MARKOV-TYPE INEQUALITIES FOR PRODUCTS OF MÜNTZ POLYNOMIALS REVISITED

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Abstract. Professor Rahman was a great expert of Markov- and Bernstein- type inequalities for various classes of functions, in particular for polynomials under various constraints on their zeros, coefficients, and so on. His books are great sources of such inequalities and related matters. Here we do not even try to survey Rahman's contributions to Markov- and Bernstein-type inequalities and related results. We focus on Markov-type inequalities for products of Müntz polynomials. Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers. We denote the linear span of $x^{\lambda_0}, x^{\lambda_1}, \dots , x^{\lambda_n}$ over $\mathbb{R}$ by $M(\Lambda_n) := \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots , x^{\lambda_n}\}$. Elements of $M(\Lambda_n)$ are called Müntz polynomials. The principal result of this paper is a Markov-type inequality for products of Müntz polynomials on intervals $[a,b] \subset (0,\infty)$ which extends a less general result proved in an earlier publication. It allows us to answer some questions asked by Thomas Bloom recently in e-mail communications. The author believes that the new results in this paper are sufficiently interesting and original to serve as a tribute to the memory of Professor Rahman in this volume.

1. Introduction and Notation

Let $\mathcal{P}_n$ denote the family of all algebraic polynomials of degree at most n with real coefficients. We use the notation

$$\|f\|_A := \|f\|_{L_\infty(A)} := \|f\|_{L_\infty A} := \sup_{t \in A} |f(t)|$$

and

$$\|f\|_{L_q A} := \|f\|_{L_q(A)} := \left(\int_a^b |f(t)|^q \, dt \right)^{1/q}, \qquad q > 0\,,$$

for measurable functions f defined on a nonempty set $A \subset \mathbb{R}$. Two classical inequalities for polynomials are the following.

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Markov Inequality. *We have*

$$\|f'\|_{[a,b]} \le \frac{2n^2}{b-a}\, \|f\|_{[a,b]}$$

for every $f \in \mathcal{P}_n$ and for every subinterval $[a,b]$ of the real line.

Bernstein Inequality. *We have*

$$|f'(y)| \le \frac{n}{\sqrt{(b-y)(y-a)}}\, \|f\|_{[a,b]}\,, \qquad y \in (a,b)\,,$$

for every $f \in \mathcal{P}_n$ and for every $[a,b] \subset \mathbb{R}$.

For proofs see [4] or [16], for example. Professor Rahman was a great expert of Markov- and Bernstein-type inequalities for various classes of functions, in particular for polynomials under various constraints on their zeros, coefficients, and so on. His books [33] and [34] are great sources of such inequalities. See also [4,19,22], for instance. Here we do not even try to survey Rahman's contributions to Markov- and Bernstein-type inequalities and related results. We focus only on Markov- and Bernstein-type inequalities for products of Müntz polynomials. Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers. We denote the linear span of $x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}$ over $\mathbb{R}$ by

$$M(\Lambda_n) := \mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}\,.$$

Elements of $M(\Lambda_n)$ are called Müntz polynomials. We denote the linear span of $e^{\lambda_0 t}, e^{\lambda_1 t}$, $\ldots, e^{\lambda_n t}$ over $\mathbb{R}$ by

$$E(\Lambda_n) := \mathrm{span}\{e^{\lambda_0 t}, e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\}\,.$$

Elements of $E(\Lambda_n)$ are called exponential sums. Observe that the substitution $x = e^t$ transforms exponential sums into Müntz polynomials and the interval $(-\infty, 0]$ onto $(0,1]$.

Newman [31] established an essentially sharp Markov-type inequality for $M(\Lambda_n)$.

Theorem 1.1 (Newman's Inequality). *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of nonnegative real numbers. We have*

$$\frac{2}{3}\sum_{j=0}^{n} \lambda_j \le \sup_{0 \not\equiv f \in M(\Lambda_n)} \frac{|f'(1)|}{\|f\|_{[0,1]}} \le \sup_{0 \not\equiv f \in M(\Lambda_n)} \frac{\|xf'(x)\|_{[0,1]}}{\|f\|_{[0,1]}} \le 11 \sum_{j=0}^{n} \lambda_j\,.$$

Frappier [28] showed that the constant 11 in Newman's inequality can be replaced by 8.29. By modifying and simplifying Newman's arguments, Borwein and Erdélyi [9] showed that the constant 11 in the above inequality can be replaced by 9. But more importantly, this modification allowed us to prove the "right" L_q version ($1 \le q \le \infty$) of Newman's inequality [9] (an L_2 version of which was proved earlier by Borwein, Erdélyi, and Zhang [13]). Note that Newman's inequality can be rewritten as

$$\frac{2}{3}\sum_{j=0}^{n} \lambda_j \le \sup_{0 \not\equiv g \in E(\Lambda_n)} \frac{|g'(0)|}{\|g\|_{(-\infty,0]}} \le \sup_{0 \not\equiv g \in E(\Lambda_n)} \frac{\|g'\|_{(-\infty,0]}}{\|g\|_{(-\infty,0]}} \le 11 \sum_{j=0}^{n} \lambda_j\,,$$

whenever $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ is a set of nonnegative real numbers.

It is non-trivial and proved by Borwein and Erdélyi [4] that under a growth condition, $\|xf'(x)\|_{[0,1]}$ in Newman's inequality can be replaced by $\|f'\|_{[0,1]}$. More precisely, the following result holds.

Theorem 1.2 (Newman's Inequality Without the Factor x). *Let*

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$$

be a set of nonnegative real numbers with $\lambda_0 = 0$ and $\lambda_j \geq j$ for each j. We have

$$\|f'\|_{[0,1]} \leq 18 \left(\sum_{j=0}^{n} \lambda_j \right) \|f\|_{[0,1]}$$

for every $f \in M(\Lambda_n)$.

It can be shown that the growth condition in Theorem 1.2 is essential. This observation is based on an example given by Len Bos (non-published communication). The statement below is proved in [18].

Example 1.3. *For every $\delta \in (0,1)$ there exists a sequence $\Lambda := (\lambda_j)_{j=0}^{\infty}$ with $\lambda_0 := 0$, $\lambda_1 \geq 1$, and*

$$\lambda_{j+1} - \lambda_j \geq \delta \,, \qquad j = 0, 1, 2, \ldots$$

such that with $\Lambda_\mu := \{\lambda_0 < \lambda_1 < \cdots < \lambda_\mu\}$ we have

$$\lim_{\mu \to \infty} \sup_{0 \not\equiv f \in M(\Lambda_\mu)} \frac{|f'(0)|}{\left(\sum_{j=0}^{\mu} \lambda_j \right) \|f\|_{[0,1]}} = \infty \,.$$

Note that the interval $[0,1]$ plays a special role in the study of Müntz polynomials. A linear transformation $y = \alpha x + \beta$ does not preserve membership in $M(\Lambda_n)$ in general (unless $\beta = 0$), that is, $f \in M(\Lambda_n)$ does not necessarily imply that $g(x) := f(\alpha x + \beta) \in M(\Lambda_n)$. Analogs of the above results on $[a,b]$, $a > 0$, cannot be obtained by a simple transformation. However, Borwein and Erdélyi [8] proved the following result.

