ON THE REAL PART OF ULTRAFLAT SEQUENCES OF UNIMODULAR POLYNOMIALS

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Abstract. Let $P_n(z) = \sum_{k=0}^n a_{k,n} z^k \in \mathbb{C}[z]$ be a sequence of unimodular polynomials ($|a_{k,n}| = 1$ for all k, n) which is ultraflat in the sense of Kahane, i.e.,

$$\lim_{n\to\infty} \max_{|z|=1} \left| (n+1)^{-1/2} |P_n(z)| - 1 \right| = 0 \,.$$

For continuous functions f defined on $[0, 2\pi]$, and for $q \in (0, \infty)$, we define

$$\|f\|_q := \left(\int_0^{2\pi} |f(t)|^q \, dt \right)^{1/q} \,.$$

We also define

$$\|f\|_\infty := \lim_{q\to\infty} \|f\|_q = \max_{t\in[0,2\pi]} |f(t)| \,.$$

We prove the following conjecture of Queffelec and Saffari, see (1.30) in [QS2]. If $q \in (0, \infty)$ and (P_n) is an ultraflat sequence of unimodular polynomials P_n of degree n, then for $f_n(t) := \mathrm{Re}(P_n(e^{it}))$ we have

$$\|f_n\|_q \sim \left(\frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}} \right)^{1/q} n^{1/2}$$

and

$$\|f_n'\|_q \sim \left(\frac{\Gamma\left(\frac{q+1}{2}\right)}{(q+1)\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}} \right)^{1/q} n^{3/2} \,,$$

where Γ denotes the usual gamma function, and the $\sim$ symbol means that the ratio of the left and right hand sides converges to 1 as $n \to \infty$. To this end we use results from [Er1] where (as well as in [Er2], [Er3], and [Er4]) we studied the structure of ultraflat polynomials and proved several conjectures of Queffelec and Saffari.

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1. Introduction and the New Result

Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D. Let

$$\mathcal{K}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^{n} a_k z^k, \;\; a_k \in \mathbb{C}\,, \;\; |a_k| = 1 \right\}.$$

The class $\mathcal{K}_n$ is often called the collection of all (complex) unimodular polynomials of degree n. Let

$$\mathcal{L}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^{n} a_k z^k, \;\; a_k \in \{-1, 1\} \right\}.$$

The class $\mathcal{L}_n$ is often called the collection of all (real) unimodular polynomials of degree n. By Parseval's formula,

$$\int_0^{2\pi} |P_n(e^{it})|^2 \, dt = 2\pi(n+1)$$

for all $P_n \in \mathcal{K}_n$. Therefore

$$\min_{z \in \partial D} |P_n(z)| \leq \sqrt{n+1} \leq \max_{z \in \partial D} |P_n(z)|\,.$$

An old problem (or rather an old theme) is the following.

Problem 1.1 (Littlewood's Flatness Problem). *How close can a unimodular polynomial $P_n \in \mathcal{K}_n$ or $P_n \in \mathcal{L}_n$ come to satisfying*

$$|P_n(z)| = \sqrt{n+1}\,, \qquad z \in \partial D? \tag{1.1}$$

Obviously (1.1) is impossible if $n \geq 1$. So one must look for less than (1.1), but then there are various ways of seeking such an "approximate situation". One way is the following. In his paper [Li1] Littlewood had suggested that, conceivably, there might exist a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ (possibly even $P_n \in \mathcal{L}_n$) such that $(n+1)^{-1/2}|P_n(e^{it})|$ converge to 1 uniformly in $t \in \mathbb{R}$. We shall call such sequences of unimodular polynomials "ultraflat". More precisely, we give the following definition.

