UPPER BOUNDS FOR THE L_q NORM OF FEKETE POLYNOMIALS ON SUBARCS

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Abstract. For a prime p the polynomial

$$f_p(z) := \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) z^k \,,$$

where the coefficients are Legendre symbols, is called the p-th Fekete polynomial. In this paper the size of the Fekete polynomials on subarcs is studied. We prove essentially sharp bounds for the average value of $|f_p(z)|^q$, $0 < q < \infty$, on subarcs of the unit circle even in the cases when the subarc is rather small. Our upper bounds are matching with the lower bounds proved in a preceding paper for the L_0 norm of the Fekete polynomials on subarcs of the unit circle.

1. Introduction

Finding polynomials in the class

$$\mathcal{L}_n := \left\{ Q : Q(z) = \sum_{k=0}^{n} a_k z^k \,, \quad a_k \in \{-1, 1\} \right\} ,$$

with small uniform norm on the unit circle raised the interest of many authors. Observe that the uniform norm of any polynomial in $\mathcal{L}_n$ on the unit circle is always at least $(n+1)^{1/2}$ since the L_2 norm of any such polynomial is $(2\pi(n+1))^{1/2}$ by the Parseval formula. It is difficult to exhibit a polynomial $Q \in \mathcal{L}_n$ with uniform norm at most $C(n+1)^{1/2}$ for all n with an absolute constant C. An example known was found by H.S. Shapiro [Sh-51] and W. Rudin [Ru-59]. A nice account of this and related problems were given by Littlewood in [Li-69, pages 25–32]. For a prime number p the p-th Fekete polynomial is defined as

$$f_p(z) := \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) z^k \,,$$

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where

$$\left(\frac{k}{p}\right) = \begin{cases} 1, & \text{if } \ x^2 \equiv k \pmod{p} \ \text{has a nonzero solution,} \\ 0, & \text{if } \ p \ \text{divides } \ k\,, \\ -1, & \text{otherwise} \end{cases}$$

is the usual Legendre symbol. Since f_p has constant coefficient 0, it is not a Littlewood polynomial, but $g_p(z) := f_p(z)/z$ is a Littlewood polynomial, and has the same modulus as f_p has on the unit circle. Fekete polynomials are examined in detail in [B-02] and [CG-00].

Let $\alpha < \beta$ be real numbers. The Mahler measure $M_0(Q, [\alpha, \beta])$ is defined for bounded measurable functions $Q(e^{it})$ defined on $[\alpha, \beta]$ as

$$M_0(Q, [\alpha, \beta]) := \exp\left(\frac{1}{\beta - \alpha}\int_\alpha^\beta \log|Q(e^{it})|\, dt\right).$$

It is well known that

$$M_0(Q, [\alpha, \beta]) = \lim_{q \to 0+} M_q(Q, [\alpha, \beta])\,,$$

where

$$M_q(Q, [\alpha, \beta]) := \left(\frac{1}{\beta - \alpha}\int_\alpha^\beta \left|Q(e^{it})\right|^q\, dt\right)^{1/q}, \qquad q > 0\,.$$

It is a simple consequence of the Jensen formula that

$$M_0(Q, [0, 2\pi]) = |c| \prod_{k=1}^n \max\{1, |z_k|\}$$

for every polynomial of the form

$$Q(z) = c \prod_{k=1}^n (z - z_k)\,, \qquad c, z_k \in \mathbb{C}\,.$$

In [M-80] Montgomery proved that there is an absolute constant c such that

$$\max_{t \in [0, 2\pi]} |f_p(e^{it})| \le c p^{1/2} \log p$$

for all primes p. In fact a closer look of his argument shows that combining Lemma 1.1 due to Gauss and the upper bound for the Lebesgue constant for trigonometric interpolation on equidistant nodes given in [CR-76, Theorem 1] implies that

$$\max_{t \in [0, 2\pi]} |f_p(e^{it})| \le p^{1/2}\left(\frac{5}{3} + \frac{2}{\pi}\log\frac{p-1}{2}\right).$$

Montgomery [Mo-80] also showed that the lower bound

$$\frac{2}{\pi} p^{1/2} \log\log p < \max_{t \in [0, 2\pi]} |f_p(e^{it})|$$

holds for all sufficiently large primes p. No better upper and lower bounds than those of Montgomery are known even today.

