RECENT PROGRESS IN THE STUDY OF POLYNOMIALS WITH CONSTRAINED COEFFICIENTS

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0. Notation

Let $\mathcal{P}_n$ be the set of all algebraic polynomials of degree at most n with real coefficients. Let $\mathcal{P}_n^c$ be the set of all algebraic polynomials of degree at most n with complex coefficients. Let

$$\mathcal{K}_n := \left\{ Q_n : Q_n(z) = \sum_{k=0}^n a_k z^k, \;\; a_k \in \mathbb{C}\,, \;\; |a_k| = 1 \right\}.$$

The class $\mathcal{K}_n$ is often called the collection of all (complex) unimodular polynomials of degree n. Let

$$\mathcal{L}_n := \left\{ Q_n : Q_n(z) = \sum_{k=0}^n a_k z^k, \;\; a_k \in \{-1, 1\} \right\}.$$

The class $\mathcal{L}_n$ is often called the collection of all (real) unimodular polynomials of degree n. Let D denote the open unit disk of the complex plane. We will denote the unit circle of the complex plane by ∂D. We define the Mahler measure of Q (geometric mean of Q on ∂D) by

$$M_0(Q) := \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log |Q(e^{it})|\, dt \right)$$

for bounded measurable functions Q on ∂D. It is well known, see [HL-52], for instance, that

$$M_0(Q) = \lim_{q \to 0+} M_q(Q)\,,$$

where

$$M_q(Q) := \left(\frac{1}{2\pi} \int_0^{2\pi} \left|Q(e^{it})\right|^q \, dt \right)^{1/q}, \qquad q > 0\,.$$

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It is also well known that for a function Q continuous on ∂D we have

$$M_\infty(Q) := \max_{t\in[0,2\pi]} |Q(e^{it})| = \max_{t\in\mathbb{R}} |Q(e^{it})| = \lim_{q\to\infty} M_q(Q)\,.$$

It is a simple consequence of the Jensen formula that

$$M_0(Q) = |c| \prod_{k=1}^{n} \max\{1, |z_k|\}$$

for every polynomial of the form

$$Q(z) = c \prod_{k=1}^{n} (z - z_k)\,, \qquad c, z_k \in \mathbb{C}\,.$$

We define the Mahler measure (geometric mean of Q on $[\alpha,\beta]$)

$$M_0(Q,[\alpha,\beta]) := \exp\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \log|Q(e^{it})|\,dt\right)$$

for $[\alpha,\beta] \subset \mathbb{R}$ and bounded measurable functions $Q(e^{it})$ on $[\alpha,\beta]$. It is well known, see [HL-52], for instance, that

$$M_0(Q,[\alpha,\beta]) = \lim_{q\to 0+} M_q(Q,[\alpha,\beta])\,,$$

where, for $q > 0$,

$$M_q(Q,[\alpha,\beta]) := \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \left|Q(e^{it})\right|^q\,dt\right)^{1/q}\,.$$

If $Q \in \mathcal{P}_n^c$ is of the form

$$Q(z) = \sum_{j=0}^{n} a_j z^j\,, \qquad a_j \in \mathbb{C}\,,$$

then its conjugate polynomial is defined by

$$Q^*(z) := z^n \overline{Q}(1/z) := \sum_{j=0}^{n} \overline{a}_{n-j} z^j\,.$$

A polynomial $Q \in \mathcal{P}_n^c$ is called conjugate-reciprocal if $Q = Q^*$.

The Lebesgue measure of a measurable set $A \subset \mathbb{R}$ will be denoted by $m(A)$ throughout the paper.

1. Ultraflat sequences of unimodular polynomials

By Parseval's formula,

$$\int_0^{2\pi} \left|P_n(e^{it})\right|^2 \, dt = 2\pi(n+1)$$

for all $P_n \in \mathcal{K}_n$. Therefore

$$\min_{t\in\mathbb{R}} |P_n(e^{it})| \leq \sqrt{n+1} \leq \max_{t\in\mathbb{R}} |P_n(e^{it})| \, .$$

An old problem (or rather an old theme) is the following.

Problem 1.1 (Littlewood's Flatness Problem). *How close can a polynomial $P_n \in \mathcal{K}_n$ or $P_n \in \mathcal{L}_n$ come to satisfying*

$$|P_n(e^{it})| = \sqrt{n+1} \, , \qquad t \in \mathbb{R}? \tag{1.1}$$

Obviously (1.1) is impossible if $n \geq 1$. So one must look for less than (1.1), but then there are various ways of seeking such an "approximate situation". One way is the following. In his paper [L-66b] Littlewood had suggested that, conceivably, there might exist a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ (possibly even $P_n \in \mathcal{L}_n$) such that $(n+1)^{-1/2}|P_n(e^{it})|$ converge to 1 uniformly in $t \in \mathbb{R}$. We shall call such sequences of unimodular polynomials "ultraflat". More precisely, we give the following definition.

Definition 1.2. *Given a positive number ε, we say that a polynomial $P_n \in \mathcal{K}_n$ is ε-flat if*

$$(1-\varepsilon)\sqrt{n+1} \leq |P_n(e^{it})| \leq (1+\varepsilon)\sqrt{n+1} \, , \qquad t \in \mathbb{R} \, .$$

Definition 1.3. *Let (n_k) be an increasing sequence of positive integers. Given a sequence (ε_{n_k}) of positive numbers tending to 0, we say that a sequence (P_{n_k}) of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is (ε_{n_k})-ultraflat if each P_{n_k} is (ε_{n_k})-flat. We simply say that a sequence (P_{n_k}) of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is ultraflat if it is (ε_{n_k})-ultraflat with a suitable sequence (ε_{n_k}) of positive numbers tending to* 0.

The existence of an ultraflat sequence of unimodular polynomials seemed very unlikely, in view of a 1957 conjecture of P. Erdős (Problem 22 in [E-57]) asserting that, for all $P_n \in \mathcal{K}_n$ with $n \geq 1$,

$$\max_{t\in\mathbb{R}} |P_n(e^{it})| \geq (1+\varepsilon)\sqrt{n+1} \, , \tag{1.2}$$

where $\varepsilon > 0$ is an absolute constant (independent of n). Yet, refining a method of Körner [K-80b], Kahane [K-80a] proved that there exists a sequence (P_n) with $P_n \in \mathcal{K}_n$ which is (ε_n)-ultraflat, where $\varepsilon_n = O\left(n^{-1/17}\sqrt{\log n}\right)$. (Kahane's paper contained though a slight error which was corrected in [QS-95].) Thus the Erdős conjecture (2.2) was disproved for the classes $\mathcal{K}_n$. For the more restricted class $\mathcal{L}_n$ the analogous Erdős conjecture is unsettled to this date. It is a common belief that the analogous Erdős conjecture for $\mathcal{L}_n$ is true, and consequently there is no ultraflat sequence of polynomials $P_n \in \mathcal{L}_n$. An interesting result related to Kahane's breakthrough is given in [B-91]. For an account of some of the work done till the mid 1960's, see Littlewood's book [L-68] and [QS-96].

Let (ε_n) be a sequence of positive numbers tending to 0. Let the sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ be (ε_n)-ultraflat. We write

$$(1.3) \qquad P_n(e^{it}) = R_n(t)e^{i\alpha_n(t)}\,, \qquad R_n(t) = |P_n(e^{it})|\,, \qquad t \in \mathbb{R}\,.$$

It is simple to show that α_n can be chosen to be in $C^\infty(\mathbb{R})$. This is going to be our understanding throughout the paper. It is easy to find a formula for $\alpha_n(t)$ in terms of P_n. We have

$$(1.4) \qquad \alpha_n'(t) = \mathrm{Re}\left(\frac{e^{it}P_n'(e^{it})}{P_n(e^{it})}\right)\,,$$

see formulas (7.1) and (7.2) on p. 564 and (8.2) on p. 565 in [S-92]. The angular function α_n^* and modulus function $R_n^* = R_n$ associated with the polynomial P_n^* are defined by

$$P_n^*(e^{it}) = R_n^*(t)e^{i\alpha_n^*(t)}\,, \qquad R_n^*(t) = |P_n^*(e^{it})|\,.$$

Similarly to α_n, the angular function α_n^* can also be chosen to be in $C^\infty(\mathbb{R})$ on $\mathbb{R}$. By applying formula (1.4) to P_n^*, it is easy to see that

$$(1.5) \qquad \alpha_n'(t) + {\alpha_n^*}'(t) = n\,, \qquad t \in \mathbb{R}\,.$$

The structure of ultraflat sequences of unimodular polynomials is studied in [E-00a], [E-00b], [E-01a], and [E-01b], where several conjectures of Saffari are proved. These are closely related to each other.

