THE UNIFORM CLOSURE OF NON-DENSE RATIONAL SPACES ON THE UNIT INTERVAL

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Abstract. Let $\mathcal{P}_n$ denote the set of all algebraic polynomials of degree at most n with real coefficients. Associated with a set of poles $\{a_1, a_2, \dots , a_n\} \subset \mathbb{R} \setminus [-1, 1]$ we define the rational function spaces

$$\mathcal{P}_n(a_1, a_2, \dots , a_n) := \left\{ f : f(x) = b_0 + \sum_{j=1}^{n} \frac{b_j}{x - a_j} \, , \quad b_0, b_1, \dots , b_n \in \mathbb{R} \right\} .$$

Associated with a set of poles $\{a_1, a_2, \dots \} \subset \mathbb{R} \setminus [-1, 1]$, we define the rational function spaces

$$\mathcal{P}(a_1, a_2, \dots) := \bigcup_{n=1}^{\infty} \mathcal{P}_n(a_1, a_2, \dots , a_n) \, .$$

It is an interesting problem to charcterize sets $\{a_1, a_2, \dots \} \subset \mathbb{R} \setminus [-1, 1]$ for which $\mathcal{P}(a_1, a_2, \dots)$ is not dense in $C[-1, 1]$, where $C[-1, 1]$ denotes the space of all continuous functions equipped with the uniform norm on $[-1, 1]$. Akhieser showed that the density of $\mathcal{P}(a_1, a_2, \dots)$ is characterized by the divergence of the series $\sum_{n=1}^{\infty} \sqrt{a_n^2 - 1}$.

In this paper we show that the so-called Clarkson-Erdős-Schwartz phenomenon occurs in the non-dense case. Namely, if $\mathcal{P}(a_1, a_2, \dots)$ is not dense in $C[-1, 1]$, then it is "very much not so". More precisely, we prove the following result.

Theorem. *Let $\{a_1, a_2, \dots \} \subset \mathbb{R} \setminus [-1, 1]$. Suppose $\mathcal{P}(a_1, a_2, \dots)$ is not dense in $C[-1, 1]$, that is,*

$$\sum_{n=1}^{\infty} \sqrt{a_n^2 - 1} < \infty \, .$$

Then every function in the uniform closure of $\mathcal{P}(a_1, a_2, \dots)$ in $C[-1, 1]$ can be extended analytically throughout the set $\mathbb{C} \setminus \{-1, 1, a_1, a_2, \dots \}$.

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Throughout this paper $\|f\|_A$ will denote the uniform norm of a continuous function f on a set $A \subset \mathbb{C}$. Let $\mathcal{P}_n$ denote the set of all algebraic polynomials of degree at most n with real coefficients. Associated with a set of poles $\{a_1, a_2, \dots, a_n\} \subset \mathbb{R} \setminus [-1,1]$ we define the rational function spaces

$$\mathcal{P}_n(a_1, a_2, \dots, a_n) := \left\{ f : f(x) = b_0 + \sum_{j=1}^{n} \frac{b_j}{x - a_j}\,, \quad b_0, b_1, \dots, b_n \in \mathbb{R} \right\}.$$

Note that every $f \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$ can be written as $f = p/q$ with

$$p \in \mathcal{P}_n \qquad \text{and} \qquad q(x) = \prod_{j=1}^{n} (x - a_j)\,.$$

Associated with a set of poles

$$\{a_1, a_2, \dots\} \subset \mathbb{R} \setminus [-1,1]\,,$$

we define the rational function spaces

$$\mathcal{P}(a_1, a_2, \dots) := \bigcup_{n=0}^{\infty} \mathcal{P}_n(a_1, a_2, \dots, a_n)\,.$$

It is an interesting problem to charcterize sets $\{a_1, a_2, \dots\} \subset \mathbb{R} \setminus [-1,1]$ for which $\mathcal{P}(a_1, a_2, \dots)$ is not dense in $C[-1,1]$, where $C[-1,1]$ denotes the space of all continuous functions equipped with the uniform norm on $[-1,1]$. Akhieser presents the answer (which is recaptured in [BE], see Corollary 4.3.4 on page 208) in his book by proving the following result.