In [EL-07] we proved that for every $\varepsilon > 0$ there is a constant c_ε such that

$$M_0(f_p, [0, 2\pi]) \geq \left(\frac{1}{2} - \varepsilon\right) p^{1/2} \tag{1.1}$$

for all primes $p \geq c_\varepsilon$. One of the key lemmas in the proof of the above theorem formulates a remarkable property of the Fekete polynomials. A simple proof of it is given in [B-02, pp. 37-38].

Lemma 1.1 (Gauss). *We have*

$$f_p(z_p^j) = \varepsilon_p \left(\frac{j}{p}\right) p^{1/2}\,, \qquad j = 1, 2, \ldots, p-1\,,$$

and $f_p(1) = 0$, *where*

$$z_p := \exp\left(\frac{2\pi i}{p}\right)$$

is the first p-th root of unity, and $\varepsilon_p \in \{-1, 1, -i, i\}$.

The choice of ε_p is more subtle. This is also a result of Gauss, see [Hua-82].

Lemma 1.2 (Gauss). *In Lemma 1.1 we have*

$$\varepsilon_p = \begin{cases} 1, & \text{if} \quad p \equiv 1 \pmod 4 \\ i, & \text{if} \quad p \equiv 3 \pmod 4\,. \end{cases}$$

In [Er-11] the author extended (1.1) to subarcs of the unit circle. Namely it is proved that there is an absolute constant $c_1 > 0$ such that

$$M_0(f_p, [\alpha, \beta]) \geq c_1 p^{1/2}$$

for all prime numbers p and for all $\alpha, \beta \in \mathbb{R}$ such that $(\log p)^{3/2} p^{-1/2} \leq \beta - \alpha \leq 2\pi$.

2. New Results

We give an upper bound for the average value of $|f_p(z)|^q$ over any subarc I of the unit circle, valid for all sufficiently large primes p and exponents $q > 0$.

Theorem 2.1. *There is a constant $c_2(q, \varepsilon)$ depending only on $q > 0$ and $\varepsilon > 0$ such that*

$$\left(\frac{1}{|I|} \int_I |f_p(z)|^q \, |dz|\right)^{1/q} \leq c_2(q, \varepsilon) p^{1/2}\,,$$

for every subarc I of the unit circle with length $|I| \geq 2p^{-1/2+\varepsilon}$.

We remark that together with the result from [Er-10] mentioned at the end of the Introduction, Theorem 2.1 shows that there is an absolute constant $c_1 > 0$ and a constant $c_2(q, \varepsilon) > 0$ depending only on $q > 0$ and $\varepsilon > 0$ such that

$$c_1 p^{1/2} \leq \left(\frac{1}{|I|} \int_I |f_p(z)|^q \, |dz|\right)^{1/q} \leq c_2(q, \varepsilon) p^{1/2}$$

for every subarc I of the unit circle with $|I| \geq 2p^{-1/2+\varepsilon} \geq (\log p)^{3/2} p^{-1/2}$.

Theorem 2.2. *For every sufficiently large prime p and for every $8\pi p^{-1/8} \leq s \leq 2\pi$ there is a closed subset $E := E_{p,s}$ of the unit circle with linear measure $|E| = s$ such that*

$$\frac{1}{|E|} \int_E |f_p(z)|\,|dz| \geq c_3\, p^{1/2} \log\log(1/s)$$

with an absolute constant $c_3 > 0$.

3. Proofs

Our proof of Theorem 2.1 is a combination of Lemma 1.1 due to Gauss, a well-known direct theorem of approximation due to Jackson, and the Marcinkiewicz-Zygmund inequality [MZ-37], [Zy-77, Theorem 7.5, Chapter X]. The Marcinkiewicz-Zygmund inequality asserts that there is a constant $c_4(q)$ depending only on q such that

$$c_4(q)^{-1}\, \frac{1}{n} \sum_{j=1}^{n} |P(z_n^j)|^q \leq \int_0^{2\pi} |P(e^{it})|^q \, dt \leq c_4(q)\, \frac{1}{n} \sum_{j=1}^{n} |P(z_n^j)|^q$$

for any polynomial P of degree at most $n-1$ and for any $1 < q < \infty$, where

$$z_n := \exp\left(\frac{2\pi i}{n}\right)$$

is the first n-th root of unity.