Conjecture 1.4 (Uniform Distribution Conjecture for the Angular Speed). *Let (P_n) be a fixed ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. With the notation (1.3), in the interval $[0, 2\pi]$, the distribution of the normalized angular speed $\alpha_n'(t)/n$ converges to the uniform distribution as $n \to \infty$. More precisely, we have*

$$(1.6) \qquad m(\{t \in [0, 2\pi] : 0 \le \alpha_n'(t) \le nx\}) = 2\pi x + o_n(x)$$

for every $x \in [0, 1]$, where $o_n(x)$ converges to 0 uniformly on $[0, 1]$. As a consequence, $|P_n'(e^{it})|/n^{3/2}$ also converges to the uniform distribution as $n \to \infty$. More precisely, we have

$$m(\{t \in [0, 2\pi] : 0 \le |P_n'(e^{it})| \le n^{3/2}x\}) = 2\pi x + o_n(x)$$

for every $x \in [0, 1]$, where $o_n(x)$ converges to 0 uniformly on $[0, 1]$.

The basis of this conjecture was that for the special ultraflat sequences of unimodular polynomials produced by Kahane [K-80a], (1.6) is indeed true.

In Section 4 of [E-00a] we prove this conjecture in general.

In the general case (1.6) can, by integration, be reformulated (equivalently) in terms of the moments of the angular speed $\alpha_n'(t)$. This was observed and recorded by Saffari [S-92]. For completeness the proof of this equivalence is presented in Section 4 of [E-00a] and we settle Conjecture 1.4 by proving the following result.

Theorem 1.5 (Reformulation of the Uniform Distribution Conjecture). *Let (P_n) be a fixed ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. For any $q > 0$ we have*

$$\frac{1}{2\pi} \int_0^{2\pi} |\alpha_n'(t)|^q \, dt = \frac{n^q}{q+1} + o_{n,q} n^q \,. \tag{1.7}$$

with suitable constants $o_{n,q}$ converging to 0 for every fixed $q > 0$.

An immediate consequence of (1.7) is the remarkable fact that for large values of $n \in \mathbb{N}$, the $L_q(\partial D)$ Bernstein factors

$$\frac{\int_0^{2\pi} \left|P_n'(e^{it})\right|^q \, dt}{\int_0^{2\pi} |P_n(e^{it})|^q \, dt}$$

of the elements of ultraflat sequences (P_n) of unimodular polynomials are essentially independent of the polynomials. More precisely Theorem 1.5 implies the following result.

Theorem 1.6 (The Bernstein Factors). *Let q be an arbitrary positive real number. Let (P_n) be a fixed ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. We have*

$$\frac{\int_0^{2\pi} \left|P_n'(e^{it})\right|^q \;, dt}{\int_0^{2\pi} |P_n(e^{it})|^q \, dt} = \frac{n^{q+1}}{q+1} + o_{n,q} n^{q+1} \,,$$

and as a limit case,

$$\frac{\max_{0 \le t \le 2\pi} |P_n'(e^{it})|}{\max_{0 \le t \le 2\pi} |P_n(e^{it})|} = n + o_n n \,.$$

with suitable constants $o_{n,q}$ and o_n converging to 0 for every fixed q.

In Section 3 of [E-00a] we prove the following result which turns out to be stronger than Theorem 1.5.

Theorem 1.7 (Negligibility Theorem for Higher Derivatives). *Let (P_n) be a fixed ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. For every integer $r \geq 2$, we have*

$$\max_{0 \le t \le 2\pi} |\alpha_n^{(r)}(t)| = o_{n,r} n^r$$

with suitable constants $o_{n,r}$ converging to 0 for every fixed $r = 2, 3, \ldots$.

We show in Section 4 of [E-00a] how Theorem 1.4 follows from Theorem 1.7.

In Section 4 of [E-00a] we also prove an an extension of Saffari's Uniform Distribution Conjecture 1.4 to higher derivatives.

Theorem 1.8 (Distribution of the Modulus of Higher Derivatives of Ultraflat Sequences of Unimodular Polynomials). *Let (P_n) be a fixed ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. The distribution of*

$$\left(\frac{|P_n^{(r)}(e^{it})|}{n^{r+1/2}} \right)^{1/r}$$

converges to the uniform distribution as $n \to \infty$. More precisely, we have

$$m\left(\left\{t \in [0,2\pi] : 0 \le |P_n^{(r)}(e^{it})| \le n^{r+1/2}x^r\right\}\right) = 2\pi x + o_{r,n}(x)$$

for every $x \in [0,1]$, where $o_{r,n}(x)$ converges to 0 unifirmly for every fixed $r = 1,2,\dots$.

In [E-03], based on the results in [E-00a], we proved yet another conjecture of Queffelec and Saffari, see (1.30) in [QS-96]. Namely we proved asymptotic formulas for the L_q norms of the real part and the derivative of the real part of ultraflat unimodular polynomials on the unit circle. A recent paper of Bombieri and Bourgain [BB-09] is devoted to the construction of ultraflat sequences of unimodular polynomials. In particular, they obtained a much improved estimate for the error term. A major part of their paper deals also with the long-standing problem of the effective construction of ultraflat sequences of unimodular polynomials.

For $\lambda \ge 0$, let

$$\mathcal{K}_n^\lambda := \left\{P_n : P_n(z) = \sum_{k=0}^n a_k k^\lambda z^k,\;\; a_k \in \mathbb{C}\,,\;\; |a_k| = 1\right\}\,.$$

Ultraflat sequences (P_n) of polynomials $P_n \in \mathcal{K}_n^\lambda$ are defined and studied thoroughly in [EN-16] where various extensions of Saffari's conjectures have been proved. Note that it is not yet known whether or not ultraflat sequences (P_n) of polynomials $P_n \in \mathcal{K}_n^\lambda$ exist for any $\lambda > 0$, in particular, it is not yet known for $\lambda = 1$.

In [E-01a] we examined how far an ultraflat unimodular polynomial is from being conjugate-reciprocal, and we proved the following three theorems.

Theorem 1.9. *Let (P_n) be a fixed ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. We have*

$$\frac{1}{2\pi}\int_0^{2\pi}\left(|P_n'(e^{it})| - |P_n^{*\prime}(e^{it})|\right)^2\,dt = \left(\frac{1}{3} + \gamma_n\right)n^3\,,$$

with some constants γ_n converging to 0.

Theorem 1.10. *Let (P_n) be a fixed ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. If*

$$P_n(z) = \sum_{k=0}^n a_{k,n}z^k\,, \qquad k = 0,1,\dots,n, \quad n = 1,2,\dots\,,$$

then

$$\sum_{k=0}^n k^2\,|a_{k,n} - \overline{a}_{n-k,n}|^2 = \frac{1}{2\pi}\int_0^{2\pi}\left|(P_n' - P_n^{*\prime})(e^{it})\right|^2\,dt \ge \left(\frac{1}{3} + h_n\right)n^3\,,$$

with some constants h_n converging to 0.

Theorem 1.11. *Let (P_n) be a fixed ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Using the notation of Theorem 1.10 we have*

$$\sum_{k=0}^{n} |a_{k,n} - \overline{a}_{n-k,n}|^2 = \frac{1}{2\pi} \int_0^{2\pi} \left|(P_n - P_n^*)(e^{it})\right|^2 \, dt \geq \left(\frac{1}{3} + h_n\right) n\,,$$

with some constants h_n (the same as in Theorem 1.10) converging to 0.

There are quite a few recent publications on or related to ultraflat sequences of unimodular polynomials. Some of them (not mentioned before) are [B-02], [S-01], [QS-95], [O-18], and [M-17].

2. More recent results on ultraflat sequences of unimodular polynomials

In a recent paper [E-19d] we revisited the topic. Theorems 2.1–2.4 and 2.6 are new in [E-19d], and Theorems 2.5 and 2.7 recapture old results.

In our results below Γ denotes the usual gamma function, and the $\sim$ symbol means that the ratio of the left and right hand sides converges to 1 as $n \to \infty$.

Theorem 2.1. *If (P_n) is an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$ and $q \in (0, \infty)$, then*

$$\frac{1}{2\pi} \int_0^{2\pi} \left|(P_n - P_n^*)(e^{it})\right|^q \, dt \sim \frac{2^q \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}} \, n^{q/2}\,.$$

Our next theorem is a special case ($q=2$) of Theorem 2.1. Compare it with Theorem 1.11.