Definition 1.2. *Given a positive number ε, we say that a polynomial $P_n \in \mathcal{K}_n$ is ε-flat if*

$$(1-\varepsilon)\sqrt{n+1} \leq |P_n(z)| \leq (1+\varepsilon)\sqrt{n+1}\,, \qquad z \in \partial D\,.$$

Definition 1.3. *Given a sequence (ε_{n_k}) of positive numbers tending to* 0*, we say that a sequence (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is (ε_{n_k})-ultraflat if each P_{n_k} is (ε_{n_k})-flat. We simply say that a sequence (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is ultraflat if it is (ε_{n_k})-ultraflat with a suitable sequence (ε_{n_k}) of positive numbers tending to* 0*.*

The existence of an ultraflat sequence of unimodular polynomials seemed very unlikely, in view of a 1957 conjecture of P. Erdős (Problem 22 in [Er]) asserting that, for all $P_n \in \mathcal{K}_n$ with $n \geq 1$,

$$\max_{z \in \partial D} |P_n(z)| \geq (1+\varepsilon)\sqrt{n+1}\,, \tag{1.2}$$

where $\varepsilon > 0$ is an absolute constant (independent of n). Yet, refining a method of Körner [Kö], Kahane [Ka] proved that there exists a sequence (P_n) with $P_n \in \mathcal{K}_n$ which is (ε_n)-ultraflat, where $\varepsilon_n = O\left(n^{-1/17}\sqrt{\log n}\right)$. (Kahane's paper contained though a slight error which was corrected in [QS2].) Thus the Erdős conjecture (1.2) was disproved for the classes $\mathcal{K}_n$. For the more restricted class $\mathcal{L}_n$ the analogous Erdős conjecture is unsettled to this date. It is a common belief that the analogous Erdős conjecture for $\mathcal{L}_n$ is true, and consequently there is no ultraflat sequence of polynomials $P_n \in \mathcal{L}_n$. An interesting result related to Kahane's breakthrough is given in [Be]. For an account of some of the work done till the mid 1960's, see Littlewood's book [Li2] and [QS2].

Let (ε_n) be a sequence of positive numbers tending to 0. Let the sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ be (ε_n)-ultraflat. We write

$$P_n(e^{it}) = R_n(t)e^{i\alpha_n(t)}\,, \qquad R_n(t) = |P_n(e^{it})|\,, \qquad t \in \mathbb{R}\,. \tag{1.3}$$

It is a simple exercise to show that α_n can be chosen so that it is differentiable on $\mathbb{R}$. This is going to be our understanding throughout the paper.

The structure of ultraflat sequences of unimodular polynomials is studied in [Er1], [Er2], [Er3], and [Er4] where several conjectures of Saffari are proved. Here, based on the results in [Er1], we prove yet another conjecture of Saffari and Queffelec, see (1.30) in [QS2].

For continuous functions f defined on $[0, 2\pi]$, and for $q \in (0, \infty)$, we define

$$\|f\|_q := \left(\int_0^{2\pi} |f(t)|^q \, dt\right)^{1/q}.$$

We also define

$$\|f\|_\infty := \lim_{q \to \infty} \|f\|_q = \max_{t \in [0, 2\pi]} |f(t)|\,.$$

Theorem 1.4. *Let $q \in (0,\infty)$. If (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$, and $q \in (0,\infty)$, then for $f_n(t) := \mathrm{Re}(P_n(e^{it}))$ we have*

$$\|f_n\|_q \sim \left(\frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}}\right)^{1/q} n^{1/2}$$

and

$$\|f_n'\|_q \sim \left(\frac{\Gamma\left(\frac{q+1}{2}\right)}{(q+1)\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}}\right)^{1/q} n^{3/2}\,,$$

where Γ denotes the usual gamma function, and the $\sim$ symbol means that the ratio of the left and right hand sides converges to 1 as $n \to \infty$.

We remark that trivial modifications of the proof of Theorem 1.4 yield that the statement of the above theorem remains true if the ultraflat sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ is replaced by an ultraflat sequence (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$, $0 < n_1 < n_2 < \dots$.

2. Proof of Theorem 1.4

To prove the theorem we need a few lemmas. The first three are from [Er1].

