve membership in $M_n(\Lambda)$ in general (unless $\beta = 0$). So the analogue of Newman's Inequality on $[a,b]$ for $a > 0$ does not seem to be obtainable in any straightforward fashion from the $[0,b]$ case.

1. Introduction and Notation

Let $\Lambda := (\lambda_j)_{j=0}^\infty$ be a sequence of distinct real numbers. The span of

$$\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$$

over $\mathbb{R}$ will be denoted by

$$M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}.$$

Elements of $M_n(\Lambda)$ are called Müntz polynomials. Newman's beautiful inequality [6] is an essentially sharp Markov-type inequality for $M_n(\Lambda)$, where $\Lambda := (\lambda_j)_{j=0}^\infty$ is a sequence of distinct nonnegative real numbers. For notational convenience, let $\|\cdot\|_{[a,b]} := \|\cdot\|_{L_\infty[a,b]}$.

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Theorem 1.1 (Newman's Inequality). *Let $\Lambda := (\lambda_j)_{j=0}^\infty$ be a sequence of distinct nonnegative real numbers. Then*

$$\frac{2}{3}\sum_{j=0}^n \lambda_j \leq \sup_{0\neq P\in M_n(\Lambda)} \frac{\|xP'(x)\|_{[0,1]}}{\|P\|_{[0,1]}} \leq 11\sum_{j=0}^n \lambda_j .$$

Frappier [4] shows that the constant 11 in Newman's Inequality can be replaced by 8.29. In [2], by modifying (and simplifying) Newman's arguments, we showed that the constant 11 in the above inequality can be replaced by 9. But more importantly, this modification allowed us to prove the following L_p version of Newman's Inequality [2] (an L_2 version of which was proved earlier in [3]).

Theorem 1.2 (Newman's Inequality in L_p). *Let $p\in[1,\infty)$. Let $\Lambda := (\lambda_j)_{j=0}^\infty$ be a sequence of distinct real numbers greater than $-1/p$. Then*

$$\|xP'(x)\|_{L_p[0,1]} \leq \left(1/p + 12\left(\sum_{j=0}^n (\lambda_j + 1/p)\right)\right)\|P\|_{L_p[0,1]}$$

for every $P\in M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$.

We believe on the basis of considerable computation that the best possible constant in Newman's Inequality is 4. (We remark that an incorrect argument exists in the literature claiming that the best possible constant in Newman's Inequality is at least $4+\sqrt{15} = 7.87\dots$.)

Conjecture (Newman's Inequality with Best Constant). *Let $\Lambda := (\lambda_j)_{j=0}^\infty$ be a sequence of distinct nonnegative real numbers. Then*

$$\|xP'(x)\|_{[0,1]} \leq 4\left(\sum_{j=0}^n \lambda_j\right)\|P\|_{[0,1]}$$

for every $P\in M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$.

It is proved in [1] that under a growth condition, which is essential, $\|xP'(x)\|_{[0,1]}$ in Newman's Inequality can be replaced by $\|P'\|_{[0,1]}$. More precisely, the following result holds.

Theorem 1.3 (Newman's Inequality Without the Factor x). *Let $\Lambda := (\lambda_j)_{j=0}^\infty$ be a sequence of distinct real numbers with $\lambda_0 = 0$ and $\lambda_j \geq j$ for each j. Then*

$$\|P'\|_{[0,1]} \leq 18\left(\sum_{j=1}^n \lambda_j\right)\|P\|_{[0,1]}$$

for every $P\in M_n(\Lambda)$.

Note that the interval $[0,1]$ plays a special role in the study of Müntz polynomials. A linear transformation $y=\alpha x+\beta$ does not preserve membership in $M_n(\Lambda)$ in general (unless $\beta=0$), that is $P\in M_n(\Lambda)$ does not necessarily imply that $Q(x) := P(\alpha x+\beta)\in M_n(\Lambda)$. Analogues of the above results on $[a,b]$, $a>0$, cannot be obtained by a simple transformation. We can, however, prove the following result.