Theorem 2.2. *Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. If*

$$P_n(z) = \sum_{k=0}^{n} a_{k,n} z^k\,, \qquad k = 0, 1, \ldots, n, \quad n = 1, 2, \ldots\,,$$

then

$$\sum_{k=0}^{n} |a_{k,n} - \overline{a}_{n-k,n}|^2 = \frac{1}{2\pi} \int_0^{2\pi} \left|(P_n - P_n^*)(e^{it})\right|^2 \, dt \sim 2n\,.$$

Our next theorem should be compared with Theorem 1.10.

Theorem 2.3. *Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Using the notation in Theorem 2.2 we have*

$$\sum_{k=0}^{n} k^2 \, |a_{k,n} - \overline{a}_{n-k,n}|^2 = \frac{1}{2\pi} \int_0^{2\pi} \left|(P_n' - P_n^{*\prime})(e^{it})\right|^2 \, dt \sim \frac{2n^3}{3}\,.$$

We also proved the following result.

Theorem 2.4. *If (P_n) is an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$ and $q \in (0,\infty)$, then*
$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{d}{dt}|(P_n - P_n^*)(e^{it})|\right|^q \, dt \sim \frac{\Gamma\left(\frac{q+1}{2}\right)}{(q+1)\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}}\, n^{3q/2}\,.$$

As a Corollary of Theorem 2.2 we have recaptured Saffari's "near orthogonality conjecture" raised in [S-92] and proved first in [E-01b].

Theorem 2.5. *Let (P_n) be a fixed ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Using the notation in Theorem 2.2 we have*
$$\sum_{k=0}^{n} a_{k,n}a_{n-k,n} = o(n)\,.$$

As a Corollary of Theorem 2.3 we have proved a new "near orthogonality" formula.

Theorem 2.6. *Let (P_n) be a fixed ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Using the notation in Theorem 2.2 we have*
$$\sum_{k=0}^{n} k^2 a_{k,n}a_{n-k,n} = o(n^3)\,.$$

Finally we have recaptured the asymptotic formulas for the real part and the derivative of the real part of ultraflat unimodular polynomials proved in [E-03] first.

Theorem 2.7. *If (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$, and $q \in (0,\infty)$, then for $f_n(t) := \mathrm{Re}(P_n(e^{it}))$ we have*
$$\frac{1}{2\pi}\int_0^{2\pi} |f_n(t)|^q \, dt \sim \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}}\, n^{q/2}$$
and
$$\frac{1}{2\pi}\int_0^{2\pi} |f_n'(t)|^q \, dt \sim \frac{\Gamma\left(\frac{q+1}{2}\right)}{(q+1)\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}}\, n^{3q/2}\,.$$

We remark that trivial modifications of the proof of Theorem 2.1–2.7 yield that the statement of the above theorem remains true if the ultraflat sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ is replaced by an ultraflat sequence (P_{n_k}) of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$, where (n_k) is an increasing sequence of positive integers.

3. Flatness of conjugate-reciprocal unimodular polynomials

There is a beautiful short argument to see that

$$M_\infty(P) \geq \sqrt{4/3}\, n^{1/2} \tag{3.1}$$

for every conjugate-reciprocal unimodular polynomial $P \in \mathcal{K}_n$. Namely, Parseval's formula gives

$$M_\infty(P') \geq M_2(P') = \left(\frac{n(n+1)(2n+1)}{6}\right)^{1/2}, \qquad P \in \mathcal{K}_n\,.$$

Combining this with Malik's extension of Lax's Bernstein-type inequality

$$M_\infty(P') \leq \frac{n}{2}\, M_\infty(P)$$

valid for all conjugate reciprocal algebraic polynomials $P \in \mathcal{P}_n^c$ (see p. 438 in [BE-95], for instance), we obtain

$$M_\infty(P) \geq \frac{2}{n}\left(\frac{n(n+1)(2n+1)}{6}\right)^{1/2} \geq \sqrt{4/3}\, n^{1/2}$$

for all conjugate-reciprocal unimodular polynomials $P \in \mathcal{K}_n$. In [E-15] we prove the following results.

Theorem 3.1. *There is an absolute constant $\varepsilon > 0$ such that*

$$M_1(P') \leq (1-\varepsilon)\sqrt{1/3}\, n^{3/2}$$

for every conjugate-reciprocal unimodular polynomial $P \in \mathcal{K}_n$ and for all sufficiently large n.

Theorem 3.2. *There is an absolute constant $\varepsilon > 0$ such that*

$$M_\infty(P') \geq (1+\varepsilon)\sqrt{1/3}\, n^{3/2}$$

for every conjugate-reciprocal unimodular polynomial $P \in \mathcal{K}_n$ and for all sufficiently large n.

Theorem 3.3. *There is an absolute constant $\varepsilon > 0$ such that*

$$M_\infty(P) \geq (1+\varepsilon)\sqrt{4/3}\, n^{1/2}$$

for every conjugate-reciprocal unimodular polynomial $P \in \mathcal{K}_n$ and for all sufficiently large n.

Theorem 3.4. *There is an absolute constant $\varepsilon > 0$ such that*

$$M_q(P') \leq \exp(\varepsilon(q-2)/q)\sqrt{1/3}\left(\frac{n(n+1)(2n+1)}{6}\right)^{1/2}, \qquad 1 \leq q < 2\,,$$

and

$$M_q(P') \geq \exp(\varepsilon(q-2)/q)\sqrt{1/3}\left(\frac{n(n+1)(2n+1)}{6}\right)^{1/2}, \qquad 2 < q\,,$$

for every conjugate-reciprocal unimodular polynomial $P \in \mathcal{K}_n$ and for all sufficiently large n.

A polynomial $P \in \mathcal{P}_n^c$ is called skew-reciprocal if $P(-z) = P^*(z)$ for all $z \in \mathbb{C}$. A polynomial $P \in \mathcal{P}_n^c$ is called self-reciprocal if $P^* = \overline{P}$, that is, $P(z) = z^n P(1/z)$ for all $z \in \mathbb{C} \setminus \{0\}$.

Problem 3.5. *Is there an absolute constant $\varepsilon > 0$ such that*

$$M_\infty(P') \geq (1+\varepsilon)\sqrt{1/3}\, n^{3/2}$$

holds for all self-reciprocal and skew-reciprocal unimodular polynomials $P \in \mathcal{K}_n$ and for all sufficiently large n?

Problem 3.6. *Is there an absolute constant $\varepsilon > 0$ such that*

$$M_\infty(P') \geq (1+\varepsilon)\sqrt{1/3}\, n^{3/2}$$

or at least

$$\max_{z\in\partial D} |P'(z)| - \min_{z\in\partial D} |P'(z)| \geq \varepsilon n^{3/2}$$

holds for all unimodular polynomials $P \in \mathcal{K}_n$ and for all sufficiently large n?

Our method to prove Theorem 3.2 does not seem to work for all unimodular polynomials $P \in \mathcal{K}_n$. In an e-mail communication several years ago Saffari speculated that the answer to Problem 3.6 is no. However, we know the answer to neither Problem 3.6 nor Problem 3.5.

Let $\mathcal{L}_n$ be the collection of all polynomials of degree n with each of their coefficients in $\{-1, 1\}$.

Problem 3.7. *Is there an absolute constant $\varepsilon > 0$ such that*

$$M_\infty(P') \geq (1+\varepsilon)\sqrt{1/3}\, n^{3/2}$$

or at least

$$\max_{z\in\partial D} |P'(z)| - \min_{z\in\partial D} |P'(z)| \geq \varepsilon n^{3/2}$$

holds for all Littlewood polynomials $P \in \mathcal{L}_n$ and for all sufficiently large n?

The following problem due to Erdős [E-57] is open for a long time.

Problem 3.8. *Is there an absolute constant $\varepsilon > 0$ such that*

$$M_\infty(P) \geq (1+\varepsilon) n^{1/2}$$

or at least

$$\max_{z\in\partial D} |P(z)| - \min_{z\in\partial D} |P(z)| \geq \varepsilon n^{1/2}$$

holds for all Littlewood polynomials $P \in \mathcal{L}_n$ and for all sufficiently large n?

The same problem may be raised only for all skew-reciprocal Littlewood polynomials $P \in \mathcal{L}_n$, and as far as we know, it is also open.

4. Average L_q norm of Littlewood polynomials on the unit circle

P. Borwein and Lockhart [BL-01] investigated the asymptotic behavior of the mean value of normalized M_q norms of Littlewood polynomials for arbitrary $q > 0$. They proved the following result.

Theorem 4.1.

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{f\in\mathcal{L}_n} \frac{(M_q(f))^q}{n^{q/2}} = \Gamma\left(1+\frac{q}{2}\right).$$

In [C-15a] we showed the following.