Proof of Theorem 2.1. It is well known that

$$\left(\frac{1}{|I|} \int_I |f_p(z)|^q \, |dz|\right)^{1/q}$$

is an increasing function of q on $(0, \infty)$. So it is sufficient to prove the theorem only for $q > \varepsilon^{-1} > 2$. Let $q > 1$, we will use $q \geq \varepsilon^{-1} > 2$ only at the end of the proof. Without loss of generality we may assume that $|I| \leq 2\pi/3$. We introduce the truncated Fekete polynomials $f_{p,m}$ by

$$f_{p,m}(z) := \sum_{k=1}^{p-(m+1)} \left(\frac{k}{p}\right) z^k \,,$$

with $m := \lfloor p^{1/2} \rfloor$. Then $f_{p,m}$ is a polynomial of degree $p-(m+1)$. Let $I = \{e^{it} : t \in [a,b]\}$ and let $3I := \{e^{it} : t \in [2a-b, 2b-a]\}$ be the arc centered at the midpoint of I with arclength $3|I|$. We define the piecewise linear function L_I on on $[2a-b, 2a-b+2\pi]$ first by

$$L_I(t) := \begin{cases} 1, \quad \text{if} \quad t \in [a,b]\,, \\ \dfrac{t-(2a-b)}{b-a}, \quad \text{if} \quad t \in [2a-b, a]\,, \\ \dfrac{(2b-a)-t}{b-a}, \quad \text{if} \quad t \in [b, 2b-a]\,, \\ 0, \quad \text{if} \quad t \in [2b-a, 2a-b+2\pi]\,, \end{cases}$$

and then we extend it as a periodic function with period 2π defined on $\mathbb{R}$. By a well-known direct theorem of approximation (see [DL-93, p. 205], for example) there is a real trigonometric polynomial T_m of degree at most $m/2$ such that

$$\max_{t\in\mathbb{R}} |L_I(t) - T_m(t)| \le \frac{c_5}{m|I|} \le \frac{1}{2} \tag{3.1}$$

with an absolute constant $c_5 > 0$. Without loss of generality we may assume that $T_m(t) \ge 0$ for every $t \in \mathbb{R}$, hence $T_m(t) = |Q_m(e^{it})|$ with an appropriate algebraic polynomial Q_m of degree at most m. Note that $\frac{1}{2} \le |Q_m(z)| \le \frac{3}{2}$ for every $z = e^{it} \in I$. Observe that

$$|f_p(z) - f_{p,m}(z)| \le m, \qquad z = e^{it}, \quad t \in \mathbb{R}\,. \tag{3.2}$$

Using Lemma 1.1 and (3.2) we can deduce that

$$|f_{p,m}(z_p^j)| \le |f_p(z_p^j)| + |f_{p,m}(z_p^j) - f_p(z_p^j)| \le p^{1/2} + m\,, \qquad j = 1, 2, \ldots, p\,. \tag{3.3}$$

Combining the inequality

$$|a+b|^q \le 2^{q-1}(|a|^q + |b|^q), \qquad a, b \in \mathbb{C}, \quad q \in [1,\infty)\,,$$

with (3.2), and then recalling that $\frac{1}{2} \le |Q_m(z)|$ for all $z = e^{it} \in I$, we obtain

$$\begin{aligned} \int_I |f_p(z)|^q\,|dz| \le & \int_I 2^{q-1}(|f_{p,m}(z)|^q + |f_p(z) - f_{p,m}(z)|^q)\,|dz| \\ =& 2^{q-1}\int_I |f_{p,m}(z)|^q\,|dz| + 2^{q-1}\int_I |f_p(z) - f_{p,m}(z)|^q\,|dz| \\ \le& 2^{q-1}\int_I |f_{p,m}(z)|^q\,|dz| + 2^{q-1}m^q|I| \\ \le& 2^{q-1}2^q \int_I |(f_{p,m}Q_m)(z)|^q\,|dz| + 2^{q-1}m^q|I|\,. \end{aligned} \tag{3.4}$$

Applying the Marcinkiewicz-Zygmund inequality to the polynomial

$$P := f_{p,m}Q_m$$

of degree at most $p-1$, then using (3.3), we obtain

$$\begin{aligned} \int_I |(f_{p,m}Q_m)(z)|^q\,|dz| &\le c_4(q)\frac{1}{p}\sum_{j=1}^{p} |(f_{p,m}Q_m)(z_p^j)|^q \\ &\le c_4(q)(p^{1/2}+m)^q\frac{1}{p}\sum_{j=1}^{p}|Q_m(z_p^j)|^q\,. \end{aligned} \tag{3.5}$$

