EXTREMAL PROPERTIES OF THE DERIVATIVES OF THE NEWMAN POLYNOMIALS

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Abstract. Let $\Lambda_{n-1} := \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be a set of n distinct positive numbers. The span of

$$\{e^{-\lambda_1 t}, e^{-\lambda_2 t}, \dots, e^{-\lambda_n t}\}$$

over $\mathbb{R}$ will be denoted by

$$E(\Lambda_{n-1}) := \operatorname{span}\{e^{-\lambda_1 t}, e^{-\lambda_2 t}, \dots, e^{-\lambda_n t}\}\,.$$

Our main result of this note is the following.

Theorem. *Suppose $0 < q \leq p \leq \infty$. Let μ be a non-negative integer. Then there are constants $c_1(p,q,\mu) > 0$ and $c_2(p,q,\mu) > 0$ depending only on p, q, and μ such that*

$$c_1(p,q,\mu)\left(\sum_{j=1}^n \lambda_j\right)^{\mu+\frac{1}{q}-\frac{1}{p}} \leq \sup_{Q \in E(\Lambda_{n-1})} \frac{\|Q^{(\mu)}\|_{L_p[0,\infty)}}{\|Q\|_{L_q[0,\infty)}} \leq c_2(p,q,\mu)\left(\sum_{j=1}^n \lambda_j\right)^{\mu+\frac{1}{q}-\frac{1}{p}}\,,$$

where the lower bound holds for all $0 < q \leq p \leq \infty$ and for all $\mu \geq 0$, while the upper bound holds when $\mu = 0$ and $0 < q \leq p \leq \infty$ and when $\mu \geq 1$, $p \geq 1$, and $0 < q \leq p \leq \infty$.

1. Introduction and Notation

Let $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ be a set of $n+1$ distinct non-negative numbers. The span of

$$\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$$

over $\mathbb{R}$ will be denoted by

$$M(\Lambda_n) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}.$$

Elements of $M(\Lambda_n)$ are called Müntz polynomials of $n+1$ terms. The span of

$$\{e^{-\lambda_0 t}, e^{-\lambda_1 t}, \dots, e^{-\lambda_n t}\}$$

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over $\mathbb{R}$ will be denoted by

$$E(\Lambda_n) := \operatorname{span}\{e^{-\lambda_0 t}, e^{-\lambda_1 t}, \ldots, e^{-\lambda_n t}\}.$$

Elements of $E(\Lambda_n)$ are called exponential sums of $n+1$ terms.

For a function f defined on a set A let

$$\|f\|_A := \sup\{|f(x)| : x \in A\}\,,$$

and let

$$\|f\|_{L_p A} := \left(\int_A |f(x)|^p \, dx \right)^{1/p}, \qquad p > 0\,,$$

whenever the Lebesgue integral exists. Newman's beautiful inequality (see [1] and [4]) is an essentially sharp Markov-type inequality for $M(\Lambda_n)$ on $[0,1]$ in the case when each λ_j is non-negative.

Theorem 1.1 (Newman's Inequality). *Let $\Lambda_n := \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ be a set of $n+1$ distinct non-negative numbers. Then*

$$\frac{2}{3} \sum_{j=0}^{n} \lambda_j \leq \sup_{0 \neq Q \in M(\Lambda_n)} \frac{\|xQ'(x)\|_{[0,1]}}{\|Q\|_{[0,1]}} \leq 9 \sum_{j=0}^{n} \lambda_j\,,$$

or equivalently

$$\frac{2}{3} \sum_{j=0}^{n} \lambda_j \leq \sup_{0 \neq Q \in E(\Lambda_n)} \frac{\|Q'\|_{[0,\infty]}}{\|Q\|_{[0,\infty]}} \leq 9 \sum_{j=0}^{n} \lambda_j\,.$$

An L_p version of this is established in [1], [2], and [3].