Theorem 4.2.

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{f\in\mathcal{L}_n} \frac{M_q(f)}{n^{1/2}} = \left(\Gamma\left(1+\frac{q}{2}\right)\right)^{1/q}$$

for every $q > 0$.

In [CE-15] we also proved the following result on the average Mahler measure of Littlewood polynomials.

Theorem 4.3. *We have*

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{f\in\mathcal{L}_n} \frac{M_0(f)}{n^{1/2}} = e^{-\gamma/2} = 0.749306\cdots,$$

where

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) = 0.577215\cdots$$

is the Euler constant.

These last two results are analogues of the results proved earlier by Choi and Mossinghoff [CM-11] for polynomials in $\mathcal{K}_n$.

5. Rudin-Shapiro Polynomials

Finding polynomials with suitably restricted coefficients and maximal Mahler measure has interested many authors. The classes $\mathcal{L}_n$ and $\mathcal{K}_n$ are two of the most important classes considered. Observe that $\mathcal{L}_n \subset \mathcal{K}_n$ and

$$M_0(Q) \le M_2(Q) = \sqrt{n+1}$$

for every $Q \in \mathcal{K}_n$.

It is open whether or not for every $\varepsilon > 0$ there is a sequence (Q_n) of polynomials $Q_n \in \mathcal{L}_n$ such that

$$M_0(Q_n) \ge (1-\varepsilon)\sqrt{n}.$$

Beller and Newman [B-73] constructed a sequence (Q_n) of unimodular polynomials $Q_n \in \mathcal{K}_n$ such that
$$M_0(Q_n) \geq \sqrt{n} - \frac{c}{\log n}\,.$$
Littlewood asked if there were $Q_{n_k} \in \mathcal{L}_{n_k}$ satisfying
$$c_1\sqrt{n_k+1} \leq |Q_{n_k}(z)| \leq c_2\sqrt{n_k+1}\,, \qquad z \in \partial D\,,$$
with some absolute constants $c_1 > 0$ and $c_2 > 0$, see [B-02, p. 27] for a reference to this problem of Littlewood. No sequence (Q_{n_k}) of Littlewood polynomials $Q_{n_k} \in \mathcal{L}_{n_k}$ is known that satisfies the lower bound. A sequence of Littlewood polynomials that satisfies just the upper bound is given by the Rudin-Shapiro polynomials. The Rudin-Shapiro polynomials appear in Harold Shapiro's 1951 thesis [S-51] at MIT and are sometimes called just Shapiro polynomials. They also arise independently in Golay's paper [G-51]. They are remarkably simple to construct and are a rich source of counterexamples to possible conjectures. The Rudin-Shapiro polynomials are defined recursively as follows:
$$\begin{aligned}
P_0(z) &:= 1\,, \qquad Q_0(z) := 1\,,\\
P_{k+1}(z) &:= P_k(z) + z^{2^k}Q_k(z)\,,\\
Q_{k+1}(z) &:= P_k(z) - z^{2^k}Q_k(z)\,, \qquad k = 0,1,2,\ldots\,.
\end{aligned}$$
Note that both P_k and Q_k are polynomials of degree $n-1$ with $n := 2^k$ having each of their coefficients in $\{-1,1\}$. In signal processing, the Rudin-Shapiro polynomials have good autocorrelation properties and their values on the unit circle are small. Binary sequences with low autocorrelation coefficients are of interest in radar, sonar, and communication systems. It is well known and easy to check by using the parallelogram law that
$$|P_{k+1}(z)|^2 + |Q_{k+1}(z)|^2 = 2(|P_k(z)|^2 + |Q_k(z)|^2)\,, \qquad z \in \partial D\,.$$
Hence
$$|P_k(z)|^2 + |Q_k(z)|^2 = 2^{k+1} = 2n\,, \qquad z \in \partial D\,.$$
It is also well known (see Section 4 of [B-02], for instance), that
$$Q_k(-z) = P_k^*(z) = z^{n-1}P_k(1/z)\,, \qquad z \in \partial D\,,$$
and hence
$$|Q_k(-z)| = |P_k(z)|\,, \qquad z \in \partial D\,.$$
P. Borwein's book [B-02] presents a few more basic results on the Rudin-Shapiro polynomials. Various properties of the Rudin-Shapiro polynomials are discussed in [B-73] by Brillhart and in [BL-76] by Brillhart, Lemont, and Morton.

As for $k \geq 1$ both P_k and Q_k have odd degree, both P_k and Q_k have at least one real zero. The fact that for $k \geq 1$ both P_k and Q_k have exactly one real zero was proved by

Brillhart in [B-73]. Another interesting observation made in [BL-76] is the fact that P_k and Q_k cannot vanish at root of unity different from -1 and 1.

Obviously

$$M_2(P_k) = 2^{k/2}$$

by the Parseval formula. In 1968 Littlewood [L-68] evaluated $M_4(P_k)$ and found that

$$M_4(P_k) \sim \left(\frac{4^{k+1}}{3}\right)^{1/4} = \left(\frac{4n^2}{3}\right)^{1/4}. \tag{5.1}$$

The M_4 norm of Rudin-Shapiro like polynomials on ∂D are studied in [BM-00].

The merit factor of a Littlewood polynomial $f \in \mathcal{L}_{n-1}$ is defined by

$$\mathrm{MF}(f) = \frac{M_2(f)^4}{M_4(f)^4 - M_2(f)^4} = \frac{n^2}{M_4(f)^4 - n^2}.$$

Observe that (5.1) implies that $\mathrm{MF}(P_k) \sim 3$.

6. Mahler measure and moments of the Rudin-Shapiro polynomials

Despite the simplicity of their definition not much is known about the Rudin-Shapiro polynomials. It has been shown in [E-16c] fairly recently that the Mahler measure (M_0 norm) and the M_∞ norm of the Rudin-Shapiro polynomials P_k and Q_k of degree $n-1$ with $n := 2^k$ on the unit circle of the complex plane have the same size, that is, the Mahler measure of the Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$ is bounded from below by $cn^{1/2}$, where $c > 0$ is an absolute constant.

Theorem 6.1. *Let P_k and Q_k be the k-th Rudin-Shapiro polynomials of degree $n-1$ with $n = 2^k$. There is an absolute constant $c_1 > 0$ such that*

$$M_0(P_k) = M_0(Q_k) \geq c_1\sqrt{n}\,, \ldots k = 1, 2, \ldots.$$

The following asymptotic formula, conjectured by Saffari in 1985, for the Mahler measure of the Rudin-Shapiro polynomials has been proved recently in [E-19a].

Theorem 6.2. *We have*

$$\lim_{n\to\infty} \frac{M_0(P_k)}{n^{1/2}} = \lim_{n\to\infty} \frac{M_0(Q_k)}{n^{1/2}} = \left(\frac{2}{e}\right)^{1/2} = 0.857763\cdots.$$

To formulate our next theorem we define

$$\widetilde{P}_k := 2^{-(k+1)/2}P_k \quad \text{and} \quad \widetilde{Q}_k := 2^{-(k+1)/2}Q_k\,.$$

By using the above normalization, we have

$$|\widetilde{P}_k(z)|^2 + |\widetilde{Q}_k(z)|^2 = 1\,, \qquad z \in \partial D\,.$$

For $q > 0$ let

$$I_q(\widetilde{P}_k) := \left(M_q(\widetilde{P}_k)\right)^q := \frac{1}{2\pi}\int_0^{2\pi} |\widetilde{P}_k(e^{it})|^q\,dt\,.$$

The following result is a simple consequence of Theorem 6.1.

Theorem 6.3. *There exists a constant $L < \infty$ independent of k such that*

$$\sum_{m=1}^{\infty} \frac{I_m(\widetilde{P}_k)}{m} < L\,, \qquad k = 0, 1, \ldots\,.$$

Consequently

$$I_m(\widetilde{P}_k) \leq \frac{L}{\log(m+1)}\,, \qquad k = 1, 2, \ldots\,, \quad m = 1, 2, \ldots\,.$$

In [E-16c] we also proved the following result.

Theorem 6.4. *There exists an absolute constant $c_2 > 0$ such that*

$$M_0(P_k, [\alpha, \beta]) \geq c_2\sqrt{n}\,, \qquad k = 1, 2, \ldots\,,$$

with $n := 2^k$ for all $\alpha, \beta \in \mathbb{R}$ such that

$$\frac{12\pi}{n} \leq \frac{(\log n)^{3/2}}{n^{1/2}} \leq \beta - \alpha \leq 2\pi\,.$$

7. Lemmas for Theorem 6.1

As the proof of Theorem 6.1 is based on interesting new properties of the Rudin-Shapiro polynomials which have been observed only recently in [E-16c], we list them in this section.