Theorem (Akhieser). *Let $\{a_1, a_2, \dots\} \subset \mathbb{R} \setminus [-1,1]$. Then $\mathcal{P}(a_1, a_2, \dots)$ is dense in $C[-1,1]$ if and only if*

$$\sum_{n=1}^{\infty} \sqrt{a_n^2 - 1} = \infty\,.$$

In this paper we show that the so-called Clarkson-Erdős-Schwartz phenomenon occurs in the non-dense case. Namely if $\mathcal{P}(a_1, a_2, \dots)$ is not dense in $C[-1,1]$, then it is "very much not so". More precisely, we prove the following result.

Theorem 1. *Let $\{a_1, a_2, \dots\} \subset \mathbb{R} \setminus [-1,1]$. Suppose $\mathcal{P}(a_1, a_2, \dots)$ is not dense in $C[-1,1]$, that is,*

$$\sum_{n=1}^{\infty} \sqrt{a_n^2 - 1} < \infty\,.$$

Then every function in the uniform closure of $\mathcal{P}(a_1, a_2, \dots)$ in $C[-1,1]$ can be extended analytically throughout the set $\mathbb{C} \setminus \{-1, 1, a_1, a_2, \dots\}$.

Theorem 1 follows immediately from our main result below.

Theorem 2. *Suppose (a_j) is a sequence with each $a_j \in \mathbb{R} \setminus [-1,1]$. Suppose*

$$\sum_{j=1}^{\infty} \sqrt{a_j^2 - 1} < \infty \,.$$

Then there is a constant C_η depending only on $\eta > 0$ and the sequence (a_j) such that

$$|f(z)| \leq C_\eta \|f\|_{[-1,1]}$$

for every $f \in \mathcal{P}(a_1, a_2, \dots)$ and $z \in \mathbb{C} \setminus \{a_1, a_2, \dots\}$ such that the distance $\mathrm{dist}(z, \{-1,1\})$ *between the point z and the set $\{-1, 1, a_1, a_2, \dots\}$ is at least $\eta > 0$.*

Theorem 2 is the key observation of this paper. Theorem 1 follows immediately from Theorem 2. Indeed, suppose the sequence (f_n) with $f_n \in \mathcal{P}(a_1, a_2, \dots)$ converges uniformly on $[-1,1]$. Then it is also uniformly Cauchy on $[-1,1]$. By Theorem 2 it remains uniformly Cauchy on any compact set $K \subset \mathbb{C} \setminus \{-1, 1, a_1, a_2, \}$. Theorem 1 now follows from the well known theorem in complex analysis stating that a uniformly convergent sequence of analytic functions on a compact set K has an analytic limit function on K.

From now on we focus on proving Theorem 2. First an extremal function for the problem is introduced and then some nice properties of the extremal function is established in Lemma 1.

Let $z_0 \in \mathbb{C} \setminus ([-1,1] \cup \{a_1, a_2, \dots, a_n\})$ be fixed. A simple compactness argument shows that there exists a function $0 \neq f^* \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$ such that

$$\text{(1)} \qquad \frac{|f^*(z_0)|}{\|f^*\|_{[-1,1]}} = \sup_{0 \leq f \in \mathcal{P}_n(a_1, a_2, \dots, a_n)} \frac{|f(z_0)|}{\|f\|_{[-1,1]}} \,.$$

Lemma 1. *Suppose $f^* \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$ satisfy (1). Then the following statements hold.*
(i) The function f^* equioscillates on $[-1,1]$ at least n times. That is, there are

$$-1 < x_1 < x_2 < \cdots < x_n < 1$$

such that

$$f^*(x_j) = \pm(-1)^j \|f^*\|_{[-1,1]} \,, \qquad j = 1, 2, \dots, n \,.$$

(ii) f^* has only real zeros. All but at most one zeros of f^* are in $(-1,1)$.

Proof. The proof of (i) can be given by a standard variational method. Assume that statement (i) of the lemma is false. Let $x_1 \in [-1,1]$ be the smallest number such that $f^*(x_1) = \pm \|f^*\|_{[-1,1]}$. Let $x_2 \in [x_1, 1]$ be the smallest value for which $f^*(x_2) = -f^*(x_1)$. Inductively, let $x_k \in [x_{k-1}, 1]$ be the smallest value such that $f^*(x_k) = -f^*(x_{k-1})$, $k = 2, 3, \dots, m$, and assume that there is no $x_{m+1} \in [x_m, 1]$ such that $f^*(x_{m+1}) = -f^*(x_m)$. By our indirect assumption, we have $m \leq n-1$. Choose $y_1, y_2, \dots, y_{m-1}$ so that