Theorem 1.4 (Newman's Inequality on $[a,b] \subset (0,\infty)$). *Let*

$$\Lambda_n := \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n\}$$

be a set of real numbers. Suppose there exists a $\varrho > 0$ such that $\lambda_j \geq \varrho j$ for each j. Suppose $a, b \in \mathbb{R}$ and $0 < a < b$. There exists a constant $c(a,b,\varrho)$ depending only on a, b, and ϱ such that

$$\|f'\|_{[a,b]} \leq c(a,b,\varrho) \left(\sum_{j=0}^{n} \lambda_j \right) \|f\|_{[a,b]}$$

for every $f \in M(\Lambda_n)$.

The above theorem is essentially sharp, as one can easily deduce it from the first inequality of Theorem 1.1 by a linear scaling. The novelty of Theorem 1.5 proved in [2] later is the fact that $\Lambda_n := \{\lambda_0 < \lambda_1 < \ldots < \lambda_n\}$ is an arbitrary set of $n+1$ distinct real numbers, not even the non-negativity of the exponents λ_j is needed.

Theorem 1.5. *Let $n \geq 1$ be an integer. Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of $n+1$ distinct real numbers. Suppose $a, b \in \mathbb{R}$ and $0 < a < b$. We have*

$$\frac{1}{3}\sum_{j=0}^{n} |\lambda_j| + \frac{1}{4\log(b/a)}(n-1)^2 \leq \sup_{0 \not\equiv f \in M(\Lambda_n)} \frac{\|xf'(x)\|_{[a,b]}}{\|f\|_{[a,b]}} \leq 11 \sum_{j=0}^{n} |\lambda_j| + \frac{128}{\log(b/a)}(n+1)^2 .$$

Remark 1.6. Of course, we can have $f'(x)$ instead of $xf'(x)$ in the above estimate, as an obvious corollary of the above theorem is

$$\frac{1}{3b}\sum_{j=0}^{n} |\lambda_j| + \frac{1}{4b\log(b/a)}(n-1)^2 \leq \sup_{0 \not\equiv f \in M(\Lambda_n)} \frac{\|f'\|_{[a,b]}}{\|f\|_{[a,b]}} \leq \frac{11}{a} \sum_{j=0}^{n} |\lambda_j| + \frac{128}{a\log(b/a)}(n+1)^2$$

for every $a, b \in \mathbb{R}$ such that $0 < a < b$. Observe also that Theorem 1.1 can be obtained from Theorem 1.5 (with the constant $1/3$ in the lower bound) as a limiting case by letting $a > 0$ tend to 0.

The following $L_q[a,b]$ version of Theorem 1.5 is also proved in [2] for $q \geq 1$.

Theorem 1.7. *Let $n \geq 1$ be an integer. Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of $n+1$ distinct real numbers. Suppose $a, b \in \mathbb{R}$, $0 < a < b$, and $1 \leq q < \infty$. There is a positive constant $c_1(a,b)$ depending only on a and b such that*

$$\sup_{0 \not\equiv f \in M(\Lambda_n)} \frac{\|f'\|_{L_q[a,b]}}{\|f\|_{L_q[a,b]}} \leq c_1(a,b)\left(n^2 + \sum_{j=0}^{n} |\lambda_j|\right) .$$

Theorem 1.7 was proved earlier under the additional assumptions that $\lambda_0 := 0$ and $\lambda_j \geq \delta j$ for each j with a constant $\delta > 0$ and with $c_1(a,b)$ replaced by $c_1(a,b,\delta)$, see [17]. The novelty of Theorem 1.7 is the fact again that $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ is an arbitrary set of $n+1$ distinct real numbers, not even the non-negativity of the exponents λ_j is needed.

In [21] the following Markov-Nikolskii-type inequality has been proved for $E(\Lambda_n)$ on $(-\infty, 0]$.

Theorem 1.8. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of nonnegative real numbers and $0 < q \leq p \leq \infty$. Let μ be a nonnegative integer. There are constants $c_2 = c_2(p,q,\mu) > 0$ and $c_3 = c_3(p,q,\mu)$ depending only on p, q, and μ such that*

$$c_2\left(\sum_{j=0}^{n} \lambda_j\right)^{\mu+\frac{1}{q}-\frac{1}{p}} \leq \sup_{0 \not\equiv f \in E(\Lambda_n)} \frac{\|f^{(\mu)}\|_{L_p(-\infty,0]}}{\|f\|_{L_q(-\infty,0]}} \leq c_3\left(\sum_{j=0}^{n} \lambda_j\right)^{\mu+\frac{1}{q}-\frac{1}{p}} ,$$

where the lower bound holds for all $0 < q \leq p \leq \infty$ and $\mu \geq 0$, while the upper bound holds when $\mu = 0$ and $0 < q \leq p \leq \infty$, and when $\mu \geq 1$, $p \geq 1$, and $0 < q \leq p \leq \infty$. Also, there are constants $c_2 = c_2(q,\mu) > 0$ and $c_3 = c_3(q,\mu)$ depending only on q and μ such that

$$c_2\left(\sum_{j=0}^{n} \lambda_j\right)^{\mu+\frac{1}{q}} \leq \sup_{0 \not\equiv f \in E(\Lambda_n)} \frac{|f^{(\mu)}(y)|}{\|f\|_{L_q(-\infty,y]}} \leq c_3\left(\sum_{j=0}^{n} \lambda_j\right)^{\mu+\frac{1}{q}}$$

for all $0 < q \le \infty$, $\mu \ge 1$, *and* $y \in \mathbb{R}$.

Motivated by a question of Michel Weber in [25] we proved the following Markov-Nikolskii-type inequalities have been proved for $E(\Lambda_n)$ on $[a,b] \subset (-\infty, \infty)$.