Lemma 7.1. *Let $k \geq 2$ be an integer, $n := 2^k$, and let*

$$z_j := e^{it_j}\,, \quad t_j := \frac{2\pi j}{n}\,, \quad j \in \mathbb{Z}\,.$$

We have

$$P_k(z_j) = 2P_{k-2}(z_j)$$

whenever j is even, and

$$P_k(z_j) = (-1)^{(j-1)/2} 2i\, Q_{k-2}(z_j)$$

whenever j is odd, where i is the imaginary unit.

Lemma 7.2. *If P_k and Q_k are the k-th Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$,*

$$\omega := \sin^2(\pi/8) = 0.146446\cdots\,,$$

and

$$z_j := e^{it_j}\,, \quad t_j := \frac{2\pi j}{n}\,, \qquad j \in \mathbb{Z}\,,$$

then

$$\max\{|P_k(z_j)|^2, |P_k(z_{j+1})|^2\} \geq \omega 2^{k+1} = 2\omega n\,, \qquad j \in \mathbb{Z}\,.$$

Lemma 7.3. *Let $n, m \geq 1$,*

$$0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m \leq 2\pi\,, \qquad \tau_0 := \tau_m - 2\pi\,, \quad \tau_{m+1} := \tau_1 + 2\pi\,,$$

$$\delta := \max\{\tau_1 - \tau_0, \tau_2 - \tau_1, \dots, \tau_m - \tau_{m-1}\}\,.$$

For every $A > 0$ there is a $B > 0$ depending only on A such that

$$\sum_{j=1}^{m} \frac{\tau_{j+1} - \tau_{j-1}}{2} \log |P(e^{i\tau_j})| \leq \int_0^{2\pi} \log |P(e^{i\tau})|\, d\tau + B$$

for all $P \in \mathcal{P}_n^c$ and $\delta \leq An^{-1}$.

8. Saffari's Conjecture on the Shapiro Polynomials

In 1980 Saffari conjectured the following.

Conjecture 8.1. *Let P_k and Q_k be the Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$. We have*

$$M_q(P_k) = M_q(Q_k) \sim \frac{(2n)^{1/2}}{(q/2+1)^{1/q}}$$

for all real exponents $q > 0$.

Conjecture 8.1*. *Equivalently to Conjecture 8.1, we have*

$$\begin{aligned}
&\lim_{k\to\infty} m\left(\left\{t \in [0,2\pi) : \left|\frac{P_k(e^{it})}{\sqrt{2^{k+1}}}\right|^2 \in [\alpha,\beta]\right\}\right) \\
&= \lim_{k\to\infty} m\left(\left\{t \in [0,2\pi) : \left|\frac{Q_k(e^{it})}{\sqrt{2^{k+1}}}\right|^2 \in [\alpha,\beta]\right\}\right) = 2\pi(\beta - \alpha)
\end{aligned}$$

whenever $0 \leq \alpha < \beta \leq 1$.

This conjecture was proved for all even values of $q \leq 52$ by Doche [D-05] and Doche and Habsieger [DH-04]. Recently B. Rodgers [R-16] proved Saffari's Conjecture 8.1 for all $q > 0$. See also [EZ-17].

An extension of Saffari's conjecture is Montgomery's conjecture below.

Conjecture 8.2. *Let P_k and Q_k be the Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$. We have*

$$\begin{aligned}
&\lim_{k\to\infty} m\left(\left\{t \in [0,2\pi) : \frac{P_k(e^{it})}{\sqrt{2^{k+1}}} \in E\right\}\right) \\
&= \lim_{k\to\infty} m\left(\left\{t \in [0,2\pi) : \frac{Q_k(e^{it})}{\sqrt{2^{k+1}}} \in E\right\}\right) = 2\mu(E)\,,
\end{aligned}$$

where $\mu(E)$ denotes the Jordan measure of a Jordan measurable set $E \subset D$.

B. Rodgers [R-16] proved Montgomery's Conjecture 8.2 as well.

9. Consequences of Saffari's Conjecture

Let P_k and Q_k be the Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$,

$$R_k(t) := |P_k(e^{it})|^2 \quad \text{or} \quad R_k(t) := |Q_k(e^{it})|^2\,,$$

$$\omega := \sin^2(\pi/8) = 0.146446\cdots\,.$$

In [E-19b] we proved Theorems 9.1–9.5 below.

Theorem 9.1. *P_k and Q_k have $o(n)$ zeros on the unit circle.*

The proof of Theorem 9.1 follows by combining the recently proved Saffari's conjecture stated as Conjecture 8.1* and the theorem below.

Theorem 9.2. *If the real trigonometric polynomial R of degree n is of the form*

$$R(t) = |P(e^{it})|^2\,,$$

where $P \in \mathcal{P}_n^c$, and P has at least k zeros in K (counting multiplicities), then

$$m(\{t \in [0,2\pi) : |R(t)| \leq \alpha \|R\|_K\}) \geq \frac{\sqrt{\alpha}}{e}\,\frac{k}{n}$$

for every $\alpha \in (0,1)$.

Theorem 9.3. *There exists an absolute constant $c > 0$ such that each of the functions $\mathrm{Re}(P_k)$, $\mathrm{Re}(Q_k)$, $\mathrm{Im}(P_k)$, and $\mathrm{Im}(Q_k)$ has at least cn zeros on the unit circle.*

Theorem 9.4. *There exists an absolute constant $c > 0$ such that the equation $R_k(t) = \eta n$ has at most $c\eta^{1/2}$ solutions in $[0,2\pi)$ for every $\eta \in (0,1]$ and sufficiently large $k \geq k_\eta$, while the equation $R_k(t) = \eta n$ has at most $c(2-\eta)^{1/2}n$ solutions in $[0,2\pi)$ for every $\eta \in [1,2)$ and sufficiently large $k \geq k_\eta$.*

Theorem 9.5. *The equation $R_k(t) = \eta n$ has at least $(1-\varepsilon)\eta n/2$ solutions in $[0,2\pi)$ for every $\eta \in (0,2\omega)$, $\varepsilon > 0$, and sufficiently large $k \geq k_{\eta,\varepsilon}$. The equation $R_k(t) = \eta n$ has at least $(1-\varepsilon)(2-\eta)n/2$ solutions in $[0,2\pi)$ for every $\eta \in (2-2\omega, 2)$, $\varepsilon > 0$, and sufficiently large $k \geq k_{\eta,\varepsilon}$.*

Theorem 9.6. *There exists an absolute constant $c > 0$ such that the equation $R_k(t) = (1+\eta)n$ has at least $cn^{0.5394282}$ distinct solutions in $[0,2\pi)$ whenever $\eta \in \mathbb{R}$ with $|\eta| < 2^{-8}$.*

In [A-19] we combined close to sharp upper bounds for the modulus of the autocorrelation coefficients of the Rudin-Shapiro polynomials with a deep theorem of Littlewood (see Theorem 1 in [L-66a]) to prove the above Theorem 9.6.

Theorem 9.7. *If*

$$|P_k(z)|^2 = P_k(z)P_k(1/z) = \sum_{j=-n+1}^{n-1} a_j z^j\,, \qquad z \in \partial D\,,$$

then

$$c_1 n^{0.7302852\cdots} \le \max_{1\le j\le n-1} |a_j| \le c_2 n^{0.7302859\cdots}$$

with an absolute constants $c_1 > 0$ *and* $c_2 > 0$.

Theorem 9.7 has been recently improved by Choi in [C-19] by showing that

$$(0.27771487\cdots)(1+o(1))\,|\lambda|^k \le \max_{1\le j\le n-1} |a_j| \le (3.78207844\cdots)\,|\lambda|^k\ ,$$

where

$$\lambda := -\frac{\left(44+3\sqrt{177}\right)^{1/3} + \left(44-3\sqrt{177}\right)^{1/3} - 1}{3} = -1.658967081916\cdots$$

is the real root ot the equation $x^3 - x^2 - 2x + 4 = 0$ and $|\lambda|^k = n^{07302852598\cdots}$. This settles a conjecture expected earlier by Saffari.