Observe that (3.1) implies that

$$|Q_m(z_p^j)|^q \le 2^q\,, \qquad z_p^j \in 3I\,,$$

$$|Q_m(z_p^j)|^q \le \left(\frac{c_5}{m|I|}\right)^q\,, \qquad z_p^j \notin 3I\,,$$

and there are at most $\dfrac{3p|I|}{2\pi}+1$ values of $j=1,2,\ldots,p$ for which $z_p^j \in 3I$. Hence

$$\begin{aligned}
\frac{1}{p}\sum_{j=1}^{p}|Q_m(z_p^j)|^q \le& \frac{1}{p}\left(2^q\left(\frac{3p|I|}{2\pi}+1\right)+\left(\frac{c_5}{m|I|}\right)^q p\right)\\
\le&\left(2^q\left(\frac{3|I|}{2\pi}+\frac{1}{p}\right)+(2c_5)^q|I|\right)\\
\le& c_6(q)|I|
\end{aligned} \tag{3.6}$$

with a constant $c_6(q)$ depending only on q, whenever

$$\left(\frac{c_5}{m|I|}\right)^q \le (2c_5)^q|I|\,,$$

that is, whenever

$$\frac{1}{m} \le 2p^{-1/2} \le 2\,|I|^{1+1/q}\,.$$

Combining (3.4), (3.5), and (3.6), and recalling that $m \le p^{1/2}$, we conclude

$$\begin{aligned}
\frac{1}{|I|}\int_I |f_p(z)|^q\,|dz| \le& \frac{4^q}{|I|}\left(\int_I |(f_{p,m}Q_m)(z)|^q\,|dz|\right)+2^q m^q\\
\le& \frac{4^q}{|I|}c_4(q)(p^{1/2}+m)^q\frac{1}{p}\left(\sum_{j=1}^{p}|Q_m(z_p^j)|^q\right)+2^q m^q\\
\le& 4^q c_4(q)2^q p^{q/2}c_6(q)+2^q m^q\\
\le& c_7(q)p^{q/2}
\end{aligned}$$

with a constant $c_7(q)$ depending only on q, whenever

$$\frac{1}{m} \le 2p^{-1/2} \le 2\,|I|^{1+1/q}\,.$$

So the theorem is proved for all $q>0$ satisfying

$$\frac{-1/2}{1+1/q} \le -\frac{1}{2}+\varepsilon\,,$$

hence for all $q>\varepsilon^{-1}>2$, with a constant $c_2(q,\varepsilon)$ depending only on q and ε. $\square$

To prove Theorem 2.2 we follow [Mo-80]. Let $e(t)=\exp(2\pi i t)$. Our first lemma is stated as Lemma 1 and proved in [Mo-80].

Lemma 3.1. *Let χ be a primitive character* (mod q)*, $q > 1$. Then*

$$\sum_{m=0}^{q-1} \chi(m)e(m\alpha) = \tau(\chi)\, q^{-1}\, e(\tfrac{1}{2}q\alpha)(\sin(\pi q\alpha))\, T(\alpha, \overline{\chi})\,,$$

where $\tau(\chi)$ is the Gauss sum

$$\tau(\chi) = \sum_{n=1}^{q} \chi(n) e\left(\frac{n}{q}\right),$$

and

$$T(\alpha, \chi) = \sum_{a=1}^{q} \chi(a) \cot\left(\pi\left(\alpha - \frac{a}{q}\right)\right).$$

Note that in the case if

$$\chi(n) = \left(\frac{n}{p}\right)$$

is the quadratic character, then Lemma 1.1 implies $\tau(\chi) = \varepsilon_p p^{1/2}$, and the content of Lemma 3.1 is just the identity obtained by expressing the Fekete polynomial f_p by the Lagrange interpolation formula associated with the p-th root of unity. In fact, in the proof of Theorem 2.2 we will need Lemma 3.1 above only in the case when χ is the quadratic character.

Our second lemma is stated as Lemma 2 and proved in [Mo-80].

Lemma 3.2. *Let p be a prime. For $k \geq 1$ let $a_1, a_2, \dots, a_k$ be integers, distinct* (mod p)*, and put $f(x) = \prod_{j=1}^{k} (x - a_j)$. Then*

$$\left|\sum_{n=1}^{p} \left(\frac{f(n)}{p}\right)\right| \leq (k-1)p^{1/2}\,.$$

Montgomery writes "This is a consequence of Weil's Riemann Hypothesis for the zeta function of a curve over a finite field: see Weil [We-45], [We-49]. The derivation of the particular bound above is given by Burgess ([Bu-57]; §2)."