Theorem 1.9. *Suppose* $0 < q \le p \le \infty$, $a, b \in \mathbb{R}$, *and* $a < b$. *There are constants* $c_4 = c_4(p,q,a,b) > 0$ *and* $c_5 = c_5(p,q,a,b)$ *depending only on* p, q, a, *and* b *such that*

$$c_4 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{\frac{1}{q} - \frac{1}{p}} \le \sup_{0 \not\equiv f \in E(\Lambda_n)} \frac{\|f\|_{L_p[a,b]}}{\|f\|_{L_q[a,b]}} \le c_5 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{\frac{1}{q} - \frac{1}{p}} .$$

Theorem 1.10. *Suppose* $0 < q \le p \le \infty$, $a, b \in \mathbb{R}$, *and* $a < b$. *There are constants* $c_6 = c_6(p,q,a,b) > 0$ *and* $c_7 = c_7(p,q,a,b)$ *depending only on* p, q, a, *and* b *such that*

$$c_6 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{1 + \frac{1}{q} - \frac{1}{p}} \le \sup_{0 \not\equiv f \in E(\Lambda_n)} \frac{\|f'\|_{L_p[a,b]}}{\|f\|_{L_q[a,b]}} \le c_7 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{1 + \frac{1}{q} - \frac{1}{p}} ,$$

where the lower bound holds for all $0 < q \le p \le \infty$, *while the upper bound holds when* $p \ge 1$ *and* $0 < q \le p \le \infty$.

We note that even more general Nikolskii-type inequalities are proved in [12] for shift invariant function spaces.

Müntz's classical theorem characterizes the sequences $\Lambda := (\lambda_j)_{j=0}^{\infty}$ with

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

for which the Müntz space

$$M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $C[0,1]$. Here $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ denotes the collection of all *finite linear combinations of the functions* $x^{\lambda_0}, x^{\lambda_1}, \dots$ *with real coefficients, and* $C(A)$ *is the space of all real-valued continuous functions on* $A \subset [0, \infty)$ *equipped with the supremum norm. If* $A := [a,b]$ *is a finite closed interval, then the notation* $C[a,b] := C([a,b])$ *is used. Müntz's Theorem states the following.*

Müntz's Theorem. *Suppose* $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$. *The space* $M(\Lambda)$ *is dense in* $C[0,1]$ *if and only if* $\sum_{j=1}^{\infty} 1/\lambda_j = \infty$.

Proofs are available in [4,14,16], for example. The original Müntz Theorem proved by Müntz [30] and Szász [38] and anticipated by Bernstein [3] was only for sequences of exponents tending to infinity. There are many generalizations and variations of Müntz's Theorem. See [4], [5], [6], [7], [27], [20], [15], [16], [39], [29], and [35] among others. There are also many problems still open today.

Somorjai [37] in 1976 and Bak and Newman [1] in 1978 proved that

$$R(\Lambda) := \{p/q : p, q \in M(\Lambda)\}$$

is always dense in $C[0,1]$ whenever $\Lambda := (\lambda_j)_{j=0}^\infty$ contains infinitely many distinct real numbers. This surprising result says that while the set $M(\Lambda)$ of Müntz polynomials may be far from dense, the set $R(\Lambda)$ of Müntz rationals is always dense in $C[0,1]$, whenever the underlying sequence Λ contains infinitely many distinct real numbers. In the light of this result, Newman [32] (p. 50) raises "the very sane, if very prosaic question". Are the functions

$$\prod_{j=1}^{k} \left(\sum_{m=0}^{n_j} a_{m,j} x^{m^2} \right), \qquad a_{m,j} \in \mathbb{R}, \quad n_j \in \mathbb{N},$$

dense in $C[0,1]$ for some fixed $k \geq 2$? In other words does the "extra multiplication" have the same power that the "extra division" has in the Bak-Newman-Somorjai result? Newman speculated that it did not.

Denote the set of the above products by H_k. Since every natural number is the sum of four squares, H_4 contains all the monomials x^n, $n = 0, 1, 2, \ldots$. However, H_k is not a linear space, so Müntz's Theorem itself cannot be applied to resolve the denseness or non-denseness of H_4 in $C[0,1]$.

Borwein and Erdélyi [4,5,10] deal with products of Müntz spaces and, in particular, the question of Newman is answered in the negative. In fact, in [6] we presented a number of inequalities each of which implies the answer to Newman's question. One of them is the following bounded Bernstein-type inequality for products of Müntz polynomials from non-dense Müntz spaces. For

$$\Lambda^{(j)} := (\lambda_{i,j})_{i=0}^\infty, \qquad 0 = \lambda_{0,j} < \lambda_{1,j} < \lambda_{2,j} < \cdots, \qquad j = 1, 2, \ldots,$$

we define the sets

$$M(\Lambda^{(1)}, \Lambda^{(2)}, \ldots, \Lambda^{(k)}) := \left\{ f = \prod_{j=1}^{k} f_j : f_j \in M(\Lambda^{(j)}) \right\}.$$

Theorem 1.11. *Suppose*

$$\Lambda^{(j)} := (\lambda_{i,j})_{i=0}^\infty, \qquad 0 = \lambda_{0,j} < \lambda_{1,j} < \lambda_{2,j} < \cdots, \qquad j = 1, 2, \ldots, k,$$

and

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_{i,j}} < \infty \qquad \text{and} \qquad \lambda_{1,j} \geq 1, \qquad j = 1, 2, \ldots, k.$$

Let $s > 0$. There exits a constant c depending only on $\Lambda^{(1)}, \Lambda^{(2)}, \ldots, \Lambda^{(k)}$, s, and k (and not on ϱ or A) such that

$$\|f'\|_{[0,\varrho]} \leq c\,\|f\|_A$$

for every $f \in M(\Lambda^{(1)}, \Lambda^{(2)}, \dots, \Lambda^{(k)})$ *and for every set* $A \subset [\varrho, 1]$ *of Lebesgue measure at least* s*.*

In [18] the right Markov-type inequalities for products of Müntz polynomials are established when the factors come from arbitrary (not necessarily non-dense) Müntz spaces. More precisely, associated with the sets

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\} \qquad \text{and} \qquad \Gamma_m := \{\gamma_0 < \gamma_1 < \cdots < \gamma_m\}$$

of real numbers we examined the magnitude of

$$K(M(\Lambda_n), M(\Gamma_m)) := \sup\left\{\frac{\|x(pq)'(x)\|_{[0,1]}}{\|pq\|_{[0,1]}} : 0 \not\equiv p \in M(\Lambda_n)\,, 0 \not\equiv q \in M(\Gamma_m)\right\}\,, \tag{1.1}$$

$$\widetilde{K}(M(\Lambda_n), M(\Gamma_m), a, b) := \sup\left\{\frac{\|(pq)'\|_{[a,b]}}{\|pq\|_{[a,b]}} : 0 \not\equiv p \in M(\Lambda_n)\,, 0 \not\equiv q \in M(\Gamma_m)\right\}\,, \tag{1.2}$$

where $[a,b] \subset [0,\infty)$, and

$$\widetilde{K}(E(\Lambda_n), E(\Gamma_m), a, b) := \sup\left\{\frac{\|(pq)'\|_{[a,b]}}{\|pq\|_{[a,b]}} : 0 \not\equiv p \in E(\Lambda_n)\,, 0 \not\equiv q \in E(\Gamma_m)\right\}\,, \tag{1.3}$$

where $[a,b] \subset (-\infty,\infty)$.