Theorem 9.8 (Littlewood). *If the real trigonometric polynomial of degree at most* n *is of the form*

$$f(t) = \sum_{m=0}^{n} a_m \cos(mt + \alpha_m)\,, \qquad a_m, \alpha_m \in \mathbb{R}\,,$$

satisfies $M_1(f) \ge c\mu$, $\mu := M_2(f)$, *where* $c > 0$ *is a constant,* $a_0 = 0$,

$$s_{\lfloor n/h \rfloor} = \sum_{m=1}^{\lfloor n/h \rfloor} \frac{a_m^2}{\mu^2} \le 2^{-9} c^6$$

for some constant $h > 0$, *and* $v \in \mathbb{R}$ *satisfies*

$$|v| \le V = 2^{-5} c^3\,,$$

then

$$\mathcal{N}(f, v) > A h^{-1} c^5 n\,,$$

where $\mathcal{N}(f, v)$ *denotes the number of real zeros of* $f - v\mu$ *in* $(-\pi, \pi)$, *and* $A > 0$ *is an absolute constant.*

In [E-19c] we improved Theorem 9.6 by showing the following two results.

Theorem 9.9. *The equation* $R_k(t) = n$ *has at least* $n/4+1$ *distinct zeros in* $[0, 2\pi)$. *Moreover, with the notation* $t_j := 2\pi j/n$, *there are at least* $n/2+2$ *values of* $j \in \{0, 1 \ldots, n-1\}$ *for which the interval* $[t_j, t_{j+1}]$ *has at least one zero of the equation* $R_k(t) = n$.

Theorem 9.10. *The equation* $R_k(t) = (1+\eta)n$ *has at least* $(1/2 - |\eta| - \varepsilon)n/2$ *distinct zeros in* $[0, 2\pi)$ *for every* $\eta \in (-1/2, 1/2)$, $\varepsilon > 0$, *and sufficiently large* $k \ge k_{\eta,\varepsilon}$.

In [E-19b] we proved the theorem below.

Theorem 9.11. *There exist absolute constants $c_1 > 0$ and $c_2 > 0$ such that both P_k and Q_k have at least $c_2 n$ zeros in the annulus*

$$\left\{ z \in \mathbb{C} : 1 - \frac{c_1}{n} < |z| < 1 + \frac{c_1}{n} \right\}.$$

A key to the proof of Theorem 9.11 is the result below.

Theorem 9.12. *Let $t_0 \in K$. There is an absolute constant $c_3 > 0$ depending only on $c > 0$ such that P_k has at least one zero in the disk*

$$\left\{ z \in \mathbb{C} : |z - e^{it_0}| < \frac{c_3}{n} \right\},$$

whenever

$$T_k'(t_0) \geq cn^2, \qquad T_k(t) = P_k(e^{it}) P_k(e^{-it}).$$

We note that for every $c \in (0,1)$ there is an absolute constant $c_4 > 0$ depending only on c such that every $U_n \in \mathcal{P}_n^c$ of the form

$$U_n(z) = \sum_{j=0}^{n} a_j z^j, \qquad |a_0| = |a_n| = 1, \quad a_j \in \mathbb{C}, \quad |a_j| \leq 1,$$

has at least cn zeros in the annulus

$$\left\{ z \in \mathbb{C} : 1 - \frac{c_4 \log n}{n} < |z| < 1 + \frac{c_4 \log n}{n} \right\}.$$

See Theorem 2.1 in [E-01c].

On the other hand, there is an absolute constant $c_4 > 0$ such that for every $n \in \mathbb{N}$ there is a polynomial $U_n \in \mathcal{K}_n$ having no zeros in the above annulus. See Theorem 2.3 in [E-01c].

So in Theorem 9.11 some special properties, in addition to being Littlewood polynomials, of the Rudin-Shapiro polynomials must be exploited.

10. Open Problems related to the Rudin-Shapiro Polynomials

Problem 10.1. *Is there an absolute constant $c > 0$ such that the equation $R_k(t) = \eta n$ has at least $c\eta n$ distinct solutions in K for every $\eta \in (0,1)$ and sufficiently large $k \geq k_\eta$? In other words, can Theorem 9.5 be extended to all $\eta \in (0,1)$?*

Problem 10.2. *Is there an absolute constant $c > 0$ such that P_k has at least cn zeros in the open unit disk?*

Problem 10.3. *Is there an absolute constant $c > 0$ such that Q_k has at least cn zeros in the open unit disk?*

Recall that

$$Q_k(-z) = P_k^*(z) = z^{n-1} P_k(1/z), \qquad z \in \partial D.$$

Problem 10.4. *Is it true that both P_k and Q_k have asymptotically half of their zeros in the open unit disk?*

Observe that $P_k(-1) = Q_k(1) = 0$ if k is odd.

Problem 10.5. *Is it true that if k is odd then P_k has a zero on the unit circle ∂D only at -1 and Q_k has a zero on the unit circle ∂D only at 1, while if k is even then neither P_k nor Q_k has a zero on the unit circle?*

11. On the size of the Fekete polynomials on the unit circle

For a prime p the p-th Fekete polynomial is defined as

$$f_p(z) := \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) z^k ,$$

where

$$\left(\frac{k}{p}\right) = \begin{cases} 1, & \text{if } \; x^2 \equiv k \pmod{p} \text{ for an } x \neq 0 , \\ 0, & \text{if } \; p \text{ divides } k , \\ -1, & \text{otherwise} \end{cases}$$

is the usual Legendre symbol. Note that $g_p(z) := f_p(z)/z$ is a Littlewood polynomial of degree $p-2$, and has the same Mahler measure as f_p.

In 1980 Montgomery [M-80] proved the following fundamental result.

Theorem 11.1. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 \sqrt{p} \log\log p \leq \max_{z \in \partial D} |f_p(z)| \leq c_2 \sqrt{p} \log p .$$

It was observed in [E-12] that Montgomery's approach can be used to prove that for every sufficiently large prime p and for every $8\pi p^{-1/8} \leq s \leq 2\pi$ there is a closed subset $E := E_{p,s}$ of the unit circle with linear measure $|E| = s$ such that

$$\frac{1}{|E|} \int_E |f_p(z)| \, |dz| \geq c_1 \, p^{1/2} \log\log(1/s)$$

with an absolute constant $c_1 > 0$.

In [BC-02] the L_4 norm of the Fekete polynomials are computed.

Theorem 11.2. *We have*

$$M_4(f_p) = \left(\frac{5p^2}{3} - 3p + \frac{4}{3} - 12(h(-p))^2\right)^{1/4} ,$$

where $h(-q)$ is the class number of $\mathbb{Q}(\sqrt{-q})$.

In [E-07] we proved the following result.

Theorem 11.3. *For every $\varepsilon > 0$ there is a constant c_ε such that*

$$M_0(f_p) \geq \left(\frac{1}{2} - \varepsilon\right)\sqrt{p}$$

for all primes $p \geq c_\varepsilon$.

From Jensen's inequality,

$$M_0(f_p) \leq M_2(f_p) = \sqrt{p-1}\,.$$

However, as it was observed in [E-07] and [E-18], a result of Littlewood [L-66a] implies that $\frac{1}{2} - \varepsilon$ in Theorem 11.2 cannot be replaced by $1 - \varepsilon$.

To prove Theorem 11.3 in [E-07] we needed to combine Theorems 11.4, 11.5 and one of Theorems 11.6 and 11.7 below. For a prime number p let

$$\zeta_p := \exp\left(\frac{2\pi i}{p}\right)$$

the first p-th root of unity. Our first lemma formulates a characteristic property of the Fekete polynomials. A simple proof is given in [B-02, pp. 37-38].

Theorem 11.4 (Gauss). *We have*

$$f_p(\zeta_p^j) = \sqrt{\left(\frac{-1}{p}\right)p}\,, \qquad j = 1, 2, \dots, p-1\,,$$

and $f_p(1) = 0$.

Theorem 11.5. *We have*

$$\left(\prod_{j=0}^{p-1} |Q(\zeta_p^j)|\right)^{1/p} \leq 2M_0(Q)$$

for all polynomials Q of degree at most p with complex coefficients.

Theorem 11.6. *There is an absolute constant $c > 0$ such that every $Q \in \mathcal{K}_n$ has at most $c\sqrt{n}$ real zeros.*

Theorem 11.7. *There is an absolute constant $c > 0$ such that every $Q \in \mathcal{L}_n$ has at most $\dfrac{c\log^2 n}{\log\log n}$ zeros at 1.*

For a proof of Theorem 11.6 see [BE-97]. For a proof of Theorem 11.7 see [B-97].

In [E-11] Theorem 11.3 was extended to subarcs of the unit circle.

Theorem 11.8. *There exists an absolute constant $c_1 > 0$ such that*

$$M_0(f_p, [\alpha, \beta]) \geq c_1 p^{1/2}$$

for all primes p and for all $\alpha, \beta \in \mathbb{R}$ such that $(\log p)^{3/2} p^{-1/2} \leq \beta - \alpha \leq 2\pi$.