Proof of Theorem 2.2. We rely heavily on Montgomery's beautiful line of proof in [Mo-80] to connect the two lemmas above to the proof of the theorem. Let $T(\alpha) := T(\alpha, \chi)$ with

$$\chi(h) = \left(\frac{h}{p}\right).$$

It follows from Lemma 1.1 that $|\tau(\chi)| = p^{1/2}$ and hence Lemma 3.1 implies that

$$\left|f_p\left(e\left(\frac{2n+\delta}{2p}\right)\right)\right| \geq \frac{1}{\sqrt{2}}\, p^{-1/2} \left|T\left(\frac{2n+\delta}{2p}\right)\right| \tag{3.7}$$

for every $n = 1, 2, \ldots, p$ and $\delta \in [\frac{1}{2}, \frac{3}{2}]$. We define

$$W(n) := W_H(n) := \prod_{h=1}^{H} \left(1 - \left(\frac{n+h}{p}\right)\right) \prod_{h=0}^{H} \left(1 + \left(\frac{n-h}{p}\right)\right), \tag{3.8}$$

and compute the size of the weighted sum

$$\sum_{n=1}^{p} T\left(\frac{2n+\delta}{2p}\right) W(n)$$

for $\delta \in [\frac{1}{2}, \frac{3}{2}]$. By multiplying the product (3.8) out, we have

$$W(n) = 1 + \sum_{f} \varepsilon_f \left(\frac{f(n)}{p}\right), \qquad \varepsilon_f \in \{-1, 1\},$$

where f runs through $2^{2H+1} - 1$ polynomials of the sort considered in Lemma 3.2. Hence, using Lemma 3.2 we can deduce that

$$\sum_{n=1}^{p} W(n) = p + O(H 2^{2H} p^{1/2}). \tag{3.9}$$

Similarly,

$$\sum_{n=1}^{p} W(n) \left(\frac{n-a}{p}\right) = c(a)p + O(H 2^{2H} p^{1/2}), \tag{3.10}$$

where $c(a) = 1$ if $0 \le a \le H$, $c(a) = 0$ if $H < a < p - H$, and $c(a) = -1$ if $p - H \le a < p$. We have

$$\begin{aligned} &\sum_{n=1}^{p} T\left(\frac{2n+\delta}{2p}\right) W(n) \\ &= \sum_{n=1}^{p} \sum_{a=1}^{p} \left(\frac{a}{p}\right) \cot\left(\pi\left(\frac{2n+\delta}{2p} - \frac{a}{p}\right)\right) W(n) \\ &= \sum_{a=1}^{p} \sum_{n=1}^{p} \left(\frac{n-a}{p}\right) W(n) \cot\left(\pi\left(\frac{2a+\delta}{2p}\right)\right) \\ &= \sum_{a=1}^{H} + \sum_{a=p-H}^{p} + \sum_{a=H+1}^{p-H-1} \end{aligned} \tag{3.11}$$

for every $\delta \in [\frac{1}{2}, \frac{3}{2}]$. Using (3.10) and the facts that

$$\cot x = -\cot(\pi - x) = \begin{cases} x^{-1} + O(x), & \text{if} \quad x \in (0, \pi/2], \\ -(\pi - x)^{-1} + O(\pi - x), & \text{if} \quad x \in [\pi/2, \pi), \end{cases}$$

and

$$\sum_{a=H+1}^{p-H-1} \cot\left(\pi\left(\frac{2a+\delta}{2p}\right)\right) = O\left(\sum_{a=H+1}^{p-H-1} \frac{p}{a}\right) = O(p\log p)\,,$$

we obtain

$$\begin{aligned}(3.12)\qquad & \sum_{a=1}^{H} + \sum_{a=p-H}^{p} + \sum_{a=H+1}^{p-H-1} \\ =& \frac{4p^2}{\pi}\sum_{a=1}^{H}\frac{1}{2a-1} + O(p^2) + O(H2^{2H}p^{1/2}p\log p) \\ =& \frac{2}{\pi}p^2\log H + O(p^2) + O(H2^{2H}p^{1/2}p\log p) \\ =& \frac{2}{\pi}p^2\log H + O(p^2)\end{aligned}$$

whenever $\delta \in [\frac{1}{2}, \frac{3}{2}]$ and $2 \le H \le \frac{1}{8}\log p$. Combining (3.11) and (3.12), we conclude

$$(3.13)\qquad \sum_{n=1}^{p} T\left(\frac{2n+\delta}{2p}\right) W(n) = \frac{2}{\pi}p^2\log H + O(p^2)$$

whenever $\delta \in [\frac{1}{2}, \frac{3}{2}]$ and $2 \le H \le \frac{1}{8}\log p$.