Lemma 2.1 (Uniform Distribution Theorem for the Angular Speed). *Suppose (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Then, with the notation (1.3), in the interval $[0,2\pi]$, the distribution of the normalized angular speed $\alpha_n'(t)/n$ converges to the uniform distribution as $n \to \infty$. More precisely, we have*

$$\mathrm{meas}(\{t \in [0,2\pi] : 0 \le \alpha_n'(t) \le nx\}) = 2\pi x + \gamma_n(x)$$

for every $x \in [0,1]$, where $\lim_{n\to\infty} \max_{x\in[0,1]} |\gamma_n(x)| = 0$. Also, (as it was first observed by Saffari [Sa]), we have

$$\delta_n n \le \alpha_n'(t) \le n - \delta_n n \tag{2.5}$$

with suitable constants δ_n converging to 0.

Lemma 2.2 (Negligibility Theorem for Higher Derivatives). *Suppose (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Then, with the notation (1.3), for every integer $r \ge 2$, we have*

$$\max_{0\le t\le 2\pi} |\alpha_n^{(r)}(t)| \le \gamma_{n,r} n^r$$

with suitable constants $\gamma_{n,r} > 0$ converging to 0 for every fixed $r = 2,3,\dots$.

Lemma 2.3. *Let $q > 0$. Suppose (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Then we have*

$$\frac{1}{2\pi} \int_0^{2\pi} |\alpha_n'(t)|^q \, dt = \frac{n^q}{q+1} + \delta_{n,q} n^q \,,$$

and as a limit case,

$$\max_{0 \le t \le 2\pi} |\alpha_n'(t)| = n + \delta_n n \,.$$

with suitable constants $\delta_{n,q}$ and δ_n converging to 0 as $n \to \infty$ for every fixed q.

Our next lemma is a special case of Lemma 4.2 from [Er1].

Lemma 2.4. *Suppose (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Using notation (1.3), we have*

$$\max_{0 \le t \le 2\pi} |R_n'(t)| = \delta_n n^{3/2} \,, \qquad m = 1, 2, \dots \,,$$

with suitable constants δ_n converging to 0 as $n \to \infty$.

The next lemma follows from the ultraflatness property (see Definition 1.3) and Lemma 2.4.

Lemma 2.5. *Let $q \in (0, \infty)$. We have*

$$\|f_n\|_q^q = \int_0^{2\pi} |n^{1/2}(1 + \delta_n(t)) \cos(\alpha_n(t))|^q \, dt$$

and

$$\|f_n'\|_q^q = \int_0^{2\pi} |n^{1/2}(1 + \eta_n(t)) \sin(\alpha_n(t)) \alpha_n'(t) + \eta_n^*(t) n^{3/2}|^q \, dt$$

with some numbers $\delta_n(t)$, $\eta_n(t)$, and $\eta_n^*(t)$ converging to 0 uniformly on $[0, 2\pi]$ as $n \to \infty$.

Finally we need the technical lemma below that follows by a simple calculation.

Lemma 2.6. *Assume that $A, B \in \mathbb{R}$, $q > 0$, and $I \subset [0, 2\pi]$ is an interval. Then*

$$\int_I |\cos(Bt + A)|^q \, dt = K(q) \text{meas}(I) + \delta_1(A, B, q)$$

and

$$\int_I |\sin(Bt + A)|^q \, dt = K(q) \text{meas}(I) + \delta_2(A, B, q) \,,$$

where, by (6.2.1), (6.2.2), and (6.1.8) from [AS] *(see pages 258 and 255), we have*

$$2\pi K(q) := \int_0^{2\pi} |\sin t|^q \, dt = \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2} + 1\right) \sqrt{\pi}}$$

and

$$|\delta_1(A,B,q)|, |\delta_2(A,B,q)| \leq \pi B^{-1}\,.$$

Proof of Theorem 1.4. By Lemma 2.5 it is sufficient to prove that

$$\int_0^{2\pi} |\cos(\alpha_n(t))|^q\,dt \sim 2\pi K(q) := \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}} \tag{2.1}$$

and

$$\int_0^{2\pi} |\sin(\alpha_n(t))n^{-1}\alpha_n'(t)|^q\,dt \sim \frac{2\pi K(q)}{q+1}\,. \tag{2.2}$$

