MARKOV-TYPE INEQUALITIES FOR PRODUCTS OF MÜNTZ POLYNOMIALS

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Abstract. Let $\Lambda := (\lambda_j)_{j=0}^\infty$ be a sequence of distinct real numbers. The span of

$$\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$$

over $\mathbb{R}$ is denoted by

$$M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}.$$

Elements of $M_n(\Lambda)$ are called Müntz polynomials. The principal result of this paper is the following Markov-type inequality for products of Müntz polynomials.

Theorem 2.1. *Let $\Lambda := (\lambda_j)_{j=0}^\infty$ and $\Gamma := (\gamma_j)_{j=0}^\infty$ be increasing sequences of nonnegative real numbers. Let*

$$K(M_n(\Lambda), M_m(\Gamma)) := \sup\left\{ \frac{\|x(pq)'(x)\|_{[0,1]}}{\|pq\|_{[0,1]}} : \ p \in M_n(\Lambda)\,, \ \ q \in M_m(\Gamma) \right\}.$$

Then

$$\frac{1}{3}\left((m+1)\lambda_n + (n+1)\gamma_m\right) \le K(M_n(\Lambda), M_m(\Gamma)) \le 18\,(n+m+1)(\lambda_n+\gamma_m)\,.$$

In particular,

$$\frac{2}{3}\,(n+1)\lambda_n \le K(M_n(\Lambda), M_n(\Lambda)) \le 36\,(2n+1)\lambda_n\,.$$

Under some necessary extra assumptions, an analog of the above Markov-type inequality is extended to the cases when the factor x is dropped, and when the interval $[0,1]$ is replaced by $[a,b] \subset (0,\infty)$.

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1. Introduction and Notation

Let $\mathcal{P}_n$ denote the family of all algebraic polynomials of degree at most n with real coefficients. A classical inequality for polynomials is the

Markov Inequality. *The inequality*

$$\|p'\|_{[a,b]} \leq \frac{2n^2}{b-a} \|p\|_{[a,b]}$$

holds for every $p \in \mathcal{P}_n$ and for every subinterval $[a,b]$ of the real line.

For proofs see, for example, Borwein and Erdélyi [3] or DeVore and Lorentz [11].

Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be a sequence of distinct real numbers. The span of

$$\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$$

over $\mathbb{R}$ will be denoted by

$$M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}\,.$$

Elements of $M_n(\Lambda)$ are called Müntz polynomials. For notational convenience, let $\|\cdot\|_{[a,b]} := \|\cdot\|_{L_\infty[a,b]}$. Newman [16] established an essentially sharp Markov-type inequality for $M_n(\Lambda)$.

Theorem 1.1 (Newman's Inequality). *Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be a sequence of distinct nonnegative real numbers. Then*

$$\frac{2}{3}\sum_{j=0}^{n}\lambda_j \leq \sup_{0\neq f\in M_n(\Lambda)} \frac{|f'(1)|}{\|f\|_{[0,1]}} \leq \sup_{0\neq f\in M_n(\Lambda)} \frac{\|xf'(x)\|_{[0,1]}}{\|f\|_{[0,1]}} \leq 11\sum_{j=0}^{n}\lambda_j\,.$$

Frappier [12] shows that the constant 11 in Newman's inequality can be replaced by 8.29. By modifying and simplifying Newman's arguments, Borwein and Erdélyi [6] showed that the constant 11 in the above inequality can be replaced by 9. But more importantly, this modification allowed us to prove the "right" L_p version $(1 \leq p \leq \infty)$ of Newman's inequality [6] (an L_2 version of which was proved earlier by Borwein, Erdélyi, and Zhang [8]).

Note that the factor x in $\|xf'(x)\|_{[0,1]}$ can be dropped from Newman's inequality if we rewrite it in terms of exponential sums (the substitution $x = e^{-t}$ transforms exponential sums into Müntz polynomials and the interval $[0,\infty)$ onto $(0,1]$). However, it is nontrivial and proved by Borwein and Erdélyi [5] that under a growth condition, $\|xf'(x)\|_{[0,1]}$ in Newman's inequality can be replaced by $\|f'\|_{[0,1]}$. More precisely, the following result holds.

Theorem 1.2 (Newman's Inequality Without the Factor x). *Let $\Lambda := (\lambda_j)_{j=0}^\infty$ be a sequence of distinct real numbers with $\lambda_0 = 0$ and $\lambda_j \geq j$ for each j. Then*

$$\|f'\|_{[0,1]} \leq 18 \left(\sum_{j=0}^{n} \lambda_j \right) \|f\|_{[0,1]}$$

for every $f \in M_n(\Lambda)$.

The following example shows that the growth condition in Theorem 1.2 is essential. It is based on an example given by Len Bos (non-published communication). This is presented in [3] with a correctable error, see E.3 b] on page 287. The correction of the mistake is made in the second edition of [3]. For completeness we present this here as well.