$$x_1 < y_1 < x_2 < y_2 < x_3 < \cdots < x_{m-1} < y_{m-1} < x_m \,.$$

We define

$$q_{m+1}(x) = (x-z_0)(x-\overline{z}_0)(x-y_1)(x-y_2)\cdots(x-y_{m-1})\,.$$

Then $q_{m+1} \in \mathcal{P}_n$, and for sufficiently small $\varepsilon > 0$ either

$$f^*(x) + \varepsilon \frac{q_{m+1}(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$$

or

$$f^*(x) - \varepsilon \frac{q_{m+1}(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$$

contradicts the extremality of f^* Hence Part (i) is proved. To see Part (ii) we can argue as follows. By using the Intermediate Value Theorem, Part (i) implies that all but at most one zero of f^* are in $(-1,1)$. Since $f^* \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$ can be written as $f^* = p/q$ with

$$p \in p \in \mathcal{P}_n \qquad \text{and} \qquad q(x) = \prod_{j=1}^{n} (x - a_j)\,,$$

we conclude that the only possibly remaining zero of f^* is also real. $\square$

Our next tool is the bounded Bernstein-type inequality below for non-dense rational spaces $\mathcal{P}(a_1, a_2, \dots)$. This is proved in [BE] (see Corollary 7.1.4 on page 323) and plays an important role in the proof Theorem 2.

Lemma 2. *Suppose* $\{a_1, a_2, \dots, a_n\} \subset \mathbb{R} \setminus [-1,1]$. *Then*

$$|f'(x)| \le \frac{1}{\sqrt{1-x^2}} \left(\sum_{j=1}^{n} \frac{\sqrt{a_j^2 - 1}}{|x - a_j|} \right) \|f\|_{[-1,1]}$$

for every $f \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$ *and* $x \in (-1,1)$.

In fact, to prove Theorem 2, we will need the following consequence of the above lemma.

Corollary 3. *Suppose* (a_j) *is a sequence with each* $a_j \in \mathbb{R} \setminus [-1,1]$. *Suppose*

$$C := \sum_{j=1}^{\infty} \sqrt{a_j^2 - 1} < \infty\,.$$

Then

$$|f'(x)| \le \frac{2C}{(1-x^2)^{3/2}} \|f\|_{[-1,1]}$$

for every $f \in \mathcal{P}(a_1, a_2, \dots)$ *and* $x \in (-1,1)$.

Now we are ready to prove Theorem 2.

Proof of Theorem 2. We fix $n \in \mathbb{N}$ and $z_0 \in \mathbb{C} \setminus ([-1,1] \cup \{a_1, a_2, \dots, a_n\})$. It is sufficient to prove the lemma for rational functions

$$f \in S_{2n}(a_1, a_2, \dots, a_n) := \mathcal{P}_{2n}(a_1, -a_1, a_2, -a_2, \dots, a_n, -a_n)\,.$$

Without loss of generality we may assume that $\mathrm{Re}(z_0) \geq 0$ and $\mathrm{Im}(z_0) \neq 0$. By Lemma 1 we may assume that $f \in S_{2n}(a_1, a_2, \dots, a_n)$ equioscillates on $[-1,1]$ at least $2n$ times. That is, there exist $-1 \leq x_1 < x_2 < \cdots < x_{2n} \leq 1$ such that

$$f(x_j) = \pm(-1)^j \|f\|_{[-1,1]} \, .$$

Hence there are $y_j \in (x_j, x_{j+1}), j = 1, 2, \dots, 2n-1$, $\alpha, y_0 \in \mathbb{R}$, and $\sigma \in \{0,1\}$ such that

$$f(x) = \alpha \, \frac{(x-y_0)^\sigma (x-y_1) \cdots (x-y_{2n-1})}{(x^2-a_1^2)(x^2-a_2^2)\cdots(x^2-a_n^2)} \, . \tag{2}$$