Theorem 1.2. *Let $1 \leq p \leq \infty$. Let Let $\Lambda_n := \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ be a set of $n+1$ distinct real numbers greater than $-1/p$. Then*

$$\|xQ'(x)\|_{L_p[0,1]} \leq \left(1/p + 9 \left(\sum_{j=0}^{n} (\lambda_j + 1/p) \right) \right) \|Q\|_{L_p[0,1]}$$

for every $Q \in M(\Lambda_n)$. This follows from the fact that if $\Lambda_n := \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ is a set of $n+1$ distinct non-negative numbers, then

$$\|Q'\|_{L_p[0,\infty)} \leq 9 \left(\sum_{j=0}^{n} \gamma_j \right) \|Q\|_{L_p[0,\infty)}$$

for every $Q \in E(\Lambda_n)$.

A simple consequence of Theorem 1.1 is the following.

Theorem 1.3 (Nikolskii-Type Inequality). *Suppose $0 < q \le p \le \infty$. Let $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ be a set of $n+1$ distinct real numbers greater than $-1/q$. Then*

$$\|x^{1/q-1/p}Q(x)\|_{L_p[0,1]} \le c(p,q)\left(\sum_{j=0}^{n}(\lambda_j + 1/q)\right)^{1/q-1/p} \|Q\|_{L_q[0,1]}$$

for every $Q \in M(\Lambda_n)$. Equivalently, if $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ is a set of $n+1$ distinct non-negative numbers, then

$$\|Q\|_{L_p[0,\infty)} \le c(p,q)\left(\sum_{j=0}^{n}\lambda_j\right)^{1/q-1/p} \|Q\|_{L_q[0,\infty)}$$

for every $Q \in E(\Lambda_n)$. In both inequalities

$$c(p,q) := (18 \cdot 2^q)^{1/q-1/p}$$

is a suitable choice.

The purpose of this note is to show that both Theorems 1.2 and 1.3 are essentially sharp.

2. New Results

The upper bound in our main theorem below follows as a combination of Theorems 1.2 ans 1.3. The novelty of this note is the establishment of the lower bound.

Theorem 2.1. *Let $\Lambda_{n-1} := \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be a set of n distinct positive numbers. Suppose $0 < q \le p \le \infty$. Let μ be a non-negative integer. Then there are constants $c_1(p,q,\mu) > 0$ and $c_2(p,q,\mu) > 0$ depending only on p, q, and μ such that*

$$c_1(p,q,\mu)\left(\sum_{j=1}^{n}\lambda_j\right)^{\mu+\frac{1}{q}-\frac{1}{p}} \le \sup_{Q\in E(\Lambda_{n-1})}\frac{\|Q^{(\mu)}\|_{L_p[0,\infty)}}{\|Q\|_{L_q[0,\infty)}} \le c_2(p,q,\mu)\left(\sum_{j=1}^{n}\lambda_j\right)^{\mu+\frac{1}{q}-\frac{1}{p}},$$

where the lower bound holds for all $0 < q \le p \le \infty$ and for all $\mu \ge 0$, while the upper bound holds when $\mu = 0$ and $0 < q \le p \le \infty$ and when $\mu \ge 1$, $p \ge 1$, and $0 < q \le p \le \infty$.

3. Proofs

Proof of Theorem 2.1. As we have already remarked, we need to prove only the lower bound. To this end without loss of generality we may assume that the elements of Λ_{n-1} satisfy $\sum_{j=1}^{n}\lambda_j = 1$; the general result follows by a linear scaling. Then the Newman "polynomial" $T_n \in E(\Lambda_{n-1})$ is defined by

$$T_n(t) := \frac{1}{2\pi i}\int_\Gamma \frac{e^{-zt}}{B_n(z)}\,dz\,, \tag{3.1}$$

where

$$\Gamma := \{z \in \mathbb{C} : |z-1| = 1\} \qquad \text{and} \qquad B_n(z) := \prod_{j=1}^{n} \frac{z-\lambda_j}{z+\lambda_j}\,.$$

