NIKOLSKII-TYPE INEQUALITIES FOR SHIFT INVARIANT FUNCTION SPACES

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Abstract. Let V_n be a vectorspace of complex-valued functions defined on $\mathbb{R}$ of dimension $n+1$ over $\mathbb{C}$. We say that V_n is shift invariant (on $\mathbb{R}$) if $f \in V_n$ implies that $f_a \in V_n$ for every $a \in \mathbb{R}$, where $f_a(x) := f(x-a)$ on $\mathbb{R}$. In this note we prove the following.

Theorem. *Let $V_n \subset C[a,b]$ be a shift invariant vectorspace of complex-valued functions defined on $\mathbb{R}$ of dimension $n+1$ over $\mathbb{C}$. Let $p \in (0,2]$. Then*

$$\|f\|_{L_\infty[a+\delta,b-\delta]} \leq 2^{2/p^2} \left(\frac{n+1}{\delta}\right)^{1/p} \|f\|_{L_p[a,b]}$$

for every $f \in V_n$ and $\delta \in \left(0, \frac{1}{2}(b-a)\right)$.

1. Introduction

The well known results of Nikolskii assert that the essentially sharp inequality

$$\|h_n\|_{L_q[-1,1]} \leq c(p,q) n^{2/p-2/q} \|h_n\|_{L_p[-1,1]}$$

holds for all algebraic polynomials h_n of degree at most n with complex coefficients and for all $0 < p < q \leq \infty$, while the essentially sharp inequality

$$\|t_n\|_{L_q[-\pi,\pi]} \leq c(p,q) n^{1/p-1/q} \|t_n\|_{L_p[-\pi,\pi]}$$

holds for all trigonometric polynomials t_n of degree at most n with complex coefficients and for all $0 < p < q \leq \infty$. The subject started with two famous papers [5] and [6]. There are quite a few related papers in the literature. A recent one, for example, is [3].

Let V_n be a vectorspace of complex-valued functions defined on $\mathbb{R}$ of dimension $n+1$ over $\mathbb{C}$. We say that V_n is shift invariant (on $\mathbb{R}$) if $f \in V_n$ implies that $f_a \in V_n$ for every $a \in \mathbb{R}$, where $f_a(x) := f(x-a)$ on $\mathbb{R}$. Let $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ be a set of distinct COMPLEX numbers. The collection of all linear combinations of $e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}$ over $\mathbb{C}$ will be denoted by

$$E(\Lambda_n) := \operatorname{span}\{e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}\}\,.$$

Elements of $E(\Lambda_n)$ are called exponential sums of $n+1$ terms. Examples of shift invariant spaces of dimension $n+1$ include $E(\Lambda_n)$. In a recent paper [4] the following essentially sharp Nikolskii-type inequality is proved.

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Theorem A. *Suppose $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ is a set of distinct real numbers, $a, b \in \mathbb{R}$, $a < b$, and $0 < p \le q \le \infty$. There are constants $c_1 = c_1(p,q,a,b) > 0$ and $c_2 = c_2(p,q,a,b) > 0$ depending only on p, q, a, and b such that*

$$c_1 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{\frac{1}{p} - \frac{1}{q}} \le \sup_{0 \ne P \in E(\Lambda_n)} \frac{\|P\|_{L_q[a,b]}}{\|P\|_{L_p[a,b]}} \le c_2 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{\frac{1}{p} - \frac{1}{q}} .$$

Using the L_∞ norm on a fixed subinterval $[a+\delta, b-\delta] \subset [a,b]$ in the numerator in the above theorem, we proved the following essentially sharp result in [2].

Theorem B. *If $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ is a set of distinct real numbers, then the inequality*

$$\|f\|_{L_\infty[a+\delta, b-\delta]} \le e 8^{1/p} \left(\frac{n+1}{\delta} \right)^{1/p} \|f\|_{L_p[a,b]}$$

holds for every $f \in E(\Lambda_n)$, $p > 0$, and $\delta \in \left(0, \frac{1}{2}(b-a)\right)$.

Having real exponents λ_j in the above theorems is essential in the proof using some Descartes system methods. In this note we prove an analogous result for complex exponents λ_j, in which case Descartes system methods cannot help us in the proof.

2. New Result

Theorem. *Let $V_n \subset C[a,b]$ be a shift invariant vectorspace of complex-valued functions defined on $\mathbb{R}$ of dimension $n+1$ over $\mathbb{C}$. Let $p \in (0,2]$. Then*

$$\|f\|_{L_\infty[a+\delta, b-\delta]} \le 2^{2/p^2} \left(\frac{n+1}{\delta} \right)^{1/p} \|f\|_{L_p[a,b]}$$

for every $f \in V_n$ and $\delta \in \left(0, \frac{1}{2}(b-a)\right)$.

Problem. Is it possible to extend a version of the theorem for ALL $p > 0$?