First we show (2.1). Let $\varepsilon > 0$ be fixed. Let $K_n := \gamma_{n,2}^{-1/4}$, where $\gamma_{n,2}$ is defined in Lemma 2.2. We divide the interval $[0, 2\pi]$ into subintervals

$$I_m := [a_{m-1}, a_m) := \left[\frac{(m-1)K_n}{n}, \frac{mK_n}{n}\right), \qquad m = 1, 2, \dots, N-1 := \left\lfloor \frac{2\pi n}{K_n} \right\rfloor,$$

and

$$I_N := [a_{N-1}, a_N) := \left[\frac{(N-1)K_n}{n}, 2\pi\right)\,.$$

For the sake of brevity let

$$A_{m-1} := \alpha_n(a_{m-1})\,, \qquad m = 1, 2, \dots, N\,,$$

and

$$B_{m-1} := \alpha_n'(a_{m-1})\,, \qquad m = 1, 2, \dots, N\,.$$

Then by Taylor's Theorem

$$|\alpha_n(t) - (A_{m-1} + B_{m-1}(t - a_{m-1}))| \leq \gamma_{n,2} n^2 (K_n/n)^2 \leq \gamma_{n,2}\gamma_{n,2}^{-1/2} \leq \gamma_{n,2}^{1/2}$$

for every $t \in I_m$, where $\lim_{n\to\infty} \gamma_{n,2}^{1/2} = 0$ by Lemma 2.2. Hence the functions

$$G_{n,q}(t) := \begin{cases} |\cos(A_0 + B_0(t - a_0))|^q, & t \in I_1\,, \\ |\cos(A_1 + B_1(t - a_0))|^q, & t \in I_2\,, \\ \quad\vdots & \quad\vdots \\ |\cos(A_{N-1} + B_{N-1}(t - a_{N-1}))|^q, & t \in I_N\,, \end{cases}$$

and

$$F_{n,q}(t) := |\cos(\alpha_n(t)|^q$$

satisfy

$$\lim_{n\to\infty} \sup_{t\in[0,2\pi)} |G_{n,q}(t) - F_{n,q}(t)| = 0\,. \tag{2.3}$$

Therefore, in order to prove (2.1), it is sufficient to prove that

$$\int_0^{2\pi} G_{n,q}(t)\,dt \sim 2\pi K(q)\,. \tag{2.4}$$

By using Lemma 2.5, if $|B_{m-1}| \geq n\varepsilon$, then

$$\left| \int_{I_m} G_{n,q}(t)\,dt - K(q)\text{meas}(I_m) \right| \leq \frac{\pi}{n\varepsilon}\,.$$

Therefore $\lim_{n\to\infty} K_n = \infty$ implies

(2.5)

$$\begin{aligned} \left| \sum_m \int_{I_m} G_{n,q}(t)\,dt - K(q) \sum_m \text{meas}(I_m) \right| &\leq N \frac{\pi}{n\varepsilon} \leq \left(\frac{2\pi n}{K_n} + 1 \right) \frac{\pi}{n\varepsilon} \\ &\leq \eta_n^*(\varepsilon)\,, \end{aligned}$$

where the summation is taken over all $m = 1, 2, \dots, N$ for which $|B_{m-1}| \geq n\varepsilon$, and where $(\eta_n^*(\varepsilon))$ is a sequence tending to 0 as $n \to \infty$. Now let

$$E_{n,\varepsilon} := \bigcup_{m\colon |B_{m-1}| \leq n\varepsilon} I_m\,.$$

If $|B_{m-1}| \leq n\varepsilon$, then we obtain by Lemma 2.2 that

$$|\alpha_n'(t)| \leq |B_{m-1}| + \frac{K_n}{n} \max_{t\in I_m} |\alpha_n''(t)| \leq |B_{m-1}| + \frac{\gamma_{n,2}^{-1/4}}{n} \gamma_{n,2} n^2 \leq 2n\varepsilon$$