Assume that $\sigma = 1$ and $y_0 \in \mathbb{R} \setminus [-1,1]$, the remaining cases are similar (in fact easier). Let k be chosen so that

$$x_1 < x_2 < \cdots < x_k < 0 \leq x_{k+1} < x_{k+2} < \cdots < x_{2n} \, .$$

Observe that $|k-n| \leq 2$, otherwise

$$f(x) - f(-x) \in S_{2n}(a_1, a_2, \dots, a_n)$$

has at least $2n+2$ zeros by counting multiplicities. By using the Mean Value Theorem and Corollary 3 we have

$$\begin{aligned} (x_{j+1}+1) - (x_j+1) = x_{j+1} - x_j &= \frac{|f(x_{j+1}) - f(x_j)|}{|f'(\xi_j)|} = \frac{2}{|f'(\xi_j)|} \\ &\geq \frac{(1-\xi_j^2)^{3/2}}{C} \geq \frac{(x_j+1)^{3/2}}{C} \, , \qquad j = 1, 2, \dots, k-1 \, , \end{aligned} \tag{3}$$

with suitable numbers $\xi_j \in (x_j, x_{j+1})$. Similarly

$$\begin{aligned} (1-x_{j+1}) - (1-x_j) = x_{j+1} - x_j &= \frac{|f(x_{j+1}) - f(x_j)|}{|f'(\xi_j)|} = \frac{2}{|f'(\xi_j)|} \\ &\geq \frac{(1-\xi_j^2)^{3/2}}{C} \geq \frac{(1-x_{j+1})^{3/2}}{C} \, , \qquad j = k+1, k+2, \dots, n \, , \end{aligned} \tag{4}$$

with suitable numbers $\xi_j \in (x_j, x_{j+1})$. Let $m \in \mathbb{N}$. It follows from (3) that the set

$$K_m := \left\{ j \in \{1, 2, \dots, k-1\} : \frac{1}{(m+1)^2} < x_j + 1 \leq \frac{1}{m^2} \right\}$$

has at most $6C+2$ elements. Indeed, if $j \in K_m$, then (3) implies

$$(x_{j+1}+1) - (x_j+1) \geq \frac{(x_j+1)^{3/2}}{C} \geq \frac{1}{C(m+1)^3} \geq \frac{1}{6C} \left(\frac{1}{m^2} - \frac{1}{(m+1)^2} \right) ,$$

and our claim follows. Therefore

$$\sum_{j=1}^{k-1}(x_j+1) < (6C+2)\sum_{m=1}^{\infty}\frac{1}{m^2} \leq 12C+4\,. \tag{5}$$

Similarly, it follows from (4) that the set

$$L_m := \left\{j \in \{k+1, k+2, \dots, n\} : \frac{1}{(m+1)^2} < 1-x_j \leq \frac{1}{m^2}\right\}$$

has at most $6C+2$ elements. Indeed, if $j \in L_m$, then (4) implies

$$(1-x_j)-(1-x_{j+1}) \geq \frac{(1-x_j)^{3/2}}{C} \geq \frac{1}{C(m+1)^3} \geq \frac{1}{6C}\left(\frac{1}{m^2}-\frac{1}{(m+1)^2}\right),$$

and our claim follows. Therefore

$$\sum_{j=k+1}^{2n}(1-x_j) < (6C+2)\sum_{m=1}^{\infty}\frac{1}{m^2} \leq 12C+4\,. \tag{6}$$

Now, combining (5), (6), and the interlacing property

$$-1 < x_1 < y_1 < x_2 < y_2 < \cdots < x_{2n-1} < y_{2n-1} < x_{2n} < 1\,,$$

we obtain

$$\sum_{j=1}^{k}(y_j+1) \leq 12C+8 \tag{7}$$

and

$$\sum_{j=k+1}^{2n-1}(1-y_j) \leq 12C+12\,. \tag{8}$$

Using the condition for the non-denseness of $\mathcal{P}(a_1, a_2, \dots)$, we have

$$\sum_{j=1}^{\infty}(a_j^2-1) \leq C_1\sum_{j=1}^{\infty}\sqrt{a_j^2-1} \leq C_2\,, \tag{9}$$

where C_1 and C_2 are constants depending only on the sequence (a_j). Observe that if $y_0 \in \mathbb{R}\setminus[-1,1]$, then $x-y_0 = A(x+1)+B(1-x)$ with some constants A and B satisfying $AB>0$. Writing the factor $x-y_0$ in (2) as the sum of the terms $A(x+1)$ and $B(1-x)$, with some constants $A>0$ and $B>0$ satisfying