2. New Results

Theorem 2.1 (Newman's Inequality on $[a,b] \subset (0,\infty)$). *Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be an increasing sequence of nonnegative real numbers. Suppose $\lambda_0 = 0$ and there exists a $\delta > 0$ so that $\lambda_j \geq \delta j$ for each j. Suppose $0 < a < b$. Then there exists a constant $c(a,b,\delta)$ depending only on a, b, and δ so that*

$$\|P'\|_{[a,b]} \leq c(a,b,\delta) \left(\sum_{j=0}^{n} \lambda_j \right) \|P\|_{[a,b]}$$

for every $P \in M_n(\Lambda)$, where $M_n(\Lambda)$ denotes the linear span of $\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$ over $\mathbb{R}$.

Theorem 2.1 is sharp up to the constant $c(a,b,\delta)$. This follows from the lower bound in Theorem 1.1 by the substitution $y = b^{-1}x$. Indeed, take a $P \in M_n(\Lambda)$ so that

$$|P'(1)| \geq \frac{2}{3} \left(\sum_{j=0}^{n} \lambda_j \right) \|P\|_{[0,1]}.$$

Then $Q(x) := P(x/b)$ satisfies

$$\begin{aligned} \|Q'\|_{[a,b]} \geq |Q'(b)| = b^{-1}|P'(1)| &\geq \frac{2}{3b} \left(\sum_{j=0}^{n} \lambda_j \right) \|P\|_{[0,1]} \\ &\geq \frac{2}{3b} \left(\sum_{j=0}^{n} \lambda_j \right) \|Q\|_{[a,b]}. \end{aligned}$$

The following example shows that the growth condition $\lambda_j \geq \delta j$ with a $\delta > 0$ in the above theorem cannot be dropped. It will also be used in the proof of Theorem 2.1.

Theorem 2.2. *Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$, where $\lambda_j = \delta j$. Let $0 < a < b$. Then*

$$\max_{0 \neq P \in M_n(\Lambda)} \frac{|P'(a)|}{\|P\|_{[a,b]}} = |Q_n'(a)| = \frac{2\delta a^{\delta-1}}{b^\delta - a^\delta}\, n^2$$

where, with $T_n(x) = \cos(n \arccos x)$,

$$Q_n(x) := T_n\left(\frac{2x^\delta}{b^\delta - a^\delta} - \frac{b^\delta + a^\delta}{b^\delta - a^\delta} \right)$$

is the Chebyshev "polynomial" for $M_n(\Lambda)$ on $[a,b]$. In particular

$$\lim_{\delta \to 0} \max_{0 \neq P \in M_n(\Lambda)} \frac{|P'(a)|}{\left(\sum_{j=0}^{n} \lambda_j \right) \|P\|_{[a,b]}} = \infty.$$

Theorem 2.2 is a well-known property of differentiable Chebyshev spaces. See, for example, [5] or [1].

3. Lemmas

The following comparison theorem for Müntz polynomials is proved in [1, E.4 f] of Section 3.3]. For the sake of completeness, in the next section we outline a short proof suggested by Pinkus. This proof assumes familarity with the basic properties of Chebyshev and Descartes systems. All of these may be found in [5].

Lemma 3.1 (A Comparison Theorem). *Let $\Lambda := (\lambda_j)_{j=0}^\infty$ and $\Gamma := (\gamma_j)_{j=0}^\infty$ be increasing sequences of nonnegative real numbers with $\lambda_0 = \gamma_0 = 0$, and $\gamma_j \leq \lambda_j$ for each j. Let $0 < a < b$. Then*

$$\max_{P \in M_n(\Gamma)} \frac{|P'(a)|}{\|P\|_{[a,b]}} \geq \max_{P \in M_n(\Lambda)} \frac{|P'(a)|}{\|P\|_{[a,b]}}.$$

The following result is essentially proved by Saff and Varga [7]. They assume that $\Lambda := (\lambda_j)_{j=0}^\infty$ is an increasing sequence of nonnegative integers and $\delta = 1$ in the next lemma, however, this assumption can be easily dropped from their theorem, see [1, E.9 of Section 6.1]. In fact, their proof remains valid almost word by word, the modifications are straightforward.