The result below proved in [18] is an essentially sharp Newman-type inequality for products of Müntz polynomials.

Theorem 1.12. *Let*

$$\Lambda_n := \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n\} \qquad \textit{and} \qquad \Gamma_m := \{0 = \gamma_0 < \gamma_1 < \cdots < \gamma_m\}\,.$$

Let $K(M(\Lambda_n), M(\Gamma_m))$ *be defined by (1.1). We have*

$$\frac{1}{3}\,((m+1)\lambda_n + (n+1)\gamma_m) \le K(M(\Lambda_n), M(\Gamma_m)) \le 18\,(n+m+1)(\lambda_n + \gamma_m)\,.$$

In particular,

$$\frac{2}{3}\,(n+1)\lambda_n \le K(M(\Lambda_n), M(\Lambda_n)) \le 36\,(2n+1)\lambda_n\,.$$

The factor x from $\|x(pq)'(x)\|_{[0,1]}$ in Theorem 1.12 can be dropped in the expense of a growth condition. The result below proved in [18] establishes an essentially sharp Markov-type inequality on $[0,1]$.

Theorem 1.13. *Let*

$$\Lambda_n := \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n\} \qquad \text{and} \qquad \Gamma_m := \{0 = \gamma_0 < \gamma_1 < \cdots < \gamma_m\}$$

with $\lambda_j \geq j$ and $\gamma_j \geq j$ for each j. Let $\widetilde{K}(M(\Lambda_n), M(\Gamma_m), 0, 1)$ be defined by (1.2). We have

$$\frac{1}{3}\left((m+1)\lambda_n + (n+1)\gamma_m\right) \leq \widetilde{K}(M(\Lambda_n), M(\Gamma_m), 0, 1) \leq 36\,(n+m+1)(\lambda_n + \gamma_m)\,.$$

In particular,

$$\frac{2}{3}\,(n+1)\lambda_n \leq \widetilde{K}(M(\Lambda_n), M(\Lambda_n), 0, 1) \leq 72\,(2n+1)\lambda_n\,.$$

Under a growth condition again, Theorem 1.13 can be extended to the interval $[0,1]$ replaced by $[a,b] \subset (0,\infty)$. The essentially sharp Markov-type inequality below is also proved in [18].

Theorem 1.14. *Let*

$$\Lambda_n := \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n\} \qquad \text{and} \qquad \Gamma_m := \{0 = \gamma_0 < \gamma_1 < \cdots < \gamma_m\}\,.$$

Suppose there exists a $\varrho > 0$ such that $\lambda_j \geq \varrho j$ and $\gamma_j \geq \varrho j$ for each j. Suppose $a, b \in \mathbb{R}$ and $0 < a < b$. Let $\widetilde{K}(M_n(\Lambda), M_m(\Gamma), a, b)$ be defined by (1.2). There is a constant $c(a,b,\varrho)$ depending only on a, b, and ϱ such that

$$\frac{b}{3}\left((m+1)\lambda_n + (n+1)\gamma_m\right) \leq \widetilde{K}(M(\Lambda_n), M(\Gamma_m), a, b) \leq c(a,b,\varrho)\,(n+m+1)(\lambda_n + \gamma_m)\,.$$

In particular,

$$\frac{2b}{3}\,(n+1)\lambda_n \leq \widetilde{K}(M(\Lambda_n), M(\Lambda_n), a, b) \leq 2\,c(a,b,\varrho)\,(2n+1)\lambda_n\,.$$

Remark 1.15 Analogs of the above three theorems dealing with products of several Müntz polynomials can also be proved by straightforward modifications.

Remark 1.16 Let $\lambda_j = \gamma_j := j^2, j = 0, 1, \ldots, n$. If we multiply pq out, where $p, q \in M(\Lambda_n)$, and we apply Newman's inequality, we get

$$K(M_n(\Lambda), M_n(\Lambda)) \leq cn^4$$

with an absolute constant c. However, if we apply Theorem 1.12, we obtain

$$K(M_n(\Lambda), M_n(\Lambda)) \leq 36\,(2n+1)n^2\,.$$

It is quite remarkable that $K(M_n(\Lambda), M_n(\Lambda))$ is of the same order of magnitude as the Markov factor $11\left(\sum_{j=0}^n j^2\right)$ in Newman's inequality for $M_n(\Lambda)$. When the exponents λ_j grow sufficiently slowly, similar improvements can be observed in each of our Theorems 1.12–1.14 compared with the "natural first idea" of "multiply out and use Newman's inequality".

The essentially sharp Bernstein-type inequality below for

$$E_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}\,,\ \ a_j, \lambda_j \in \mathbb{R} \right\} = \bigcup E(\Lambda_n)$$

is proved in [8] (the union above is taken for all $\Lambda_n = \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ for which $0 \in \Lambda_n$).

Theorem 1.17. *We have*

$$\frac{1}{e-1}\,\frac{n-1}{\min\{y-a,b-y\}} \leq \sup_{0\not\equiv f\in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}} \leq \frac{2n-1}{\min\{y-a,b-y\}}\,, \qquad y\in(a,b)\,.$$

We note that pointwise Remez- and Nikolskii-type inequalities for E_n are also proved in [11].

2. New Results

The results of this section were motivated by e-mail communications with Thomas Bloom who was interested in Corollaries 2.3–2.6 in particular.

We examine what happens when in Theorem 1.14 we drop the growth condition "there exists a $\varrho > 0$ such that $\lambda_j \geq \varrho j$ and $\gamma_j \geq \varrho j$ for each j".

Modifying the proof of Theorem 1.14 we can prove the result below.