By the residue theorem

$$T_n(t) = \sum_{j=1}^{n} (B_n'(\lambda_j))^{-1} e^{-\lambda_j t}\,,$$

and hence $T_n \in E(\Lambda_{n-1})$. We claim that

$$|B_n(z)| \geq \frac{1}{3}\,, \qquad z \in \Gamma\,. \tag{3.2}$$

Indeed, it is easy to see that $0 \leq \lambda_j \leq 1$ implies

$$\left|\frac{z-\lambda_j}{z+\lambda_j}\right| \geq \frac{2-\lambda_j}{2+\lambda_j} = \frac{1-\frac{1}{2}\lambda_j}{1+\frac{1}{2}\lambda_j}\,, \qquad z \in \Gamma\,.$$

So, for $z \in \Gamma$,

$$|B_n(z)| \geq \prod_{j=1}^{n} \frac{1-\frac{1}{2}\lambda_j}{1+\frac{1}{2}\lambda_j} = \frac{1-\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}\,,$$

where the inequality

$$\begin{aligned}\frac{1-x}{1+x}\,\frac{1-y}{1+y} &= \frac{1-(x+y)}{1+(x+y)} + \frac{2xy(x+y)}{(1+x)(1+y)(1+(x+y))} \\ &\geq \frac{1-(x+y)}{1+(x+y)}\,, \qquad x, y \geq 0\,,\end{aligned}$$

was used. We will examine $T_{n,k} \in E(\Lambda_{n-1})$ defined by

$$T_{n,k}(t) := T_n^{(k)}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{(-z)^k \exp(-zt)}{B_n(z)}\, dz\,.$$

Note that the circle Γ can be parametrized as

$$\Gamma := \{-\exp(iu) + 1\,,\ \ u \in [-\pi, \pi)\}\,,$$

where for $z = -\exp(iu) + 1$, $u \in [-\pi, \pi)$, we have

$$|z| = |-\exp(iu) + 1| \leq |u|$$

and

$$|\exp(-zt)| \leq \exp(\mathrm{Re}(-zt)) = \exp((-1+\cos u)t) \leq \exp\left(-\frac{tu^2}{12}\right)\,.$$

Using the above inequalities together with (3.2), we obtain that

$$
\begin{aligned}
|T_{n,k}(t)| &\leq \left| \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{(-(-\exp(iu)+1))^k \exp(-(-\exp(iu)+1)t)}{B_n(-\exp(iu)+1)} \, i\exp(iu)\,du \right| \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|-\exp(iu)+1|^k \, |\exp(-(-\exp(iu)+1)t)|}{|B_n(-\exp(iu)+1)|} \, |\exp(iu)|\,du \\
&\leq \frac{3}{2\pi} \int_{-\pi}^{\pi} |u|^k \exp\left(-\frac{tu^2}{12}\right) du \\
&\leq \frac{3}{2\pi} \int_{0}^{2\pi} \left(\frac{u\sqrt{t}}{\sqrt{12}}\right)^k \exp\left(-\left(\frac{u\sqrt{t}}{\sqrt{12}}\right)^2\right) \frac{\sqrt{t}}{\sqrt{12}} \left(\frac{\sqrt{12}}{\sqrt{t}}\right)^{k+1} du \\
&\leq \frac{3}{2\pi} \left(\int_0^\infty v^k \exp(-v^2)\,dv\right) \left(\frac{\sqrt{12}}{\sqrt{t}}\right)^{k+1} \\
&\leq c(k) 12^{(k+1)/2} t^{-(k+1)/2}
\end{aligned}
$$

with a constant $c(k)$ depending only on k. Also,

$$
\begin{aligned}
|T_{n,k}(t)| &\leq \left| \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{(-(-\exp(iu)+1))^k \exp(-(-\exp(iu)+1)t)}{B_n(-\exp(iu)+1)} \, i\exp(iu)\,du \right| \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|-\exp(iu)+1|^k \, |\exp(-(-\exp(iu)+1)t)|}{|B_n(-\exp(iu)+1)|} \, |\exp(iu)|\,du \\
&\leq \frac{3}{2\pi} 2\pi 2^k \leq 3 \cdot 2^k \,.
\end{aligned}
$$