In [E-12] we gave an upper bound for the average value of $|f_p(z)|^q$ over any subarc I of the unit circle, valid for all sufficiently large primes p and all exponents $q > 0$.

Theorem 11.9. *There exists a constant $c_2(q, \varepsilon)$ depending only on $q > 0$ and $\varepsilon > 0$ such that*

$$M_q(f_p, [\alpha, \beta]) \leq c_2(q, \varepsilon) p^{1/2} \,,$$

for all primes p and for all $\alpha, \beta \in \mathbb{R}$ such that $2p^{-1/2+\varepsilon} \leq \beta - \alpha \leq 2\pi$.

We remark that a combination of Theorems 11.8 and 11.9 shows that there is an absolute constant $c_1 > 0$ and a constant $c_2(q, \varepsilon) > 0$ depending only on $q > 0$ and $\varepsilon > 0$ such that

$$c_1 p^{1/2} \leq M_q(f_p, [\alpha, \beta]) \leq c_2(q, \varepsilon) p^{1/2}$$

for all primes p and for all $\alpha, \beta \in \mathbb{R}$ such that $(\log p)^{3/2} p^{-1/2} \leq 2p^{-1/2+\varepsilon} \leq \beta - \alpha \leq 2\pi$.

The L_q norm of polynomials related to Fekete polynomials were studied in several recent papers. See [B-01a], [B-02], [BC-02], [BC-04], [G-16], [J-13a], and [J-13b], for example. An interesting extremal property of the Fekete polynomials is proved in [BC-01b].

Fekete might have been the first one to study analytic properties of the Fekete polynomials. He had an idea of proving non-existence of Siegel zeros (that is, real zeros "especially close to 1") of Dirichlet L-functions from the positivity of Fekete polynomials on the interval $(0, 1)$, where the positivity of Fekete polynomials is often referred to as the Fekete Hypothesis.

There were many mathematicians trying to understand the zeros of Fekete polynomials including Fekete and Pólya [F-12], Pólya [P-19], Chowla [C-35], Bateman, Purdy, and Wagstaff [B-75], Heilbronn [H-37], Montgomery [M-80], Baker and Montgomery [B-90], and Jung and Shen [J-16].

Baker and Montgomery [B-90] proved that f_p has a large number of zeros in $(0, 1)$ for almost all primes p, that is, the number of zeros of f_p in $(0, 1)$ tends to ∞ as p tends to ∞, and it seems likely that there are, in fact, about $\log \log p$ such zeros.

Conrey, Granville, Poonen, and Soundararajan [C-00] showed that f_p has asymptotically $\kappa_0 p$ zeros on the unit circle, where

$$0.500668 < \kappa_0 < 0.500813 \,.$$

An interesting recent paper [B-17] studies power series approximations to Fekete polynomials. In [E-18] we improved Theorem 11.2 by showing the following result.

Theorem 11.10. *There is an absolute constant $c > 1/2$ such that*

$$M_0(f_p) \geq c\sqrt{p}$$

for all sufficiently large primes.

However, there is not even a conjecture in the literature about what the asymptotically sharp constant c in Theorem 11.10 could be.

12. Unimodular zeros of self-reciprocal polynomials with coefficients in a finite set

Research on the distribution of the zeros of algebraic polynomials has a long and rich history. A few papers closely related this section include [BC-15], [BP-32], [BE-01], [BE-99], [BE-13], [BE-08a], [BE-08b], [D-99], [D-14], [E-08a], [E-08b], [E-16a], [E-02], [E-50], [L-61], [L-64], [N-16], [O-93], [P-99], [P-14], [S-19b], [Sch-32], [Sch-33], [Sz-34], [T-15], [T-07], and [T-93]. The number of real zeros trigonometric polynomials and the number of unimodular zeros (that is, zeros lying on the unit circle of the complex plane) of algebraic polynomials with various constraints on their coefficients are the subject of quite a few of these. We do not try to survey these in this section.

Let $S \subset \mathbb{C}$. Let $\mathcal{P}_n(S)$ be the set of all algebraic polynomials of degree at most n with each of their coefficients in S. An algebraic polynomial P of the form

$$P(z) = \sum_{j=0}^{n} a_j z^j \,, \qquad a_j \in \mathbb{C}\,, \tag{12.1}$$

is called conjugate-reciprocal if

$$\overline{a}_j = a_{n-j}\,, \qquad j = 0, 1, \ldots, n\,. \tag{12.2}$$

Functions T of the form

$$T(t) = \alpha_0 + \sum_{j=1}^{n} (\alpha_j \cos(jt) + \beta_j \sin(jt))\,, \qquad \alpha_j, \beta_j \in \mathbb{R}\,,$$

are called real trigonometric polynomials of degree at most n. It is easy to see that any real trigonometric polynomial T of degree at most n can be written as $T(t) = P(e^{it})e^{-int}$, where P is a conjugate-reciprocal algebraic polynomial of the form

$$P(z) = \sum_{j=0}^{2n} a_j z^j \,, \qquad a_j \in \mathbb{C}\,. \tag{12.3}$$

Conversely, if P is conjugate-reciprocal algebraic polynomial of the form (12.3), then there are $\theta_j \in \mathbb{R}$, $j = 1, 2, \ldots n$, such that

$$T(t) := P(e^{it})e^{-int} = a_n + \sum_{j=1}^{n} 2|a_{j+n}| \cos(jt + \theta_j)$$

is a real trigonometric polynomial of degree at most n. A polynomial P of the form (12.1) is called self-reciprocal if

$$a_j = a_{n-j}, \qquad j = 0, 1, \ldots, n\,. \tag{12.4}$$

If a conjugate-reciprocal algebraic polynomial P has only real coefficients, then it is obviously self-reciprocal. If the algebraic polynomial P of the form (12.3) is self-reciprocal, then

$$T(t) := P(e^{it})e^{-int} = a_n + \sum_{j=1}^{n} 2a_{j+n} \cos(jt)\,.$$

In this section, whenever we write "$P \in \mathcal{P}_n(S)$ is conjugate-reciprocal" we mean that P is of the form (12.1) with each $a_j \in S$ satisfying (12.2). Similarly, whenever we write "$P \in \mathcal{P}_n(S)$ is self-reciprocal" we mean that P is of the form (12.1) with each $a_j \in S$ satisfying (12.4). This is going to be our understanding even if the degree of $P \in \mathcal{P}_n(S)$ is less than n. It is easy to see that $P \in \mathcal{P}_n(S)$ is self-reciprocal and n is odd, then $P(-1) = 0$. We call any subinterval $[a, a+2\pi)$ of the real number line $\mathbb{R}$ a period. Associated with an algebraic polynomial P of the form (12.1) we introduce the number

$$\mathrm{NC}(P) := |\{j \in \{0,1,\dots,n\} : a_j \neq 0\}|\,.$$

Here, and in what follows $|A|$ denotes the number of elements of a finite set A. Let $\mathrm{NZ}(P)$ denote the number of real zeros (by counting multiplicities) of an algebraic polynomial P on the unit circle. Associated with an even trigonometric polynomial (cosine polynomial) of the form

$$T(t) = \sum_{j=0}^{n} a_j \cos(jt)$$

we introduce the number

$$\mathrm{NC}(T) := |\{j \in \{0,1,\dots,n\} : a_j \neq 0\}|\,.$$

Let $\mathrm{NZ}(T)$ denote the number of real zeros (by counting multiplicities) of a trigonometric polynomial T in a period. Let $\mathrm{NZ}^*(T)$ denote the number of sign changes of a trigonometric polynomial T in a period. The quotation below is from [BE-08a].

"Let $0 \le n_1 < n_2 < \cdots < n_N$ be integers. A cosine polynomial of the form $T(\theta) = \sum_{j=1}^{N} \cos(n_j\theta)$ must have at least one real zero in a period. This is obvious if $n_1 \neq 0$, since then the integral of the sum on a period is 0. The above statement is less obvious if $n_1 = 0$, but for sufficiently large N it follows from Littlewood's Conjecture simply. Here we mean the Littlewood's Conjecture proved by S. Konyagin [K-81] and independently by McGehee, Pigno, and Smith [Mc-81] in 1981. See also [D-93, pages 285–288] for a book proof. It is not difficult to prove the statement in general even in the case $n_1 = 0$ without using Littlewood's Conjecture. One possible way is to use the identity

$$\sum_{j=1}^{n_N} T\left(\frac{(2j-1)\pi}{n_N}\right) = 0\,.$$

See [K-04], for example. Another way is to use Theorem 2 of [M-06a]. So there is certainly no shortage of possible approaches to prove the starting observation of this section even in the case $n_1 = 0$.