Example 1.3. *For every $\delta \in (0,1)$ there exists a sequence $\Lambda := (\lambda_j)_{j=0}^\infty$ with $\lambda_0 = 0$, $\lambda_1 \geq 1$, and*

$$\lambda_{j+1} - \lambda_j \geq \delta\,, \qquad j = 0, 1, 2, \ldots$$

such that

$$\lim_{\mu \to \infty} \sup_{0 \neq p \in M_\mu(\Lambda)} \frac{|p'(0)|}{\left(\sum_{j=0}^{\mu} \lambda_j\right) \|p\|_{[0,1]}} = \infty\,.$$

Proof. Let Q_n be the Chebyshev polynomial T_n transformed linearly from $[-1,1]$ to $[0,1]$, that is,

$$Q_n(x) = \cos(n \arccos(2x-1))\,, \qquad x \in [0,1]\,.$$

Choose natural numbers u and v so that $\delta < u/v < 1$. Let $\Lambda := (\lambda_k)_{k=0}^\infty$ be defined by $\lambda_0 := 0$, $\lambda_1 := 1$, and

$$\lambda_j := 1 + \frac{(j-1)u}{v}\,, \qquad j = 2, 3, \ldots\,.$$

Let

$$p_n(x) := x^{1-u}\left(Q_n(x^{u/v}) - (-1)^n\right)^v \in M_{nv-v+1}(\Lambda)\,.$$

Then

$$|p_n'(0)| = \left(2n^2\right)^v.$$

Without loss of generality we may assume that $1/2 < \delta < u/v < 1$. Observe that p_n is of the form

$$p_n(x) = x r_{nv-v}(x^{u/v}) = (x^{u/v})^{v/u} r_{nv-v}(x^{u/v})$$

with an $r_{nv-v} \in \mathcal{P}_{nv-v}$. Use Theorem A.4.8 (Markov Inequality for GAP_N) from [3] to deduce that there is an absolute constant $c_1 > 0$ such that

$$\|p_n\|_{[0,1]} = \|p_n\|_{[y,1]}$$

with

$$y := (c_1 v^2 n^2)^{-v/u}\,.$$

Hence, using the definition of p_n, we obtain

$$\|p_n\|_{[0,1]} \le 2^v y^{1-u} = 2^v (c_1 v^2 n^2)^{(u-1)v/u} .$$

Therefore

$$\frac{|p_n'(0)|}{\left(\sum_{j=0}^{nv-v+1} \lambda_j\right) \|p_n\|_{[0,1]}} \ge \frac{(2n^2)^v}{\left(\sum_{j=0}^{nv-v} \left(1+\frac{ju}{v}\right)\right) 2^v (c_1 v^2 n^2)^{(u-1)v/u}}$$
$$\ge \frac{(n^2)^{v/u}}{(1+nu)nv} (c_1 v^2)^{(1-u)v/u} \to \infty$$

as $n \to \infty$. $\square$

Note that the interval $[0,1]$ plays a special role in the study of Müntz polynomials. A linear transformation $y = \alpha x + \beta$ does not preserve membership in $M_n(\Lambda)$ in general (unless $\beta = 0$), that is, $f \in M_n(\Lambda)$ does not necessarily imply that $g(x) := f(\alpha x + \beta) \in M_n(\Lambda)$. Analogs of the above results on $[a,b]$, $a > 0$, cannot be obtained by a simple transformation. However, Borwein and Erdélyi [5] proved the following result.

Theorem 1.4 (Newman's Inequality on $[a,b] \subset (0,\infty)$). *Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be an increasing sequence of nonnegative real numbers. Suppose $\lambda_0 = 0$ and there exists a $\varrho > 0$ such that $\lambda_j \ge \varrho j$ for each j. Suppose $0 < a < b$. Then there exists a constant $c(a,b,\varrho)$ depending only on a, b, and ϱ such that*

$$\|f'\|_{[a,b]} \le c(a,b,\varrho) \left(\sum_{j=0}^{n} \lambda_j\right) \|f\|_{[a,b]}$$

for every $f \in M_n(\Lambda)$.

The above theorem is essentially sharp, as one can easily deduce it from the first inequality of Theorem 1.1 by a linear scaling.

Müntz's classical theorem characterizes sequences $\Lambda := (\lambda_j)_{j=0}^{\infty}$ with

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

for which the Müntz space $M(\Lambda) = \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $C[0,1]$. Here

$$\text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

denotes the collection of finite linear combinations of the functions $x^{\lambda_0}, x^{\lambda_1}, \dots$ with real coefficients and $C(A)$ is the space of all real-valued continuous functions on $A \subset [0,\infty)$ equipped with the uniform norm. If $A := [a,b]$ is a finite closed interval, then the notation $C[a,b] := C([a,b])$ is used. Müntz's Theorem states the following.

Müntz's Theorem. *Suppose* $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$. *Then* $M(\Lambda)$ *is dense in* $C[0,1]$ *if and only if* $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$.

Proofs are available in, for example, Cheney [9], DeVore and Lorentz [11], and Borwein and Erdélyi [3]. The original Müntz Theorem proved by Müntz [15] and by Szász [21] and anticipated by Bernstein [2] was only for sequences of exponents tending to infinity. There are many generalizations and variations of Müntz's Theorem. See, for example, Borwein and Erdélyi [3], [4], [5], [6], and [7], Clarkson and Erdős [10], DeVore and Lorentz [11], von Golitschek [22], Lorentz, von Golitschek, and Makovoz [14], and Schwartz [18]. There are also still many open problems.

Somorjai [20] in 1976 and Bak and Newman [1] in 1978 proved that

$$R(\Lambda) := \{p/q : p, q \in M(\Lambda)\}$$

is always dense in $C[0,1]$ whenever $\Lambda := (\lambda_j)_{j=0}^{\infty}$ contains infinitely many distinct real numbers. This surprising result says that while the set $M(\Lambda)$ of Müntz polynomials may be far from dense, the set $R(\Lambda)$ of Müntz rationals is always dense in $C[0,1]$, no matter what the underlying sequence Λ. In light of this result, Newman [17] (p. 50) raises "the very sane, if very prosaic question." Are the functions

$$\prod_{j=1}^{k} \left(\sum_{i=0}^{n_j} a_{i,j} x^{i^2} \right), \qquad a_{i,j} \in \mathbb{R}, \quad n_j \in \mathbb{N},$$

dense in $C[0,1]$ for some fixed $k \geq 2$? In other words does the "extra multiplication" have the same power that the "extra division" has in the Bak-Newman-Somorjai result? Newman speculated that it did not.