Theorem 2.1. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ and $\Gamma_m := \{\gamma_0 < \gamma_1 < \cdots < \gamma_m\}$ be sets of real numbers such that $\lambda_0 \leq 0 \leq \lambda_n$ and $\gamma_0 \leq 0 \leq \gamma_m$. Suppose $a,b\in\mathbb{R}$ and $0<a<b$. Let $\widetilde{K}(M(\Lambda_n),M(\Gamma_m),a,b)$ be defined by (1.2). We have*

$$\widetilde{K}(M(\Lambda_n),M(\Gamma_m),a,b) \leq 22(n+m+1)(\lambda_n-\lambda_0+\gamma_m-\gamma_0) + \frac{512}{\log(b/a)}(n+m+1)^2\,.$$

If, in addition, $\lambda_0=\gamma_0=0$, then

$$\frac{1}{6}\left((m+1)\lambda_n+(n+1)\gamma_m\right) + \frac{1}{16\log(b/a)}(n+m-2)^2 \leq \widetilde{K}(M(\Lambda_n),M(\Gamma_m),a,b)\,.$$

Corollary 2.2. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers such that $\lambda_0 \leq 0 \leq \lambda_n$. Suppose $a,b\in\mathbb{R}$ and $0<a<b$. Let*

$$\widetilde{K}(M(\Lambda_n),M(\Lambda_n),a,b)$$

be defined by (1.2). We have

$$\widetilde{K}(M(\Lambda_n),M(\Lambda_n),a,b) \leq 44(2n+1)(\lambda_n-\lambda_0) + \frac{512}{\log(b/a)}(2n+1)^2\,.$$

If, in addition, $\lambda_0=0$, then

$$\frac{1}{3}(n+1)\lambda_n + \frac{1}{4}\log(b/a)(n-1)^2 \leq \widetilde{K}(M(\Lambda_n),M(\Lambda_n),a,b)\,.$$

By using the substitution $x=e^t$ it is easy to see that the theorem below is equivalent to Theorem 2.1.

Theorem 2.1*. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ and $\Gamma_m := \{\gamma_0 < \gamma_1 < \cdots < \gamma_m\}$ be sets of real numbers such that $\lambda_0 \le 0 \le \lambda_n$, $\gamma_0 \le 0 \le \gamma_m$. Suppose $a, b \in \mathbb{R}$ and $a < b$. Let*

$$\widetilde{K}(E(\Lambda_n), E(\Gamma_m), a, b)$$

be defined by (1.3). We have

$$\widetilde{K}(E(\Lambda_n), E(\Gamma_m), a, b) \le 22(n+m+1)(\lambda_n - \lambda_0 + \gamma_m - \gamma_0) + \frac{512}{b-a}(n+m+1)^2 \,.$$

If, in addition, $\lambda_0 = \gamma_0 = 0$, then

$$\frac{1}{6}\left((m+1)\lambda_n + (n+1)\gamma_m\right) + \frac{1}{16(b-a)}(n+m-2)^2 \le \widetilde{K}(E(\Lambda_n), E(\Gamma_m), a, b) \,.$$

Corollary 2.2*. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers such that $\lambda_0 \le 0 \le \lambda_n$. Suppose $a, b \in \mathbb{R}$ and $a < b$. Let*

$$\widetilde{K}(E(\Lambda_n), E(\Lambda_n), a, b)$$

be defined by (1.3). We have

$$\widetilde{K}(E(\Lambda_n), E(\Lambda_n), a, b) \le 22(2n+1)\lambda_n + \frac{512}{b-a}(2n+1)^2 \,.$$

If, in addition, $\lambda_0 = 0$, then

$$\frac{1}{3}(n+1)\lambda_n + \frac{1}{4}(b-a)(n-1)^2 \le \widetilde{K}(E(\Lambda_n), E(\Lambda_n), a, b) \,.$$

Theorem 2.1 gives the size of

$$(2.1) \qquad \widetilde{K}(\mathcal{P}_n, \mathcal{P}_m, a, b, \alpha, \beta) := \sup\left\{ \frac{\left\| \frac{d}{dx}(p(x^\alpha)q(x^\beta)) \right\|_{[a,b]}}{\|p(x^\alpha)q(x^\beta)\|_{[a,b]}} : \ p \in \mathcal{P}_n\,, \ \ q \in \mathcal{P}_m \right\}$$

immediately for real numbers $0 < a < b$, $\alpha > 0$, and $\beta > 0$.

Corollary 2.3. *Suppose $a, b, \alpha, \beta \in \mathbb{R}$, $0 < a < b$, $\alpha > 0$, and $\beta > 0$. Let*

$$\widetilde{K}(\mathcal{P}_n, \mathcal{P}_m, a, b, \alpha, \beta)$$

be defined by (2.1). We have

$$\widetilde{K}(\mathcal{P}_n, \mathcal{P}_m, a, b, \alpha, \beta) \le 22(n+m+1)(n\alpha + m\beta) + \frac{512}{b-a}(n+m+1)^2$$

and

$$\frac{1}{6}\left((m+1)n\alpha + (n+1)m\beta\right) + \frac{1}{16(b-a)}(n+m-2)^2 \le \widetilde{K}(\mathcal{P}_n, \mathcal{P}_m, a, b, \alpha, \beta) \,.$$

Corollary 2.4. *Suppose $a, b, \alpha, \beta \in \mathbb{R}$, $0 < a < b$, $\alpha > 0$, and $\beta > 0$. Let*

$$\widetilde{K}(\mathcal{P}_n, \mathcal{P}_m, a, b, \alpha, \beta)$$

be defined by (2.1). We have

$$\widetilde{K}(\mathcal{P}_n, \mathcal{P}_m, a, b, \alpha, \beta) \sim (n+m)^2 \,,$$

where $x \sim y$ means that $c_1 \leq x/y \leq c_2$ with some constants $c_1 > 0$ and $c_2 > 0$ depending only on a, b, α, and β.

Finding the size of

$$\widetilde{K}(E(\Lambda_n), \mathcal{P}_m, a, b) := \sup \left\{ \frac{\|pq)'\|_{[a,b]}}{\|pq\|_{[a,b]}} : \; p \in E(\Lambda_n)\,, \; q \in \mathcal{P}_m \right\}$$

can also be viewed as a special case of Theorem 2.1*.

Corollary 2.5. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers such that $\lambda_0 \leq 0 \leq \lambda_n$. Suppose $a, b \in \mathbb{R}$ and $a < b$. We have*

$$\widetilde{K}(E(\Lambda_n), \mathcal{P}_m, a, b) \leq 22(n+m+1)(\lambda_n - \lambda_0) + \frac{512}{b-a}(n+m+1)^2 \,.$$

If, in addition, $\lambda_0 = 0$, then

$$\frac{1}{6}\,(m+1)\lambda_n + \frac{1}{16(b-a)}(n+m-2)^2 \leq \widetilde{K}(E(\Lambda_n), \mathcal{P}_m, a, b) \,.$$

As a special case of Corollary 2.5 we record the following.

Corollary 2.6. *Suppose $a, b \in \mathbb{R}$ and $a < b$. Let $\Lambda_n := \{0, 1, \ldots, n\}$. We have*

$$\widetilde{K}(E(\Lambda_n), \mathcal{P}_m, a, b) \sim (n+m)^2 \,,$$

where $x \sim y$ means that $c_1 \leq x/y \leq c_2$ with some constants $c_1 > 0$ and $c_2 > 0$ depending only on a and b.