Proof. Since V_n is shift invariant, it is sufficient to prove only that

$$|f(0)| \le 2^{2/p^2 - 1/p} (n+1)^{1/p} \|f\|_{L_p[-2,2]}$$

for every $f \in V_n$. Take an orthonormal basis $(L_k)_{k=0}^{n}$ on $[-\frac{1}{2}, \frac{1}{2}]$ so that

$$L_k \in V_n \,, \qquad k = 0, 1, \dots, n \,, \tag{1}$$

and

$$\int_{-1/2}^{1/2} L_j(x) \overline{L_k(x)} \, dx = \delta_{j,k} \,, \qquad 0 \le j \le k \le n \,, \tag{2}$$

where $\delta_{j.k}$ is the Kronecker symbol. On writing $f \in V_n$ as a linear combination of $L_0, L_1, \dots L_n$, and using the Cauchy-Schwarz inequality and the orthonormality of $(L_k)_{k=0}^n$ on $[-\frac{1}{2}, \frac{1}{2}]$, we obtain in a standard fashion that

$$\max_{0 \neq f \in V_n} \frac{|f(t_0)|}{\|f\|_{L_2[-1/2,1/2]}} = \left(\sum_{k=0}^n |L_k(t_0)|^2 \right)^{1/2}, \qquad t_0 \in \mathbb{R}.$$

Since

$$\int_{-1/2}^{1/2} \sum_{k=0}^n |L_k(x)|^2 \, dx = n+1,$$

there exists a $t_0 \in [-\frac{1}{2}, \frac{1}{2}]$ such that

$$\max_{0 \neq f \in V_n} \frac{|f(t_0)|}{\|f\|_{L_2[-1/2,1/2]}} = \left(\sum_{k=0}^n |L_k(t_0)|^2 \right)^{1/2} \leq \sqrt{n+1}.$$

Observe that if $f \in V_n$, then g defined by $g(t) := f(t - t_0)$ is also in V_n, so

$$\max_{0 \neq f \in V_n} \frac{|f(0)|}{\|f\|_{L_2[-1,1]}} \leq \sqrt{n+1}. \tag{3}$$

We introduce

$$\widetilde{V}_n := \{ g : g(t) = f(\lambda t), \quad f \in V_n, \, \lambda \in [-2,2] \}.$$

It follows from (3) that

$$\max_{0 \neq f \in \widetilde{V}_n} \frac{|f(0)|}{\|f\|_{L_2[-1,1]}} \leq \sqrt{n+1}.$$

Let

$$C := \max_{0 \neq f \in \widetilde{V}_n} \frac{|f(0)|}{\|f\|_{L_p[-2,2]}}.$$

Let $0 \neq f \in \widetilde{V}_n$. We define $g \in \widetilde{V}_n$ by $g(t) = f(t/2 + y)$. Then

$$\frac{|f(y)|}{\|f\|_{L_p[-2,2]}} \leq \frac{|f(y)|}{\|f\|_{L_p[y-1,y+1]}} \leq \frac{|g(0)|}{\|g\|_{L_p[-2,2]}} 2^{1/p} \leq 2^{1/p} C, \qquad y \in [-1,1].$$

Hence

$$\max_{0 \neq f \in \widetilde{V}_n} \frac{|f(y)|}{\|f\|_{L_p[-2,2]}} \leq 2^{1/p} C, \qquad y \in [-1,1].$$

Therefore, for every $f \in \widetilde{V}_n$,

$$\begin{aligned}
|f(0)| &\leq \sqrt{n+1}\, \|f\|_{L_2[-1,1]} \\
&\leq \sqrt{n+1} \left(\|f\|_{L_p[-1,1]}^p \|f\|_{L_\infty[-1,1]}^{2-p} \right)^{1/2} \\
&\leq \sqrt{n+1} \left(\|f\|_{L_p[-1,1]}^p \left(2^{1/p} C \right)^{2-p} \|f\|_{L_p[-2,2]}^{2-p} \right)^{1/2} \\
&\leq \sqrt{n+1} \left(2^{1/p} C \right)^{1-p/2} \|f\|_{L_p[-2,2]} \\
&\leq 2^{1/p-1/2} \sqrt{n+1}\, C^{1-p/2} \|f\|_{L_p[-2,2]}.
\end{aligned}$$

Hence

$$C = \max_{0 \neq f \in \widetilde{V}_n} \frac{|f(0)|}{\|f\|_{L_p[-2,2]}} \leq 2^{1/p-1/2}\sqrt{n+1}\, C^{1-p/2}$$

and we conclude that

$$C \leq 2^{2/p^2-1/p}(n+1)^{1/p}\,.$$

So

$$|f(0)| \leq 2^{2/p^2-1/p}(n+1)^{1/p}\|f\|_{L_p[-2,2]}$$

for every $f \in \widetilde{V}_n$, and the result follows. $\square$

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