Lemma 3.2 (The Interval Where the Norm of a Müntz Polynomial Lives). *Let $\Lambda := (\lambda_j)_{j=1}^\infty$ be an increasing sequence of nonnegative real numbers. Let $0 \neq P \in M_n(\Lambda)$ and $Q(x) := x^{k\delta}P(x)$, where k is a nonnegative integer and δ is a positive real number. Let $\xi \in [0,1]$ be a point so that $|Q(\xi)| = \|Q\|_{[0,1]}$. Suppose $\lambda_j \geq \delta j$ for each j. Then*

$$\left(\frac{k}{k+n}\right)^{2/\delta} \leq \xi.$$

The above result is sharp in a certain limiting sense which is described in detail in Saff and Varga [7].

4. Proofs

Proof of Lemma 3.1. It can be proved by a standard perturbation argument (see, for example, [5]) that

$$\sup_{0 \neq P \in M_n(\Lambda)} \frac{|P'(a)|}{\|P\|_{[a,b]}} = \frac{|T_n'(a)|}{\|T_n\|_{[a,b]}}$$

where T_n is the Chebyshev polynomial for the Chebyshev space $M_n(\Lambda)$. In particular, T_n has n distinct zeros in (a,b) and

$$|T_n(a)| = |T_n(b)| = \|T_n\|_{[a,b]} = 1\,.$$

Let

$$T_n(x) =: \sum_{j=0}^n c_j x^{\lambda_j}\,, \qquad c_j \in \mathbb{R}\,.$$

Since

$$T_n'(x) = \sum_{j=1}^{n} c_j \gamma_j x^{\lambda_j - 1}$$

and since

$$(x^{\sigma_0}, x^{\sigma_1}, \ldots, x^{\sigma_n})$$

is a Descartes system on $[a,b]$ for any choice of $\sigma_0 < \sigma_1 < \cdots < \sigma_n$, it follows that T_n' has exactly $n-1$ zeros in $[a,b]$, and thus if we normalize T_n so that $T_n'^{(a)} > 0$, then $T_n(a) < 0$. Under this normalization,

$$c_j(-1)^{j+1} > 0, \qquad j = 0, 1, \ldots, n\,.$$

Now let $k \in \{1, 2, \ldots n\}$ be fixed. Let $(\gamma_j)_{j=0}^n$ be such that

$$0 = \gamma_0 < \gamma_1 < \cdots < \gamma_n, \qquad \gamma_j = \lambda_j, \quad j \neq k, \qquad \text{and} \qquad \lambda_{k-1} < \gamma_k < \lambda_k.$$

To prove the lemma it is sufficient to study the above case since the general case follows from this by a finite number of pairwise comparisons.

Choose $Q_n \in M_n(\Gamma)$ of the form

$$Q_n(x) = \sum_{j=0}^{n} d_j x^{\gamma_j}, \qquad d_j \in \mathbb{R}$$

so that

$$Q_n(t_i) = T_n(t_i), \qquad i = 0, 1, \ldots, n$$

where $t_0 := a$ and $t_1 < t_2 < \cdots < t_n$ are the n zeros of T_n in (a,b). By the unique interpolation property of Chebyshev spaces, Q_n is uniquely determined, has n zeros (the points $t_1, t_2, \ldots, t_n$), and is negative at a. (Thus $(-1)^{j+1} d_j > 0$ for each $j = 0, 1, \ldots, n$.)