Denote the set of the above products by H_k. Since every natural number is the sum of four squares, H_4 contains all the monomials x^n, $n = 0, 1, 2, \ldots$. However, H_k is not a linear space, so Müntz's Theorem itself cannot be applied to resolve the denseness or non-denseness of H_4 in $C[0,1]$.

Borwein and Erdélyi [3], [4], and [7] deal with products of Müntz spaces and, in particular, the question of Newman is answered in the negative. In fact, in [7] we presented a number of inequalities each of which implies the answer to Newman's question. One of them is the following bounded Bernstein-type inequality for products of Müntz polynomials from non-dense Müntz spaces. For

$$\Lambda_j := (\lambda_{i,j})_{i=0}^{\infty}, \qquad 0 = \lambda_{0,j} < \lambda_{1,j} < \lambda_{2,j} < \cdots, \qquad j = 1, 2, \ldots,$$

we define the sets

$$M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k) := \left\{ p = \prod_{j=1}^{k} p_j : p_j \in M(\Lambda_j) \right\}.$$

Theorem 1.5. *Suppose*

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_{i,j}} < \infty \qquad \text{and} \qquad \lambda_{1,j} \geq 1\,, \qquad j = 1, 2, \dots, k\,.$$

Let $s > 0$. Then there exits a constant c depending only on $\Lambda_1, \Lambda_2, \dots, \Lambda_k$, s, and k (and not on ϱ or A) such that

$$\|p'\|_{[0,\varrho]} \leq c\, \|p\|_A$$

for every $p \in M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$ and for every set $A \subset [\varrho, 1]$ of Lebesgue measure at least s.

The purpose of this paper is to establish the right Markov-type inequalities for products of Müntz polynomials when the factors come from arbitrary (not necessarily non-dense) Müntz spaces. More precisely, we examine the magnitude of

$$\sup \left\{ \frac{\|x(pq)'(x)\|_{[0,1]}}{\|pq\|_{[0,1]}} : \;\; p \in M_n(\Lambda)\,, \;\; q \in M_m(\Gamma) \right\}$$

and

$$\sup \left\{ \frac{\|(pq)'\|_{[a,b]}}{\|pq\|_{[a,b]}} : \;\; p \in M_n(\Lambda)\,, \;\; q \in M_m(\Gamma) \right\}$$

for $[a, b] \subset (0, \infty)$.

2. New Results

Our first result is an essentially sharp Newman-type inequality for products of Müntz polynomials.

Theorem 2.1. *Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ and $\Gamma := (\gamma_j)_{j=0}^{\infty}$ be increasing sequences of nonnegative real numbers. Let*

$$K(M_n(\Lambda), M_m(\Gamma)) := \sup \left\{ \frac{\|x(pq)'(x)\|_{[0,1]}}{\|pq\|_{[0,1]}} : \;\; p \in M_n(\Lambda)\,, \;\; q \in M_m(\Gamma) \right\}.$$

Then

$$\frac{1}{3}\,((m+1)\lambda_n + (n+1)\gamma_m) \leq K(M_n(\Lambda), M_m(\Gamma)) \leq 18\,(n+m+1)(\lambda_n + \gamma_m)\,.$$

In particular,

$$\frac{2}{3}\,(n+1)\lambda_n \leq K(M_n(\Lambda), M_n(\Lambda)) \leq 36\,(2n+1)\lambda_n\,.$$

Our next theorem drops the factor x from $\|x(pq)'(x)\|_{[0,1]}$ in Theorem 2.1 in the expense of a growth condition and establishes an essentially sharp Markov-type inequality on $[0, 1]$.

Theorem 2.2. *Let $\Lambda := (\lambda_j)_{j=0}^\infty$ and $\Gamma := (\gamma_j)_{j=0}^\infty$ be increasing sequences of nonnegative real numbers with $\lambda_0 = \gamma_0 = 0$ and with $\lambda_j \geq j$ and $\gamma_j \geq j$ for each j. Let*

$$\widetilde{K}(M_n(\Lambda), M_m(\Gamma), 0, 1) := \sup \left\{ \frac{\|(pq)'\|_{[0,1]}}{\|pq\|_{[0,1]}} : \ p \in M_n(\Lambda)\,, \ \ q \in M_m(\Gamma) \right\} .$$

Then

$$\frac{1}{3}\left((m+1)\lambda_n + (n+1)\gamma_m\right) \leq \widetilde{K}(M_n(\Lambda), M_m(\Gamma), 0, 1) \leq 36\,(n+m+1)(\lambda_n + \gamma_m)\,.$$

In particular,

$$\frac{2}{3}\,(n+1)\lambda_n \leq \widetilde{K}(M_n(\Lambda), M_n(\Lambda), 0, 1) \leq 72\,(2n+1)\lambda_n\,.$$

Under a growth condition again, we can extend Theorem 2.2 to the interval $[0, 1]$ replaced by $[a, b] \subset (0, \infty)$ and an essentially sharp Markov-type inequality is established on $[a, b]$.