Let $\Gamma_m := \{0 = \gamma_0 < \gamma_1 < \cdots < \gamma_m\}$ be a set of nonnegative real numbers. We denote the collection of all linear combinations of

$$1,\, \cosh(\gamma_1 t),\, \cosh(\gamma_2 t),\, \ldots\,,\, \cosh(\gamma_m t)$$

over $\mathbb{R}$ by

$$G(\Gamma_m) := \operatorname{span}\{1\,,\, \cosh(\gamma_1 t)\,,\, \cosh(\gamma_2 t)\,,\, \ldots\,,\, \cosh(\gamma_m t)\} \,.$$

Our next result is a Bernstein-type inequality for product of exponential sums. It would be desirable to replace $G(\Gamma_m)$ with $E(\Gamma_m)$ in the theorem below but our method of proof does not seem to allow us to do so.

Theorem 2.7. *Let*

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\} \qquad \text{and} \qquad \Gamma_m := \{0 = \gamma_0 < \gamma_1 < \cdots < \gamma_m\}$$

be sets of real numbers. We have

$$|f'(0)| \leq (2n+2m+1)\|f\|_{[-1,1]}$$

for all f of the form

$$f = pq, \qquad p \in E(\Lambda_n),\;\; q \in G(\Gamma_m) \,.$$

3. Lemmas for Theorem 2.1*

Our first four lemmas have been stated as Lemmas 3.1–3.4 in [22], where their proofs are also presented. Our first lemma can be proved by a simple compactness argument and may be viewed as a simple exercise.

Lemma 3.1. *Let $\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\}$ be a set of real numbers. Let $a, b, c \in \mathbb{R}$ and $a < b$. Let $0 \not\equiv w$ be a continuous function defined on $[a,b]$. Let $q \in (0,\infty]$. There exists a $0 \not\equiv T \in E(\Delta_n)$ such that*

$$\frac{|T(c)|}{\|Tw\|_{L_q[a,b]}} = \sup\left\{\frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} : \quad 0 \not\equiv P \in E(\Delta_n)\right\},$$

and there exists a $0 \not\equiv S \in E(\Delta_n)$ such that

$$\frac{|S'(c)|}{\|Sw\|_{L_q[a,b]}} = \sup\left\{\frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}} : \quad 0 \not\equiv P \in E(\Delta_n)\right\}.$$

Our next lemma is an essential tool in proving our key lemmas, Lemmas 3.3 and 3.4.

Lemma 3.2. *Let $\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\}$ be a set of real numbers. Let $a, b, c \in \mathbb{R}$ and $a < b < c$. Let $q \in (0,\infty]$. Let T and S be the same as in Lemma 3.1. The function T has exactly n zeros in $[a,b]$ by counting multiplicities. Under the additional assumption $\delta_n \geq 0$, the function S also has exactly n zeros in $[a,b]$ by counting multiplicities.*

The heart of the proof of our theorems is the following pair of comparison lemmas. Lemmas 3.3 and 3.4 below are proved in [24] based on Descartes' Rule of Sign and a technique used earlier by Pinkus and P.W. Smith [36].

Lemma 3.3. *Let $\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\}$ and $\Gamma_n := \{\gamma_0 < \gamma_1 < \cdots < \gamma_n\}$ be sets of real numbers satisfying $\delta_j \leq \gamma_j$ for each $j = 0, 1, \ldots, n$. Let $a, b, c \in \mathbb{R}$ and $a < b \leq c$. Let $0 \not\equiv w$ be a continuous function defined on $[a,b]$. Let $q \in (0,\infty]$. We have*

$$\sup\left\{\frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} : \quad 0 \not\equiv P \in E(\Delta_n)\right\} \leq \sup\left\{\frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} : \quad 0 \not\equiv P \in E(\Gamma_n)\right\}.$$

Under the additional assumption $\delta_n \geq 0$ we also have

$$\sup\left\{\frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}} : \quad 0 \not\equiv P \in E(\Delta_n)\right\} \leq \sup\left\{\frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}} : \quad 0 \not\equiv P \in E(\Gamma_n)\right\}.$$

Lemma 3.4. *Let $\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\}$ and $\Gamma_n := \{\gamma_0 < \gamma_1 < \cdots < \gamma_n\}$ be sets of real numbers satisfying $\delta_j \leq \gamma_j$ for each $j = 0, 1, \ldots, n$. Let $a, b, c \in \mathbb{R}$ and $c \leq a < b$. Let $0 \not\equiv w$ be a continuous function defined on $[a,b]$. Let $q \in (0,\infty]$. We have*

$$\sup\left\{\frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} : \quad 0 \not\equiv P \in E(\Delta_n)\right\} \geq \sup\left\{\frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} : \quad 0 \not\equiv P \in E(\Gamma_n)\right\}.$$

Under the additional assumption $\gamma_0 \leq 0$ we also have

$$\sup\left\{\frac{|Q'(c)|}{\|Qw\|_{L_q[a,b]}} : \quad 0 \not\equiv Q \in E(\Delta_n)\right\} \geq \sup\left\{\frac{|Q'(c)|}{\|Qw\|_{L_q[a,b]}} : \quad 0 \not\equiv Q \in E(\Gamma_n)\right\}.$$

We will also need the following result which may be obtained from Theorem 1.5 by a substitution $x = e^t$.

Lemma 3.5. *Let $n \ge 1$ be an integer. Let $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ be a set of $n+1$ distinct real numbers. Let $a, b \in \mathbb{R}$ and $0 < a < b$. We have*

$$\frac{1}{3}\sum_{j=0}^{n} |\lambda_j| + \frac{1}{4(b-a)}(n-1)^2 \le \sup_{0 \not\equiv P \in E(\Lambda_n)} \frac{\|P'\|_{[a,b]}}{\|P\|_{[a,b]}} \le 11 \sum_{j=0}^{n} |\lambda_j| + \frac{128}{b-a}(n+1)^2 \,.$$

4. Lemmas for Theorem 2.7

Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of distinct positive real numbers. We denote the collection of all linear combinations of

$$\sinh(\lambda_0 t),\ \sinh(\lambda_1 t),\ \dots,\ \sinh(\lambda_n t)$$

over $\mathbb{R}$ by

$$H(\Lambda_n) := \operatorname{span}\{\sinh(\lambda_0 t),\ \sinh(\lambda_1 t),\ \dots,\ \sinh(\lambda_n t)\}\,.$$

The first lemma is stated and proved in Section 4 of [23].