Now let $A := A_{p,H}$ be the union of all intervals

$$\left[\frac{2n+\frac{1}{2}}{2p}, \frac{2n+\frac{3}{2}}{2p}\right]$$

with $W(n) := W_H(n) \neq 0$, $n = 1, 2, \dots, p$. We define $B = B_{p,H} := \{e(t) : t \in A\}$. Note that

$$(3.14)\qquad W(n) \in \{2^{2H}, 2^{2H+1}, 0\}\,, \qquad n = 1, 2, \dots, p\,.$$

This, together with (3.9), implies that the linear measure of B is

$$(3.15)\qquad |B| \le \frac{p}{2^{2H}}\frac{2\pi}{2p} + O(Hp^{-1/2}) = (\pi + O(p^{-1/4}\log p))2^{-2H}$$

whenever $2 \le H \le \frac{1}{8}\log p$. Also $|B| \le 2\pi 2^{-2H}$ for all sufficiently large primes p and for all integers $2 \le H \le \frac{1}{8}\log p$. Using (3.7) we obtain

$$\begin{aligned}
\int_B |f_p(z)|\,|dz| =&2\pi \int_A |f_p(e(t))|\,dt \\
=&\frac{\pi}{p} \sum_{\substack{n=1 \\ W(n)\neq 0}}^{p} \int_{1/2}^{3/2} \left| f_p\left(e\left(\frac{2n+\delta}{2p}\right)\right)\right| \,d\delta \\
\ge&\frac{\pi}{p}\,\frac{1}{\sqrt{2}}\,p^{-1/2} \sum_{\substack{n=0 \\ W(n)\neq 0}}^{p} \int_{1/2}^{3/2} \left| T\left(\frac{2n+\delta}{2p}\right)\right| \,d\delta \\
\ge&\frac{\pi}{\sqrt{2}}\,p^{-3/2} \int_{1/2}^{3/2} \left(\sum_{\substack{n=1 \\ W(n)\neq 0}}^{p} T\left(\frac{2n+\delta}{2p}\right)\right) d\delta\,.
\end{aligned} \tag{3.16}$$

Using (3.14) and (3.13) we can continue as

$$\begin{aligned}
&\frac{\pi}{\sqrt{2}}\,p^{-3/2} \int_{1/2}^{3/2} \left(\sum_{\substack{n=1 \\ W(n)\neq 0}}^{p} T\left(\frac{2n+\delta}{2p}\right)\right) d\delta \\
\ge&\frac{\pi}{\sqrt{2}}\,p^{-3/2}2^{-(2H+1)} \int_{1/2}^{3/2} \left(\sum_{\substack{n=1 \\ W(n)\neq 0}}^{p} T\left(\frac{2n+\delta}{2p}\right) W(n)\right) d\delta \\
\ge&\frac{\pi}{\sqrt{2}}\,p^{-3/2}2^{-(2H+1)} \left(\frac{2}{\pi}p^2\log H + O(p^2)\right) \\
\ge&\frac{\pi}{\sqrt{2}}\,2^{-(2H+1)} \left(\frac{2}{\pi}p^{1/2}\log H + O(p^{1/2})\right)\,.
\end{aligned} \tag{3.17}$$

Thus (3.16) and (3.17) imply

$$\int_B |f_p(z)|\,|dz| \ge \frac{\pi}{\sqrt{2}}\,2^{-(2H+1)} \left(\frac{2}{\pi}p^{1/2}\log H + O(p^{1/2})\right)\,. \tag{3.18}$$

Now let $8\pi p^{-1/8} \le s \le 2\pi$ be fixed. Without loss of generality we may assume that $s \le 1$. Let $H \ge 2$ be the (only) integer such that

$$s/4 < 2\pi\,2^{-2H} \le s\,. \tag{3.19}$$

Then

$$H \le \frac{\log p}{16\log 2} \le \frac{1}{8}\log p\,.$$

As $|B_{p,H}| \le 2\pi\,2^{-2H}$ for all sufficiently large primes p and for all integers $2 \le H \le \frac{1}{8}\log p$, there is a closed subset $E := E_{p,s}$ of the unit circle with linear measure s containing $B := B_{p,H}$. Then (3.18) and (3.19) imply that

$$\frac{1}{s}\int_E |f_p(z)|\,|dz| \ge c(p^{1/2}\log\log(1/s) + O(p^{1/2}))$$

with an absolute constant $c > 0$. $\square$

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