Theorem 2.3. *Let $\Lambda := (\lambda_j)_{j=0}^\infty$ and $\Gamma := (\gamma_j)_{j=0}^\infty$ be increasing sequences of nonnegative real numbers. Suppose $\lambda_0 = \gamma_0 = 0$ and there exists a $\varrho > 0$ such that $\lambda_j \geq \varrho j$ and $\gamma_j \geq \varrho j$ for each j. Suppose $0 < a < b$. Let*

$$\widetilde{K}(M_n(\Lambda), M_m(\Gamma), a, b) := \sup \left\{ \frac{\|(pq)'\|_{[a,b]}}{\|pq\|_{[a,b]}} : \ p \in M_n(\Lambda)\,, \ \ q \in M_m(\Gamma) \right\} .$$

Then there is a constant $c(a, b, \varrho)$ depending only on a, b, and ϱ such that

$$\frac{b}{3}\left((m+1)\lambda_n + (n+1)\gamma_m\right) \leq \widetilde{K}(M_n(\Lambda), M_m(\Gamma), a, b) \leq c(a, b, \varrho)\,(n+m+1)(\lambda_n + \gamma_m)\,.$$

In particular,

$$\frac{2b}{3}\,(n+1)\lambda_n \leq \widetilde{K}(M_n(\Lambda), M_n(\Lambda), a, b) \leq 2\,c(a, b, \varrho)\,(2n+1)\lambda_n\,.$$

Remark 1. Analogs of the above three theorems dealing with products of several Müntz polynomials can also be proved by straightforward modifications.

Remark 2. Let $\Lambda := (\lambda_j)_{j=0}^\infty$ with $\lambda_j = j^2$. If we multiply pq out, where $p, q \in M_n(\Lambda)$, and we apply Newman's inequality, we get

$$K(M_n(\Lambda), M_n(\Lambda)) \leq cn^4$$

with an absolute constant c. However, if we apply Theoren 2.1, we obtain

$$K(M_n(\Lambda), M_n(\Lambda)) \leq 36\,(2n+1)n^2\,.$$

It is quite remarkable that $K(M_n(\Lambda), M_n(\Lambda))$ is of the same order of magnitude as the Markov factor $11\left(\sum_{j=0}^n j^2\right)$ in Newman's inequality for $M_n(\Lambda)$. When the exponents λ_j grow sufficiently slowly, similar improvements can be observed in all of our theorems compared with the "natural first idea" of "multiply out and use Newman's inequality."

3. Lemmas

Our first lemma is no more than a simple exercise.

Lemma 3.1. *Let $0 < a < b$ and $c \geq 0$. If $c = 0$, assume in addition that $\lambda_1 \geq 1$ (to guarantee differentiability at 0). Then there are $P \in M_n(\Lambda)$ and $Q \in M_m(\Gamma)$ such that*

$$\frac{|(P'Q)(c)|}{\|PQ\|_{[a,b]}} = \sup\left\{\frac{|(p'q)(c)|}{\|pq\|_{[a,b]}} : \ p \in M_n(\Lambda)\,, \ q \in M_m(\Gamma)\right\}.$$

Our next lemma is an essential tool in proving our key lemmas, Lemmas 3.3 and 3.4.

Lemma 3.2. *Let $c > b$ or $0 < c < a$ in Lemma 3.1. Then P changes sign exactly n times in (a,b); and Q changes sign exactly m times in (a,b).*

The heart of the proof of our theorems is the following pair of comparison lemmas. The proof of the next couple of lemmas is based on basic properties of Descartes systems, in particular on Descartes' Rule of Sign, and on a technique used earlier by P.W. Smith and Pinkus. Lorentz ascribes this result to Pinkus, although it was P.W. Smith [19] who published it. I have learned about the proofs of these lemmas from Peter Borwein, who also ascribes the short proof to Pinkus. This is the proof we present here.

Lemma 3.3. *Let $\Lambda := (\lambda_j)_{j=0}^\infty$, $\widetilde{\Lambda} := (\widetilde{\lambda}_j)_{j=0}^\infty$, $\Gamma := (\gamma_j)_{j=0}^\infty$, and $\widetilde{\Gamma} := (\widetilde{\gamma}_j)_{j=0}^\infty$ be increasing sequences of nonnegative real numbers satisfying $\lambda_j \leq \widetilde{\lambda}_j$ and $\gamma_j \leq \widetilde{\gamma}_j$ for each j. Let $0 \leq a < b \leq c$. Then*

$$\sup\left\{\frac{|(p'q)(c)|}{\|pq\|_{[a,b]}} : \ p \in M_n(\Lambda)\,, \ q \in M_m(\Gamma)\right\}$$
$$\leq \sup\left\{\frac{|(p'q)(c)|}{\|pq\|_{[a,b]}} : \ p \in M_n(\widetilde{\Lambda})\,, \ q \in M_m(\widetilde{\Gamma})\right\}.$$

Lemma 3.4. *Let $\Lambda := (\lambda_j)_{j=0}^\infty$, $\widetilde{\Lambda} := (\widetilde{\lambda}_j)_{j=0}^\infty$, $\Gamma := (\gamma_j)_{j=0}^\infty$, and $\widetilde{\Gamma} := (\widetilde{\gamma}_j)_{j=0}^\infty$ be increasing sequences of nonnegative real numbers satisfying*

$$\lambda_0 = \widetilde{\lambda}_0 = \gamma_0 = \widetilde{\gamma}_0 = 0$$

and $\lambda_j \leq \widetilde{\lambda}_j$ and $\gamma_j \leq \widetilde{\gamma}_j$ for each j. Let $0 \leq c \leq a < b$. If $c = 0$, assume in addition that $\lambda_1 \geq 1$ (to guarantee differentiability at 0). Then

$$\sup\left\{\frac{|(p'q)(c)|}{\|pq\|_{[a,b]}} : \ p \in M_n(\Lambda)\,, \ q \in M_m(\Gamma)\right\}$$
$$\geq \sup\left\{\frac{|(p'q)(c)|}{\|pq\|_{[a,b]}} : \ p \in M_n(\widetilde{\Lambda})\,, \ q \in M_m(\widetilde{\Gamma})\right\}.$$

4. Proofs

Proof of Lemma 3.1. Although the argument is slightly more than a standard compactness argument it is no more than an exercise. We omit the details. □

Proof of Lemma 3.2. Assume that $c > b$, the case $0 < c < a$ is similar. We show that P changes sign exactly n times in (a, b). Since $M_n(\Lambda)$ is a Chebyshev space of dimension $n+1$ on $[a, b]$, it is sufficient to show that P changes sign at least n times in (a, b). To show that Q changes sign exactly m times in (a, b) is a straightforward modification of the argument below, so we omit that part of the proof.