Lemma 4.1. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ and $\Gamma_n := \{\gamma_0 < \gamma_1 < \cdots < \gamma_n\}$ be sets of positive real numbers satisfying $\lambda_j \le \gamma_j$ for each $j = 0, 1, \dots, n$. Let $a, b \in \mathbb{R}$ and $0 \le a < b$. Let $0 \not\equiv w$ be a continuous function defined on $[a, b]$. Let $q \in (0, \infty]$. We have*

$$\sup\left\{\frac{|P'(0)|}{\|Pw\|_{L_q[a,b]}} :\ \ P \in H(\Gamma_n)\right\} \le \sup\left\{\frac{|P'(0)|}{\|Pw\|_{L_q[a,b]}} :\ \ P \in H(\Lambda_n)\right\}.$$

As before, associated with $\Lambda_n := \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n\}$, we denote the collection of all linear combinations of

$$1\,,\ \cosh(\lambda_1 t)\,,\ \cosh(\lambda_2 t)\,,\ \dots\,,\ \cosh(\lambda_n t)$$

over $\mathbb{R}$ by

$$G(\Lambda_n) := \operatorname{span}\{1\,,\ \cosh(\lambda_1 t)\,,\ \cosh(\lambda_2 t)\,,\ \dots\,,\ \cosh(\lambda_n t)\}\,.$$

The next lemma is stated and proved in Section 3 of [26].

Lemma 4.2. *Let*

$$\Lambda_n := \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n\} \qquad \text{and} \qquad \Gamma_n := \{0 = \gamma_0 < \gamma_1 < \cdots < \gamma_n\}$$

be sets of nonnegative real numbers satisfying $\lambda_j \le \gamma_j$ for each $j = 0, 1, \dots, n$. Let $a, b \in \mathbb{R}$ and $0 \le a < b$. Let $0 \not\equiv w$ be a continuous function defined on $[a, b]$. Let $q \in (0, \infty]$. We have

$$\sup\left\{\frac{|P(0)|}{\|Pw\|_{L_q[a,b]}} :\ \ P \in G(\Gamma_n)\right\} \le \sup\left\{\frac{|P(0)|}{\|Pw\|_{L_q[a,b]}} :\ \ P \in G(\Lambda_n)\right\}.$$

4. Proofs

*Proof of Theorem 2.1**. First we prove the lower bound of the theorem. The lower bound of Lemma 3.5 guarantees a

$$0 \not\equiv f \in \operatorname{span}\{e^{(\lambda_n+\gamma_0)t}, e^{(\lambda_n+\gamma_1)t}, \dots, e^{(\lambda_n+\gamma_m)t}\}$$

such that

$$\begin{aligned}\|f'\|_{[a,b]} &\geq \left(\frac{1}{3}\sum_{j=0}^{m}(\lambda_n+\gamma_j)+\frac{1}{4(b-a)}(m-1)^2\right)\|f\|_{[a,b]}\\ &\geq \left(\frac{1}{3}(m+1)\lambda_n+\frac{1}{4(b-a)}(m-1)^2\right)\|f\|_{[a,b]}\,.\end{aligned}$$

Observe that $f=pq$ with $p\in E(\Lambda_n)$ defined by $p(x):=e^{\lambda_n t}$ and with some $q\in E(\Gamma_m)$.

Similarly, the lower bound of Lemma 3.5 guarantees a

$$0 \not\equiv f \in \operatorname{span}\{e^{(\gamma_m+\lambda_0)t}, e^{(\gamma_m+\lambda_1)t}, \dots, e^{(\gamma_m+\lambda_n)t}\}$$

such that

$$\begin{aligned}\|f'\|_{[a,b]} &\geq \left(\frac{1}{3}\sum_{j=0}^{n}(\gamma_m+\lambda_j)+\frac{1}{4(b-a)}(n-1)^2\right)\|f\|_{[a,b]}\\ &\geq \left(\frac{1}{3}(n+1)\gamma_m+\frac{1}{4(b-a)}(n-1)^2\right)\|f\|_{[a,b]}\,.\end{aligned}$$

Observe that $f=pq$ with some $p\in E(\Lambda_n)$ and with $q\in E(\Gamma_m)$ defined by $q(x):=e^{\gamma_m t}$. Hence the lower bound of the theorem is proved.

We now prove the upper bound of the theorem. We want to prove that

$$|(p'q)(y)| \leq 11(n+m+1)(\lambda_n-\lambda_0+\gamma_m-\gamma_0)+\frac{256}{b-a}(n+m+1)^2\,\|pq\|_{[a,b]} \tag{4.1}$$

for every $p\in E(\Lambda_n)$, $q\in E(\Gamma_m)$, and $y\in[a,b]$. The rest follows from the product rule of differentiation (the role of Λ_n and Γ_m can be interchanged). For $\alpha<\beta$ let

$$M(n,m,\alpha,\beta) := 11(n+m+1)(\lambda_n-\lambda_0+\gamma_m-\gamma_0)+\frac{128}{\beta-\alpha}(n+m+1)^2\,.$$

Let $d:=(a+b)/2\in(a,b)$.

First let $y\in[d,b]$. We show that

$$|(p'q)(y)| \leq M(n,m,a,y)\,\|pq\|_{[a,y]} \tag{4.2}$$

for every $p \in E(\Lambda_n)$ and $q \in E(\Gamma_m)$. To show (4.4), it is sufficient to prove that

$$|p'q)(y)| \leq (1+\eta) M(n,m,a,y)\, \|pq\|_{[a,y-\delta]} \tag{4.3}$$

for every $p \in E(\Lambda_n)$ and $q \in E(\Gamma_m)$, where η denotes a quantity that tends to 0 as $\delta \in (0, y-a)$ tends to 0. The rest follows by taking the limit when $\delta \in (0, y-a)$ tends to 0.

To see (4.3), by Lemmas 3.3 and 3.4 we may assume that

$$\lambda_j := \lambda_n - (n-j)\varepsilon\,, \qquad j = 0,1,\ldots,n\,,$$
$$\gamma_j := \gamma_m - (m-j)\varepsilon\,, \qquad j = 0,1,\ldots,m\,,$$

for some $\varepsilon > 0$. By Lemma 3.2 we may also assume that p has n zeros in $(a, y-\delta)$ and q has m zeros in $(a, y-\delta)$. We normalize p and q so that $p(y) > 0$ and $q(y) > 0$. Then, using the information on the zeros of p and q, we can easily see that $p'(y) > 0$ and $q'(y) > 0$. Therefore

$$|(p'q)(b)| \leq |(pq)'(b)|\,.$$

Now observe that $f := pq \in E(\Omega_k)$, where $k := n+m$ and $\Omega_k := \{\omega_1 < \omega_2 < \cdots < \omega_k\}$ with

$$\omega_j := \lambda_n + \gamma_m - (n+m-j)\varepsilon\,, \qquad j = 0,1,\ldots,k\,.$$

Hence Lemma 3.5 implies

$$|(p'q)(y)| \leq |(pq)'(y)| = |f'(y)| \leq M(n,m,a,y)\,\|f\|_{[a,y]} = M(n,m,a,y)\,\|pq\|_{[d,b]}\,.$$

By this (4.3), and hence (4.2), is proved. Combining (4.2) with

$$\begin{aligned} M(n,m,a,y) &= 11(n+m+1)(\lambda_n - \lambda_0 + \gamma_m - \gamma_0) + \frac{128}{y-a}(n+m+1)^2 \\ &= 11(n+m+1)(\lambda_n - \lambda_0 + \gamma_m - \gamma_0) + \frac{256}{b-a}(n+m+1)^2\,, \end{aligned}$$

we conclude (4.3) for all $y \in [d,b]$.