for every $t \in I_m$ if n is sufficiently large. So

$$E_{n,\varepsilon} \subset \{t \in [0, 2\pi] : |\alpha_n'(t)| \leq 2n\varepsilon\}$$

for every sufficiently large n. Hence we obtain by Lemma 2.1 that

$$\text{meas}(E_{n,\varepsilon}) \leq 4\pi\varepsilon + \eta_n^{**}(\varepsilon)\,,$$

where $(\eta_n^{**}(\varepsilon))$ is a sequence tending to 0 as $n \to \infty$. Combining this with $0 \leq G_{n,q}(t) \leq 1$, $t \in [0, 2\pi)$, we obtain

$$\left| \sum_m \int_{I_m} G_{n,q}(t)\,dt - K(q) \sum_m \text{meas}(I_m) \right| \leq (4\pi\varepsilon + \eta_n^{**}(\varepsilon))(1 + K(q))\,, \tag{2.6}$$

where n is sufficiently large and the summation is taken over all $m = 1, 2, \dots, N$ for which $|B_{m-1}| < n\varepsilon$. Since $\varepsilon > 0$ is arbitrary, (2.4) follows from (2.5) and (2.6). The proof of (2.1) is now finished.

Now we prove (2.2). Let $\varepsilon > 0$ be fixed. Let the intervals I_m and the numbers A_m and B_m, $m = 1, 2, \dots, N$, as in the proof of (2.1). We define

$$G_{n,q}(t) := \begin{cases} |\sin(A_0 + B_0(t - a_0))|^q, & t \in I_1\,, \\ |\sin(A_1 + B_1(t - a_0))|^q, & t \in I_2\,, \\ \quad\vdots & \quad\vdots \\ |\sin(A_{N-1} + B_{N-1}(t - a_{N-1}))|^q, & t \in I_N \end{cases}$$

and

$$F_{n,q}(t) := |\sin(\alpha_n(t)|^q\,.$$

Similarly to the corresponding argument in the proof of (2.1), we obtain (2.3). Let

$$G^*_{n,q}(t) := \begin{cases} |\sin(A_0 + B_0(t-a_0))|^q |n^{-1}B_0|^q, & t \in I_1\,, \\ |\sin(A_1 + B_1(t-a_0))|^q |n^{-1}B_1|^q, & t \in I_2\,, \\ \quad\vdots & \vdots \\ |\sin(A_{N-1} + B_{N-1}(t-a_{N-1}))|^q |n^{-1}B_{N-1}|^q, & t \in I_N \end{cases}$$

and

$$F^*_{n,q}(t) := |\sin(\alpha_n(t))|^q |n^{-1}\alpha_n'(t)|^q\,.$$

We have

$$G^*_{n,q}(t) = G_{n,q}(t) H_{n,q}(t)\,, \tag{2.7}$$

where

$$H_{n,q}(t) := \begin{cases} |n^{-1}B_0|^q, & t \in I_1\,, \\ |n^{-1}B_1|^q, & t \in I_2\,, \\ \quad\vdots & \vdots \\ |n^{-1}B_{N-1}|^q, & t \in I_N\,. \end{cases}$$

It follows from Lemma 2.2 that

$$\begin{aligned} \big||n^{-1}\alpha_n'(t)| - |n^{-1}B_{m-1}|\big| &= \big||n^{-1}\alpha_n'(t)| - |n^{-1}\alpha_n'(a_{m-1})|\big| \\ &\leq \frac{K_n}{n} \max_{t\in I_m} |n^{-1}\alpha_n''(t)| \leq \frac{\gamma_{n,2}^{-1/4}}{n} n^{-1}\gamma_{n,2} n^2 = \gamma_{n,2}^{3/4} \end{aligned}$$

for every $t \in I_m$. Since $\lim_{n\to\infty} \gamma_{n,2}^{3/4} = 0$, we obtain that

$$\lim_{n\to\infty} \sup_{t\in[0,2\pi)} |H_{n,q}(t) - |n^{-1}\alpha_n'(t)|^q| = 0\,. \tag{2.8}$$