Suppose to the contrary that P changes sign exactly at

$$(a <) x_1 < x_2 < \cdots < x_k (< b)$$

on (a, b), where $k < n$. Without loss of generality we may assume that $P(x) \geq 0$ for $x \in [x_k, b]$. Let

$$P_1 \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_k}\}$$

change sign exactly at $x_1, x_2, \dots, x_k$ and be normalized so that $P_1(c) > 0$, therefore $P_1'(c) > 0$. Let

$$P_2 \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_{k+1}}\} \subset M_n(\Lambda)$$

change sign exactly at $x_1, x_2, \dots, x_k$ and b and be normalized so that $P_1(c) < 0$, therefore $P_1'(c) < 0$. The existence of such P_1 and P_2 follows from the elementary properties of the Chebyshev space $M_n(\Lambda)$ on $[a, b]$. Let

$$R_\varepsilon := P - \varepsilon P_1 \in M_n(\Lambda) \qquad \text{and} \qquad S_\varepsilon := P - \varepsilon P_2 \in M_n(\Lambda)\,.$$

Observe that for sufficiently small $\varepsilon > 0$,

$$\|R_\varepsilon Q\|_{[a,b]} < \|PQ\|_{[a,b]}$$

and

$$\|S_\varepsilon Q\|_{[a,b]} < \|PQ\|_{[a,b]}\,.$$

Also, for sufficiently small $\varepsilon > 0$, either $|P'(c)| \leq |R_\varepsilon'(c)|$ or $|P'(c)| \leq |S_\varepsilon'(c)|$. Therefore either $R_\varepsilon Q$ or $S_\varepsilon Q$ contradicts the extremality of PQ. This contradiction shows that $k \geq n$, so P changes sign at least (hence exactly) n times in (a, b), indeed. □

The following comparison theorem for Müntz polynomials is similar to the one in Borwein and Erdélyi [3] (see E.4 f] of Section 3.3). Its proof assumes familiarity with the basic properties of Chebyshev and Descartes systems. All of these may be found in Borwein and Erdélyi [3] or Karlin and Studden [13].

Proof of Lemma 3.3. We may assume that $0 < a < b < c$. The general case when $0 \leq a < b \leq c$ follows by a standard continuity argument. We study the following two cases:

Case 1. Let $k \in \{0, 1, \dots, n\}$ be fixed. Let $(\widetilde{\gamma}_j)_{j=0}^m = (\gamma_j)_{j=0}^m$, and let $(\widetilde{\lambda}_j)_{j=0}^n$ be such that

$$0 \le \widetilde{\lambda}_0 < \widetilde{\lambda}_1 < \cdots < \widetilde{\lambda}_n \,, \qquad \widetilde{\lambda}_j = \lambda_j \,, \quad j \neq k \,, \qquad \text{and} \qquad \lambda_k < \widetilde{\lambda}_k < \lambda_{k+1} \,.$$

Case 2. Let $k \in \{0, 1, \dots, m\}$ be fixed. Let $(\widetilde{\lambda}_j)_{j=0}^n = (\lambda_j)_{j=0}^n$, and let $(\widetilde{\gamma}_j)_{j=0}^m$ be such that

$$0 \le \widetilde{\gamma}_0 < \widetilde{\gamma}_1 < \cdots < \widetilde{\gamma}_m \,, \qquad \widetilde{\gamma}_j = \gamma_j, \quad j \neq k \,, \qquad \text{and} \qquad \gamma_k < \widetilde{\gamma}_k < \gamma_{k+1} \,.$$

To prove the lemma it is sufficient to study the above cases since the general case follows from this by a finite number of pairwise comparisons.

Case 2 can be handled by a straightforward modification of the arguments given in Case 1. Therefore we present the details only in Case 1.

By Lemmas 3.1 and 3.2, there are $P \in M_n(\Lambda)$ and $Q \in M_m(\Gamma)$ such that

$$\frac{|(P'Q)(c)|}{\|PQ\|_{[a,b]}} = \sup\left\{ \frac{|(p'q)(c)|}{\|pq\|_{[a,b]}} : \ p \in M_n(\Lambda) \,, \ q \in M_m(\Gamma) \right\} ,$$

where P has exactly n zeros in (a,b); Q has exactly m zeros in (a,b). Let $t_1 < t_2 < \cdots < t_n$ denote the n zeros of P in (a,b) and let $t_0 := 0$ and $t_{n+1} := c$. Let

$$P(x) =: \sum_{j=0}^{n} c_j x^{\lambda_j} , \qquad c_j \in \mathbb{R} \,.$$

Without loss of generality we may assume that $P(c) > 0$. (Note that $P(c) \neq 0$ since $P \in M_n(\Lambda)$, $M_n(\Lambda)$ is a Chebyshev space of dimension $n+1$ on $[a,b]$, P has exactly n zeros in (a,b), and $c > b$.) We have $\lim_{x \to \infty} P(x) = \infty$, otherwise, in addition to its n zeros in (a,b), P would have one more zero in (c, ∞), which is impossible, since $0 \neq P$ comes from a Chebyshev space of dimension $n+1$.