So with $k := \lfloor 4/q \rfloor$ we have

(3.3)

$$
\begin{aligned}
\|T_{n,k}\|_{L_q[0,\infty)} &\leq \left(\int_1^\infty |T_{n,k}(t)|^q\,dt + \int_0^1 |T_{n,k}(t)|^q\,dt\right)^{1/q} \\
&\left(\int_1^\infty c(k) 12^{(k+1)q/2} t^{-(k+1)q/2}\,dt + 3^q \cdot 2^{kq}\right)^{1/q} \\
&\leq \left(c(k) 12^{(\lfloor 4/q \rfloor+1)/2} \int_1^\infty t^{-2}\,dt + 3^q \cdot 2^4\right)^{1/q} \leq C(q) < \infty
\end{aligned}
$$

with a constant $C(q)$ depending only on q, and

(3.4) $$\|T_{n,k}\|_{[0,\infty)} \leq C(\infty) < \infty \,.$$

Now observe that

$$|T_{n,k}^{(\mu)}(0)| = \frac{1}{2\pi i} \int_\Gamma \frac{(-z)^{k+\mu}}{B_n(z)}\,dz \,.$$

Here, for all $z \in \mathbb{C}$ with $|z| > 1 \geq \max_{1 \leq j \leq n} |\lambda_j|$, we have

$$\begin{aligned}\frac{z^{k+\mu}}{B_n(z)} &= z^{k+\mu} \prod_{j=1}^{n} \frac{1+\lambda_j/z}{1-\lambda_j/z} = z^{k+\mu} \prod_{j=1}^{n} \left(1 + 2 \sum_{m=1}^{\infty} \left(\frac{\lambda_j}{z}\right)^m\right) \\ &= z^{k+\mu-1} \left(z + 2 + \sum_{m=1}^{\infty} \frac{A_m}{z^m}\right)\end{aligned}$$

with a constant $A_m \geq 1/m!$. Therefore

$$|T_{n,k}^{(\mu)}(0)| \geq \frac{1}{(k+\mu)!} \,. \tag{3.5}$$

Pick a point $y \in [0, \infty)$ so that

$$|T_{n,k}^{(\mu)}(y)| = \|T_{n,k}^{(\mu)}\|_{[0,\infty)} \,.$$

Note that $T_{n,k}^{(\mu)} \in E(\Lambda_{n-1})$. Combining the upper bound of Theorem 1.1 (Newman's inequality) with the Mean Value Theorem, we obtain that

$$|T_{n,k}^{(\mu)}(t)| \geq \frac{1}{2} \|T_{n,k}^{(\mu)}\|_{[0,\infty)} \,, \qquad t \in I := \left[y, y + \frac{1}{18}\right] \,.$$

Hence (3.5) and $k := \lfloor 4/q \rfloor$ implies that

$$\begin{aligned}\|T_{n,k}^{(\mu)}\|_{L_p[0,\infty)} &\geq \left(\frac{1}{18} \left(\frac{1}{2} \|T_{n,k}^{(\mu)}\|_{[0,\infty)}\right)^p\right)^{1/p} \\ &\geq \left(\frac{1}{18} \left(\frac{1}{2} \frac{1}{(k+\mu)!}\right)^p\right)^{1/p} \\ &\geq c(p,q,\mu) > 0\end{aligned} \tag{3.6}$$

and

$$\|T_{n,k}^{(\mu)}\|_{[0,\infty)} \geq \frac{1}{(k+\mu)!} \geq c(\infty, q, \mu) \tag{3.7}$$

Combining (3.3)–(3.7) finishes the proof. □

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