$$AB > 0\,, \tag{10}$$

we obtain

$$f(x) = f_1(x) + f_2(x)\,, \tag{11}$$

where

$$f_1(x) = \alpha A\, \frac{(x+1)(x-y_1)\cdots(x-y_{2n-1})}{(x^2-a_1^2)(x^2-a_2^2)\cdots(x^2-a_n^2)}\,, \tag{12}$$

and

$$f_2(x) = \alpha B\, \frac{(1-x)(x-y_1)\cdots(x-y_{2n-1})}{(x^2-a_1^2)(x^2-a_2^2)\cdots(x^2-a_n^2)}\,, \tag{13}$$

and $AB > 0$ implies

$$|f_1(x)| \le |f(x)| \qquad \text{and} \qquad |f_2(x)| \le |f(x)|\,, \qquad x \in [-1,1]\,.$$

Assume now that $\|f\|_{[-1,1]} \le 1$. Then $\|f_1\|_{[-1,1]} \le 1$ and $\|f_2\|_{[-1,1]} \le 1$. By E.7 on page 153 in [BE], for the factors $A\alpha$ in (11) and $B\alpha$ in (12), we have

$$\alpha A \le C_3\|f_1\|_{[-1,1]} \le C_3\|f\|_{[-1,1]} \le C_3\,, \tag{14}$$

and

$$\alpha B \le C_3\|f_2\|_{[-1,1]} \le C_3\|f\|_{[-1,1]} \le C_3\,, \tag{15}$$

with a constant $C_3 > 0$ depending only on the sequence (a_j) (this exercise can be easily solved by using the explicit formula for the Chebyshev "polynomial" for the space $\mathcal{P}_n(a_1, a_2, \dots, a_n)$ on $[-1,1]$ and by observing that for every fixed $k = 0, 1, \dots, n$, in the extremal problem

$$\sup_f \frac{|b_k|}{\|f\|_{[-1,1]}}\,,$$

where the supremum is taken for all "polynomials" $f \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$ of the form

$$f(x) = b_0 + \sum_{j=1}^{n} \frac{b_j}{x-a_j}\,, \qquad b_0, b_1, \dots, b_n \in \mathbb{R}\,,$$

the extremal "polynomial" is the Chebyshev "polynomial" for the space $\mathcal{P}_n(a_1, a_2, \dots, a_n)$ on $[-1,1]$ (in fact, we need this observation only when $k = 0$). This latter observation can be easily seen by a standard zero-counting argument by noting that if one drops an element from the system

$$\left\{1\,, \frac{1}{x-a_1}\,, \frac{1}{x-a_2}\,, \cdots\,, \frac{1}{x-a_n}\right\}\,, \tag{16}$$

then the remaining elements form a Chebyshev system on $[-1,1]$ (αA and αB are the coefficients of the basis element 1 in f_1 and f_2, respectively if one writes them as the linear combinations of the basis elements in (16)).

Observe that (7), (8), and $|k-n| \leq 2$ imply

$$\prod_{j=1}^{k} |z_0 - y_j| = \prod_{j=1}^{k} |(z_0+1) - (y_j+1)| \leq |z_0+1|^{k+1} \prod_{j=1}^{k} \left(1 + \left|\frac{y_j+1}{z_0+1}\right|\right) \leq |z_0+1|^{n+3} C_4 \tag{17}$$

and

$$\prod_{j=k+1}^{2n-1} |z_0 - y_j| = \prod_{j=1}^{k} |(1-z_0) - (1-y_j)| \leq |1-z_0|^{n+3} \prod_{j=k+1}^{2n-1} \left(1 + \left|\frac{1-y_j}{1-z_0}\right|\right) \leq |1-z_0|^{n+3} C_4 \tag{18}$$

with some constant $C_4 > 0$ depending only on the sequence (a_j) and $|1 - z_0^2|$. Further, it follows from (9) that

$$\prod_{j=1}^{n} |z_0^2 - a_j^2| = \prod_{j=1}^{n} |(z_0^2 - 1) - (a_j^2 - 1)| = |z_0^2 - 1|^n \prod_{j=1}^{n} \left|1 - \frac{a_j^2 - 1}{z_0^2 - 1}\right| \geq C_5 |z_0^2 - 1|^n \tag{19}$$

with some constant $C_5 > 0$ depending only on (a_j) and the distance between z_0 and $\{-1, 1, \pm a_1, \pm a_2, \dots\}$. The theorem now follows from (2) and (10)–(19). □

References

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