Because of the extremal property of P , $P'(c) \neq 0$. We show that $P'(c) > 0$. To see this observe that Rolle's Theorem implies that

$$P' \in \text{span}\{x^{\lambda_0 - 1}, x^{\lambda_1 - 1}, \dots, x^{\lambda_n - 1}\}$$

has at least $n-1$ zeros in (t_1, t_n). If $P'(c) < 0$, then $P(t_n) = 0$ and $\lim_{x \to \infty} P(x) = \infty$ imply that P' has at least 2 more zeros in (t_n, ∞). Thus $P'(c) < 0$ would imply that P' has at least $n+1$ zeros in $[a, \infty)$, which is impossible, since $0 \neq P'$ comes from a Chebyshev space of dimension $n+1$. Hence $P'(c) > 0$, indeed.

Since

$$(x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n})$$

is a Descartes system on $(0, \infty)$ it follows from Descartes' Rule of Signs that

$$(-1)^{n-j} c_j > 0 \,, \qquad j = 0, 1, \dots, n \,.$$

Choose $R \in M_n(\widetilde{\Lambda})$ of the form

$$R(x) = \sum_{j=0}^{n} d_j x^{\widetilde{\lambda}_j}, \qquad d_j \in \mathbb{R},$$

so that

$$R(t_i) = P(t_i)\,, \qquad i = 1, 2, \ldots, n+1$$

By the unique interpolation property of Chebyshev spaces, R is uniquely determined, has n zeros (the points $t_1, t_2, \ldots, t_n$) in (a, b), and is positive at c. Since

$$(x^{\widetilde{\lambda}_0}, x^{\widetilde{\lambda}_1}, \ldots, x^{\widetilde{\lambda}_n})$$

is a Descartes system on $(0, \infty)$, by Descartes' Rule of Signs,

$$(-1)^{n-j} d_j > 0\,, \qquad j = 0, 1, \ldots, n\,.$$

We have

$$(P - R)(x) = c_k x^{\lambda_k} - d_k x^{\widetilde{\lambda}_k} + \sum_{j=0, j\neq k}^{n} (c_j - d_j) x^{\lambda_j}\,.$$

The function $P - R$ changes sign in $(0, \infty)$ strictly at the points $t_1, t_2, \ldots, t_{n+1}$, and has no other zeros. Since

$$(x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_k}, x^{\widetilde{\lambda}_k}, x^{\lambda_{k+1}}, \ldots, x^{\lambda_n})$$

is a Descartes system on $(0, \infty)$, by Descartes' Rule of Signs, the sequence

$$(c_0 - d_0,\ \ c_1 - d_1,\ \ \ldots,\ \ c_{k-1} - d_{k-1},\ \ c_k,\ \ -d_k,\ \ c_{k+1} - d_{k+1},\ \ \ldots,\ \ c_n - d_n)$$

strictly alternates in sign. Since $(-1)^{n-k} c_k > 0$, this implies that $c_n - d_n < 0$ so

$$(P - R)(x) < 0\,, \qquad x > t_{n+1}\,.$$

Thus for $x \in (t_j, t_{j+1})$, we have

$$(-1)^{n-j} P(x) > (-1)^{n-j} R(x) > 0\,, \qquad j = 0, 1, \ldots, n\,.$$

In addition, we recall that $R(c) = P(c) > 0$.

The observations above imply that

$$\|RQ\|_{[a,b]} \leq \|PQ\|_{[a,b]} \qquad \text{and} \qquad R'(c) \geq P'(c) > 0\,.$$

(Note that even $|(RQ)(x)| \leq |(PQ)(x)|$ holds for all $x \in [a, b] \subset [a, c]$.)

Thus

$$\frac{|(R'Q)(c)|}{\|RQ\|_{[a,b]}} \geq \frac{|(P'Q)(c)|}{\|PQ\|_{[a,b]}} = \sup\left\{\frac{|(p'q)(c)|}{\|pq\|_{[a,b]}}: \ p \in M_n(\Lambda)\,, \ q \in M_m(\Gamma)\right\}.$$

Since $R \in M_n(\widetilde{\Lambda})$, the desired conclusion follows from this. This finishes the proof in Case 1. □

Proof of Lemma 3.4. The proof is a straightforward modification of the arguments in the proof of Lemma 3.3. We omit the details. □

Proof of Theorem 2.1. To prove the upper bound of the theorem, it is sufficient to prove that

$$(4.1) \qquad |(p'q)(1)| \leq (9+\eta)(n+m+1)(\lambda_n + \gamma_m)\|pq\|_{[\delta, 1-\delta]}$$

for every $p \in M_n(\Lambda)$ and $q \in M_m(\Gamma)$, where η denotes a quantity that tends to 0 as $\delta \in \left(0, \frac{1}{4}\right)$ tends to 0. The rest follows by the product rule of differentiation (the role of Λ and Γ can be interchanged), by taking the limit when $\delta \in \left(0, \frac{1}{4}\right)$ tends to 0, and by a linear scaling. To prove the above inequality, by Lemma 3.3 we may assume that

$$\lambda_j := \lambda_n - (n-j)\varepsilon\,, \qquad j = 0, 1, \ldots, n$$
$$\gamma_j := \gamma_m - (m-j)\varepsilon\,, \qquad j = 0, 1, \ldots, m$$

for some $\varepsilon > 0$. By Lemma 3.2 we may also assume that p has n zeros in $(\delta, 1-\delta)$ and q has m zeros in $(\delta, 1-\delta)$. We normalize p and q so that $p(1) > 0$ and $q(1) > 0$. Then, using the information on the zeros of p and q, we can easily see that $p'(1) > 0$ and $q'(1) > 0$. Therefore

$$|(p'q)(1)| \leq |(pq)'(1)|\,.$$

Now observe that $f := pq \in M_k(\Omega)$, where $k := n+m$ and $\Omega := (\omega_j)_{j=0}^{\infty}$ with

$$\omega_j := \lambda_n + \gamma_m - (n+m-j)\varepsilon\,.$$

Hence by Newman's inequality (see also the remark after it),

$$\begin{aligned} |(p'q)(1)| \leq |(pq)'(1)| = |f'(1)| &\leq 9\,(n+m+1)(\lambda_n + \gamma_m)\,\|f\|_{[0,1]} \\ &= 9\,(n+m+1)(\lambda_n + \gamma_m)\,\|pq\|_{[0,1]} \\ &\leq (9+\eta)\,(n+m+1)(\lambda_n + \gamma_m)\,\|pq\|_{[\delta,1-\delta]} \end{aligned}$$

with $\eta \to 0$ as $\delta \to 0$. The proof of the upper bound of the theorem is now finished.