Now let $y \in [a,d]$. We show that

$$|(p'q)(y)| \leq K(n,m,y,b)\|pq\|_{[y,b]} \tag{4.4}$$

for every $p \in E(\Lambda_n)$ and $q \in E(\Gamma_m)$. To show (4.4), it is sufficient to prove that

$$|(p'q)(y)| \leq (1+\eta) M(n,m,y,b)\,\|pq\|_{[y+\delta,b]} \tag{4.5}$$

for every $p \in E(\Lambda_n)$ and $q \in E(\Gamma_m)$, where η denotes a quantity that tends to 0 as $\delta \in (0, b-y)$ tends to 0. The rest follows by taking the limit when $\delta \in (0, b-y)$ tends to 0.

To see (4.5), by Lemmas 3.3 and 3.4 we may assume that

$$\lambda_j := \lambda_0 + \varepsilon j\,, \qquad j = 0, 1, \dots, n\,,$$
$$\gamma_j := \gamma_0 + \varepsilon j\,, \qquad j = 0, 1, \dots, m\,,$$

with a sufficiently small $\varepsilon > 0$. By Lemma 3.2 we may also assume that p has n zeros in $(y+\delta, b)$ and q has m zeros in $(y+\delta, b)$. We normalize p and q so that $p(y) > 0$ and $q(y) > 0$. Then, using the information on the zeros of p and q, we can easily see that $p'(y) < 0$ and $q'(y) < 0$. Therefore

$$|(p'q)(y)| \le |(pq)'(y)|\,.$$

Now observe that $f := pq \in E(\Omega_k)$, where $k := n+m$ and $\Omega_k := \{\omega_1 < \omega_2 < \cdots < \omega_k\}$ with

$$\omega_j := \lambda_0 + \gamma_0 + j\varepsilon\,, \qquad j = 0, 1, \dots, k\,.$$

Hence Lemma 3.5 implies

$$|(p'q)(y)| \le |(pq)'(y)| = |f'(y)| \le M(n,m,y,b)\,\|f\|_{[y,b]} = M(n,m,y,b)\,\|pq\|_{[y,b]}\,.$$

By this (4.5), and hence (4.4), is proved. Combining (4.4) with

$$\begin{aligned} M(n,m,y,b) &= 11(n+m+1)(\lambda_n - \lambda_0 + \gamma_m - \gamma_0) + \frac{128}{b-y}(n+m+1)^2 \\ &\le 11(n+m+1)(\lambda_n - \lambda_0 + \gamma_m - \gamma_0) + \frac{256}{b-a}(n+m+1)^2\,, \end{aligned}$$

we conclude (4.1) for all $y \in [a,d]$. The proof of the theorem is now complete. $\square$

Corollaries 2.3 and 2.4 follow from Theorem 2.1 immediately.

Proof of Corollary 2.5. Observe that

$$t = \lim_{\varepsilon \to 0+} \frac{e^{\varepsilon t} - 1}{\varepsilon}\,,$$

hence every $q \in \mathcal{P}_m$ and $\eta > 0$ there is a sufficiently small $\varepsilon > 0$ and a

$$q_\varepsilon \in E(\Gamma_{m,\varepsilon}) := \mathrm{span}\{0, \varepsilon, 2\varepsilon, \dots, m\varepsilon\}$$

such that

$$\|q_\varepsilon - q\|_{[a,b]} < \eta \qquad \text{and} \qquad \|q'_\varepsilon - q'\|_{[a,b]} < \eta\,.$$

Therefore the corollary follows from Theorem 2.1* as a limit case. $\square$

Corollary 2.6 follows from Corollary 2.5 immediately.

Proof of Theorem 2.7. Let

$$f = pq, \qquad p \in E(\Lambda_n),\ \ q \in G(\Gamma_m)\,.$$

Observe that $q \in G(\Gamma_m)$ is even, hence $q(t) = q(-t)$ for all t, and $q'(0) = 0$. Hence, replacing p with $\tilde{p}$ defined by $\tilde{p}(t) := (p(t) - p(-t))/2$ we have $(\tilde{p}q)'(0) = (pq)'(0)$ and

$$\|\tilde{p}q\|_{[-1,1]} \leq \|pq\|_{[-1,1]}\,,$$

without loss of generality we may assume that

$$\Lambda_{n+1} = \{\lambda_0 < \lambda_1 < \cdots < \lambda_{n+1}\} \subset (0, \infty)$$

and $p \in H(\Lambda_{n+1})$. So let $f = pq$ with $p \in H(\Lambda_{n+1})$ and $q \in G(\Gamma_m)$, where

$$\Lambda_{n+1} := \{\lambda_0 < \lambda_1 < \cdots < \lambda_{n+1}\} \subset (0, \infty) \qquad \text{and} \qquad \Gamma_m := \{0 = \gamma_0 < \gamma_1 < \cdots < \gamma_m\}\,.$$

As $\|pq\|_{[-1,1]} = \|pq\|_{[0,1]}$, we want to prove that

$$|(pq)'(0)| = |p'q)(0)| \leq (2n + 2m + 1)\|pq\|_{[0,1]} \tag{4.6}$$

for all $p \in H(\Lambda_{n+1})$ and $q \in G(\Gamma_m)$. To prove (4.6), by Lemmas 4.1 and 4.2 we may assume that

$$\begin{aligned} \lambda_j &:= j\varepsilon\,, \qquad j = 0, 1, \ldots, n+1\,, \\ \gamma_j &:= j\varepsilon\,, \qquad j = 0, 1, \ldots, m\,, \end{aligned}$$

for some $\varepsilon > 0$. Now observe that $f := pq \in H(\Omega_k)$, where $k := n + m + 1$ and

$$\Omega_k := \{\omega_1 < \omega_2 < \cdots < \omega_k\}$$

with

$$\omega_j := j\varepsilon\,, \qquad j = 0, 1, \ldots, k\,.$$

Hence Theorem 1.17 implies (4.6). $\square$

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