We have

$$(T_n - Q_n)(x) = \sum_{j=0, j\neq k}^{n} (c_j - d_j) x^{\lambda_j} + c_k x^{\lambda_k} - d_k x^{\gamma_k}\,.$$

The function $T_n - Q_n$ changes sign on $(0,\infty)$ strictly at the points t_i, $i = 0, 1, \ldots, n$, and has no other zeros. As such a sequence,

$$c_0 - d_0,\ c_1 - d_1,\ \ldots,\ c_{k-1} - d_{k-1},\ -d_k,\ c_k,\ c_{k+1} - d_{k+1},\ \ldots,\ c_n - d_n$$

strictly alternates in sign. Since $(-1)^{k+1} c_k > 0$, this implies that

$$(-1)^{n+1}(T_n - Q_n)(x) > 0 \qquad \text{for } x > t_n\,.$$

Thus for $x \in (t_{j-1}, t_j)$ we have

$$(-1)^j T_n(x) > (-1)^j Q_n(x) > 0\,.$$

In addition, we recall that $Q_n(a) = T_n(a) < 0$.

The observations above imply that

$$\|Q_n\|_{[a,b]} \leq \|T_n\|_{[a,b]} = 1 \qquad \text{and} \qquad Q_n'(a) \geq T_n'(a) > 0.$$

Thus

$$\frac{|Q_n'(a)|}{\|Q_n\|_{[a,b]}} \geq \frac{|T_n'(a)|}{\|T_n\|_{[a,b]}} = \sup_{0 \neq P \in M_n(\Lambda)} \frac{|P'(a)|}{\|P\|_{[a,b]}}.$$

The desired conclusion follows from this. □

Proof of Theorem 2.1. Let $P \in M_n(\Lambda)$. We want to estimate $|P'(y)|$ for every $y \in [a,b]$. First let $y \in \left[\frac{1}{2}(a+b), b\right]$. We define $Q(x) := x^{mn\delta}P(x)$, where m is the smallest positive integer satisfying

$$a \leq \frac{a+b}{2}\left(\frac{m}{m+1}\right)^{2/\delta}.$$

Scaling Newman's Inequality from $[0,1]$ to $[0,y]$, then using Lemma 3.2, we obtain

$$\begin{aligned}
|Q'(y)| &\leq \frac{9}{y}\sum_{j=0}^{n}(\lambda_j + mn\delta)\|Q\|_{[0,y]} \\
&= \frac{9}{y}\sum_{j=0}^{n}(\lambda_j + mn\delta)\|Q\|_{\left[y\left(\frac{m}{m+1}\right)^{2/\delta},y\right]} \\
&\leq c_1(a,b,\delta)\left(\sum_{j=0}^{n}\lambda_j\right)\|Q\|_{[a,y]}
\end{aligned}$$

with a constant $c_1(a,b,\delta)$ depending only on a, b, and δ. Hence

$$\begin{aligned}
|P'(y)| &\leq \left|Q'(y)y^{-mn\delta}\right| + \frac{mn\delta}{y}|P(y)| \\
&\leq y^{-mn\delta}c_1(a,b,\delta)\left(\sum_{j=0}^{n}\lambda_j\right)\|Q\|_{[a,y]} + \frac{mn\delta}{y}\|P\|_{[a,y]} \\
&\leq c_2(a,b,\delta)\left(\sum_{j=0}^{n}\lambda_j\right)\|P\|_{[a,y]} \\
&\leq c_2(a,b,\delta)\left(\sum_{j=0}^{n}\lambda_j\right)\|P\|_{[a,b]}
\end{aligned}$$

with a constant $c_2(a,b,\delta)$ depending only on a, b, and δ.

Now let $y \in \left[a, \frac{1}{2}(a+b)\right]$. Then, by Lemma 3.1 and Theorem 2.2, we can deduce that

$$\begin{aligned}
|P'(y)| &\leq \frac{2\delta y^{\delta-1}}{b^\delta - y^\delta}n^2\|P\|_{[y,b]} \\
&\leq c_3(a,b,\delta)n^2\|P\|_{[y,b]} \\
&\leq c_4(a,b,\delta)\left(\sum_{j=0}^{n}\lambda_j\right)\|P\|_{[y,b]}
\end{aligned}$$

with constants $c_3(a,b,\delta)$ and $c_4(a,b,\delta)$ depending only on a, b, and δ. This finishes the proof. $\square$

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