The proof of the lower bound of the theorem can be easily reduced to the lower bound in Newman's inequality. Because of symmetry, we may assume that

$$(m+1)\lambda_n \leq (n+1)\gamma_m\,.$$

The lower bound of Newman's inequality guarantees a

$$0 \neq f \in \operatorname{span}\{x^{\lambda_0+\gamma_m}, x^{\lambda_1+\gamma_m}, \dots, x^{\lambda_n+\gamma_m}\}$$

such that

$$\begin{aligned}|f'(1)| &\geq \frac{2}{3}\left(\sum_{j=0}^{n}(\lambda_j+\gamma_m)\right)\|f\|_{[0,1]} \geq \frac{2}{3}(n+1)\gamma_m\|f\|_{[0,1]} \\ &\geq \frac{1}{3}((m+1)\lambda_n+(n+1)\gamma_m)\|f\|_{[0,1]}\,.\end{aligned}$$

Now observe that $f = pq$ with some $p \in M_n(\Lambda)$ and with $q \in M_m(\Gamma)$ defined by $q(x) := x^{\gamma_m}$. This finishes the proof of the lower bound in the theorem. □

Proof of Theorem 2.2. The lower bound of the theorem was shown in the proof of Theorem 2.1. We now prove the upper bound of the theorem. We want to prove that

$$|(p'q)(y)| \leq 18\,(n+m+1)(\lambda_n+\gamma_m)\,\|pq\|_{[0,1]} \tag{4.2}$$

for every $p \in M_n(\Lambda)$, $q \in M_m(\Gamma)$, and $y \in [0,1]$. The rest follows by the product rule of differentiation (the role of Λ and Γ can be interchanged). When $y \in [1/2,1]$, (4.2) follows from (4.1) by a linear scaling. Now let $y \in (0,1/2]$. To prove (4.2) for $y \in (0,1/2]$, we show that

$$|(p'q)(y)| \leq (18+\eta)\,(n+m+1)(\lambda_n+\gamma_m)\,\|pq\|_{[y+\delta,1]}\,,$$

where η denotes a quantity that tends to 0 as $\delta \in \left(0,\frac{1}{4}\right)$ tends to 0 (the rest follows by taking the limit when $\delta \in \left(0,\frac{1}{4}\right)$ tends to 0).

To see this, by Lemma 3.4 we may assume that

$$\begin{aligned}\lambda_j &:= j\,, \qquad j = 0,1,\dots,n\,,\\ \gamma_j &:= j\,, \qquad j = 0,1,\dots,m\,.\end{aligned}$$

By Lemma 3.2 we may also assume that p has n zeros in $(y+\delta,1)$ and q has m zeros in $(y+\delta,1)$. We normalize p and q so that $p(y) > 0$ and $q(y) > 0$. Then, using the information on the zeros of p and q, we can easily see that $p'(y) < 0$ and $q'(y) < 0$. Therefore

$$|(p'q)(y)| \leq |(pq)'(y)|\,.$$

Now observe that $f := pq \in M_k(\Omega)$, where $k := n+m$ and $\Omega := (\omega_j)_{j=0}^{\infty}$ with $\omega_j := j$. Hence by Markov's inequality,

$$\begin{aligned}|(p'q)(y)| \leq |(pq)'(y)| = |f'(y)| &\leq \frac{2}{1-y}\,(n+m)^2\,\|f\|_{[y,1]}\\ &\leq 4\,(n+m)^2\,\|f\|_{[y,1]}\\ &\leq 18\,(n+m+1)(\lambda_n+\gamma_m)\,\|pq\|_{[y,1]}\\ &\leq (18+\eta)\,(n+m+1)(\lambda_n+\gamma_m)\,\|pq\|_{[y+\delta,1]}\end{aligned}$$

with $\eta \to 0$ as $\delta \to 0$. The proof of the upper bound of the theorem is now finished. $\square$

Proof of Theorem 2.3. The lower bound of the theorem can be obtained by considering $g(x) := f(x/b)$, where f is the product that shows the lower bound in Theorem 2.1. We now prove the upper bound of the theorem. We want to prove that

$$|(p'q)(y)| \le c(a,b,\varrho)\,(n+m+1)(\lambda_n+\gamma_m)\,\|pq\|_{[a,b]} \tag{4.3}$$

for every $p \in M_n(\Lambda)$, $q \in M_m(\Gamma)$, and $y \in [a,b]$. The rest follows by the product rule of differentiation (the role of Λ and Γ can be interchanged). Let $d := \frac{2ab}{a+b} < b$.

First we show that

$$|(p'q)(b)| \le c_1(a,b,\varrho)\,(n+m+1)(\lambda_n+\gamma_m)\,\|pq\|_{[d,b]} \tag{4.4}$$

for every $p \in M_n(\Lambda)$ and $q \in M_m(\Gamma)$. To show (4.4), it is sufficient to prove that

$$|(p'q)(b)| \le (c_1(a,b,\varrho)+\eta)\,(n+m+1)(\lambda_n+\gamma_m)\,\|pq\|_{[d,b-\delta]} \tag{4.5}$$

for every $p \in M_n(\Lambda)$ and $q \in M_m(\Gamma)$, where η denotes a quantity that tends to 0 as $\delta \in (0,b-d)$ tends to 0. The rest follows by taking the limit when $\delta \in (0,b-d)$ tends to 0.