Now observe that

$$\sup_{t\in[0,2\pi)} |\sin(\alpha_n(t)|^q \leq 1 \tag{2.9}$$

and by Lemma 2.1 we have

$$\sup_{t\in[0,2\pi)} |n^{-1}\alpha_n'(t)|^q \leq 2^q \tag{2.10}$$

for all sufficiently large n. Now (2.3), (2.8), (2.9), (2.10), and (2.7) imply

$$\lim_{n\to\infty} \sup_{t\in[0,2\pi)} |G^*_{n,q}(t) - F^*_{n,q}(t)| = 0\,.$$

Therefore, in order to prove (2.2), it is sufficient to prove that

$$\int_0^{2\pi} G^*_{n,q}(t)\,dt \sim \frac{2\pi K(q)}{q+1}\,. \tag{2.11}$$

As a special case of (2.10), we have

$$|n^{-1}B_{m-1}|^q \le 2^q\,, \qquad m = 1,2,\dots,N\,,$$

for all sufficiently large n. Hence, if n is sufficiently large and $|B_{m-1}| \ge n\varepsilon$, then, with the help of Lemma 2.6, we obtain that

$$\left|\int_{I_m} G^*_{n,q}(t)\,dt - K(q)\mathrm{meas}(I_m)|n^{-1}B_{m-1}|^q\right| \le 2^q\frac{\pi}{n\varepsilon}\,.$$

Therefore $\lim_{n\to\infty} K_n = \infty$ implies

(2.12)
$$\begin{aligned}\left|\sum_m \int_{I_m} G_{n,q}(t)\,dt - K(q)\sum_m \mathrm{meas}(I_m)|n^{-1}B_{m-1}|^q\right| &\le 2^q N\frac{\pi}{n\varepsilon}\\ &\le 2^q\left(\frac{2\pi n}{K_n}+1\right)\frac{\pi}{n\varepsilon}\\ &\le \eta^*_{n,q}(\varepsilon)\,,\end{aligned}$$

where the summation is taken over all $m = 1,2,\dots,N$ for which $|B_{m-1}| \ge n\varepsilon$, and where $(\eta^*_{n,q}(\varepsilon))$ is a sequence tending to 0 as $n \to \infty$. Now let

$$E_{n,\varepsilon} := \bigcup_{m:\,|B_{m-1}|\le n\varepsilon} I_m\,.$$

As in the proof of (2.1) we have

$$\mathrm{meas}(E_{n,\varepsilon}) \le 4\pi\varepsilon + \eta^{**}_n(\varepsilon)\,,$$

where $(\eta^{**}_n(\varepsilon))$ is a sequence tending to 0 as $n \to \infty$. Combining this with (2.9) and (2.10), and recalling the definition of $G^*_{n,q}$, we obtain

$$\begin{aligned}&\left|\sum_m \int_{I_m} G^*_{n,q}(t)\,dt - K(q)\sum_m \mathrm{meas}(I_m)|n^{-1}B_{m-1}|^q\right|\\ &\le (4\pi\varepsilon + \eta^{**}_n(\varepsilon))2^q(1+K(q))\,,\end{aligned} \tag{2.13}$$

where n is sufficiently large and the summation is taken over all $m = 1,2,\dots,N$ for which $|B_{m-1}| < n\varepsilon$. Since $\varepsilon > 0$ is arbitrary, from (2.12) and (2.13) we obtain that

$$\int_0^{2\pi} G^*_{n,q}(t)\,dt \sim K(q)\int_0^{2\pi} H_{n,q}(t)\,dt \tag{2.14}$$

However (2.8) and Lemma 2.3 imply that

$$\int_0^{2\pi} H_{n,q}(t)\,dt \sim n^{-q}\int_0^{2\pi} |\alpha'_n(t)|^q\,dt \sim \frac{2\pi}{q+1} \tag{2.15}$$

The statement under (2.11) now follows by combining (2.14), and (2.15). As we have remarked before, (2.11) implies (2.2). □

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