To prove the above inequality, by Lemma 3.3 we may assume that

$$\lambda_j := \lambda_n - (n-j)\varepsilon\,, \qquad j = 0,1,\dots,n\,,$$
$$\gamma_j := \gamma_m - (m-j)\varepsilon\,, \qquad j = 0,1,\dots m\,,$$

for some $\varepsilon > 0$. By Lemma 3.2 we may also assume that p has n zeros in $(d,b-\delta)$ and q has m zeros in $(d,b-\delta)$. We normalize p and q so that $p(b)>0$ and $q(b)>0$. Then, using the information on the zeros of p and q, we can easily see that $p'(b)>0$ and $q'(b)>0$. Therefore

$$|(p'q)(b)| \le |(pq)'(b)|\,.$$

Now observe that $f := pq \in M_k(\Omega)$, where $k := n+m$ and $\Omega := (\omega_j)_{j=0}^{\infty}$ with

$$\omega_j := \lambda_n + \gamma_m - (n+m-j)\varepsilon\,.$$

Hence Theorem 1.4 (Newman's Inequality on $[a,b] \subset (0,\infty)$) implies

$$\begin{aligned} |(p'q)(b)| \le |(pq)'(b)| = |f'(b)| &\le c_1(a,d,\varrho)\,(n+m+1)(\lambda_n+\gamma_m)\,\|f\|_{[d,b]} \\ &= c_1(a,b,\varrho)\,(n+m+1)(\lambda_n+\gamma_m)\,\|pq\|_{[d,b]}\,. \end{aligned}$$

By this (4.5), and hence (4.4), is proved.

Now let $y \in \left[\frac{1}{2}(a+b), b\right]$. Applying (4.4) with $p \in M_n(\Lambda)$ and $q \in M_m(\Gamma)$ replaced by $P \in M_n(\Lambda)$ and $Q \in M_m(\Gamma)$ defined by $P(x) := p(\eta x)$ and $Q(x) := q(\eta x)$ with $\eta := y/b$, we obtain that

$$\begin{aligned} |(p'q)(y)| = \frac{b}{y}\,|(P'Q)(b)| &\le \frac{b}{y}\,c_1(a,b,\varrho)\,(n+m+1)(\lambda_n+\gamma_m)\,\|PQ\|_{[d,b]} \\ &\le c_2(a,b,\varrho)\,(n+m+1)(\lambda_n+\gamma_m)\,\|pq\|_{[d\eta,y]} \\ &\le c_2(a,b,\varrho)\,(n+m+1)(\lambda_n+\gamma_m)\,\|pq\|_{[a,b]}\,. \end{aligned}$$

Note that $d\eta = dy/b \geq a$ for $y \in \left[\frac{1}{2}(a+b), b\right]$. So (4.3) is proved for all $y \in \left[\frac{1}{2}(a+b), b\right]$.

Now let $y \in \left[a, \frac{1}{2}(a+b)\right]$. We show that

$$|(p'q)(y)| \leq c_3(a,b,\varrho)\,(n+m+1)(\lambda_n+\gamma_m)\,\|pq\|_{[y,b]} \tag{4.6}$$

for every $p \in M_n(\Lambda)$ and $q \in M_m(\Gamma)$. To show (4.6), it is sufficient to prove that

$$|(p'q)(y)| \leq (c_3(a,b,\varrho)+\eta)\,(n+m+1)(\lambda_n+\gamma_m)\,\|pq\|_{[y+\delta,b]} \tag{4.7}$$

for every $p \in M_n(\Lambda)$ and $q \in M_m(\Gamma)$, where η denotes a quantity that tends to 0 as $\delta > 0$ tends to 0. The rest follows by taking the limit when $\delta > 0$ tends to 0.

To see (4.7), by Lemma 3.3 we may assume that

$$\begin{aligned} \lambda_j &:= \varrho j\,, \qquad j = 0,1,\dots,n\,, \\ \gamma_j &:= \varrho j\,, \qquad j = 0,1,\dots,m\,. \end{aligned}$$

By Lemma 3.2 we may also assume that p has n zeros in $(y+\delta, b)$ and q has m zeros in $(y+\delta, b)$. We normalize p and q so that $p(y) > 0$ and $q(y) > 0$. Then, using the information on the zeros of p and q, we can easily see that $p'(y) < 0$ and $q'(y) < 0$. Therefore

$$|(p'q)(y)| \leq |(pq)'(y)|\,.$$

Now observe that $f := pq \in M_k(\Omega)$, where $k := n+m$ and $\Omega := (\omega_j)_{j=0}^{\infty}$ with $\omega_j := \varrho j$. Hence by Markov's inequality,

$$\begin{aligned} |(p'q)(y)| \leq |(pq)'(y)| = |f'(y)| &\leq \frac{2\varrho\, y^{\varrho-1}}{b^{\varrho}-y^{\varrho}}\,(n+m)^2\,\|f\|_{[y,b]} \\ &\leq c_3(a,b,\varrho)\,(n+m)^2\,\|f\|_{[y,b]} \\ &\leq c_3(a,b,\varrho)\,(n+m+1)(\lambda_n+\gamma_m)\,\|pq\|_{[y,b]}\,. \end{aligned}$$

So (4.7), and hence (4.6), is proved for all $y \in \left[a, \frac{1}{2}(a+b)\right]$.

The proof of the theorem is now complete. □

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