It seems likely that the number of zeros of the above sums in a period must tend to ∞ with N. In a private communication B. Conrey asked how fast the number of real zeros of the above sums in a period tends to ∞ as a function N. In [C-00] the authors observed that for an odd prime p the Fekete polynomial

$$f_p(z) = \sum_{k=0}^{p-1} \left(\frac{k}{p}\right) z^k$$

(the coefficients are Legendre symbols) has $\sim \kappa_0 p$ zeros on the unit circle, where $0.500813 > \kappa_0 > 0.500668$. Conrey's question in general does not appear to be easy. Littlewood in his 1968 monograph 'Some Problems in Real and Complex Analysis' [L-68, problem 22] poses the following research problem, which appears to still be open: 'If the n_m are integral and all different, what is the lower bound on the number of real zeros of $\sum_{m=1}^N \cos(n_m\theta)$? Possibly $N-1$, or not much less.' Here real zeros are counted in a period. In fact no progress appears to have been made on this in the last half century. In a recent paper [BE-08a] we showed that this is false. There exist cosine polynomials $\sum_{m=1}^N \cos(n_m\theta)$ with the n_m integral and all different so that the number of its real zeros in a period is $O(N^{9/10}(\log N)^{1/5})$ (here the frequencies $n_m = n_m(N)$ may vary with N). However, there are reasons to believe that a cosine polynomial $\sum_{m=1}^N \cos(n_m\theta)$ always has many zeros in a period."

Let, as before,

$$\mathcal{L}_n := \left\{ P : \ P(z) = \sum_{j=0}^n a_j z^j \, , \ \ a_j \in \{-1,1\} \right\} .$$

Elements of $\mathcal{L}_n$ are often called Littlewood polynomials of degree n. Let

$$\mathcal{H}_n := \left\{ P : \ P(z) = \sum_{j=0}^n a_j z^j \, , \ \ a_j \in \mathbb{C}, \ \ |a_0| = |a_n| = 1, \ \ |a_j| \le 1 \right\} .$$

Observe that $\mathcal{L}_n \subset \mathcal{H}_n$. In [BE-08b] we proved that any polynomial $P \in \mathcal{K}_n$ has at least $8n^{1/2}\log n$ zeros in any open disk centered at a point on the unit circle with radius $33n^{-1/2}\log n$. Thus polynomials in $\mathcal{H}_n$ have quite a few zeros near the unit circle. One may naturally ask how many unimodular roots a polynomial in $\mathcal{H}_n$ can have. Mercer [M-06a] proved that if a Littlewood polynomial $P \in \mathcal{L}_n$ of the form (12.1) is skew-reciprocal, that is, $a_j = (-1)^j a_{n-j}$ for each $j = 0, 1, \ldots, n$, then it has no zeros on the unit circle. However, by using different elementary methods it was observed in both [E-02] and [M-06a] that if a Littlewood polynomial P of the form (12.1) is self-reciprocal, then it has at least one zero on the unit circle. Mukunda [M-06b] improved this result by showing that every self-reciprocal Littlewood polynomial of odd degree has at least 3 zeros on the unit circle. Drungilas [D-08] proved that every self-reciprocal Littlewood polynomial of odd degree $n \ge 7$ has at least 5 zeros on the unit circle and every self-reciprocal Littlewood polynomial of even degree $n \ge 14$ has at least 4 zeros on the unit circle. In [BE-08a] we proved that the average number of zeros of self-reciprocal Littlewood polynomials of degree n is at least $n/4$. However, it is much harder to give decent lower bounds for the quantities $\mathrm{NZ}_n := \min_P \mathrm{NZ}(P)$, where $\mathrm{NZ}(P)$ denotes the number of zeros of a polynomial P lying on the unit circle and the minimum is taken for all self-reciprocal Littlewood polynomials $P \in \mathcal{L}_n$. It has been conjectured for a long time that $\lim_{n\to\infty} \mathrm{NZ}_n = \infty$. In [E-16b] we showed that $\lim_{n\to\infty} \mathrm{NZ}(P_n) = \infty$ whenever $P_n \in \mathcal{L}_n$ is self-reciprocal and $\lim_{n\to\infty} |P_n(1)| = \infty$. This follows as a consequence of a more general result, see Corollary 2.3 in [E-16b], stated as Corollary 12.5 here, in which the coefficients of the self-reciprocal polynomials P_n of degree at most n belong to a fixed finite set of real numbers. In [BE-07] we proved the following result.

Theorem 12.1. *If the set $\{a_j : j \in \mathbb{N}\} \subset \mathbb{R}$ is finite, the set $\{j \in \mathbb{N} : a_j \neq 0\}$ is infinite, the sequence (a_j) is not eventually periodic, and*

$$T_n(t) = \sum_{j=0}^{n} a_j \cos(jt)\,,$$

then $\lim_{n\to\infty} \mathrm{NZ}(T_n) = \infty$.

In [BE-07] Theorem 12.1 is stated without the assumption that the sequence (a_j) is not eventually periodic. However, as the following example shows, Lemma 3.4 in [BE-07], dealing with the case of eventually periodic sequences (a_j), is incorrect. Let

$$\begin{aligned} T_n(t) :=& \cos t + \cos((4n+1)t) + \sum_{k=0}^{n-1} (\cos((4k+1)t) - \cos((4k+3)t)) \\ =& \frac{1+\cos((4n+2)t)}{2\cos t} + \cos t\,. \end{aligned}$$

It is easy to see that $T_n(t) \neq 0$ on $[-\pi, \pi] \setminus \{-\pi/2, \pi/2\}$ and the zeros of T_n at $-\pi/2$ and $\pi/2$ are simple. Hence T_n has only two (simple) zeros in a period. So the conclusion of Theorem 12.1 above is false for the sequence (a_j) with $a_0 := 0$, $a_1 := 2$, $a_3 := -1$, $a_{2k} := 0$, $a_{4k+1} := 1$, $a_{4k+3} := -1$ for every $k = 1, 2, \ldots$. Nevertheless, Theorem 12.1 can be saved even in the case of eventually periodic sequences (a_j) if we assume that $a_j \neq 0$ for all sufficiently large j. See Lemma 3.11 in [E-16b] where Theorem 1 in [BE-07] is corrected as

Theorem 12.2. *If the set $\{a_j : j \in \mathbb{N}\} \subset \mathbb{R}$ is finite, $a_j \neq 0$ for all sufficiently large j, and*

$$T_n(t) = \sum_{j=0}^{n} a_j \cos(jt)\,,$$

then $\lim_{n\to\infty} \mathrm{NZ}(T_n) = \infty$.

It was expected that the conclusion of the above theorem remains true even if the coefficients of T_n do not come from the same sequence, that is,

$$T_n(t) = \sum_{j=0}^{n} a_{j,n} \cos(jt)\,,$$

where the set

$$S := \{a_{j,n} : j \in \{0,1,\dots,n\},\, n \in \mathbb{N}\} \subset \mathbb{R}$$

is finite and

$$\lim_{n\to\infty} |\{j \in \{0,1,\dots,n\},\, a_{j,n} \neq 0\}| = \infty\,.$$

Associated with an algebraic polynomial

$$P(z) = \sum_{j=0}^{n} a_j z^j\,, \qquad a_j \in \mathbb{C}\,,$$

let

$$\mathrm{NC}_k(P) := |\{u : 0 \le u \le n-k+1,\;\; a_u + a_{u+1} + \cdots + a_{u+k-1} \neq 0\}|\,.$$

In [E-16b] we proved the following results.

Theorem 12.3. *If $S \subset \mathbb{R}$ is a finite set, $P_{2n} \in \mathcal{P}_{2n}(S)$ are self-reciprocal polynomials,*

$$T_n(t) := P_{2n}(e^{it})e^{-int}\,,$$

and

$$\lim_{n\to\infty} \mathrm{NC}_k(P_{2n}) = \infty$$

for every $k = 1, 2, \dots,$ then

$$\lim_{n\to\infty} \mathrm{NZ}(P_{2n}) = \lim_{n\to\infty} \mathrm{NZ}(T_n) = \infty\,.$$

Some of the most important consequences of the above theorem obtained in [E-16b] are stated below.

Corollary 12.4. *If $S \subset \mathbb{R}$ is a finite set, $P_n \in \mathcal{P}_n(S)$ are self-reciprocal polynomials, and*

$$\lim_{n\to\infty} |P_n(1)| = \infty\,,$$

then

$$\lim_{n\to\infty} \mathrm{NZ}(P_n) = \infty\,.$$

Corollary 12.5. *Suppose the finite set $S \subset \mathbb{R}$ has the property that*

$$s_1 + s_2 + \cdots + s_k = 0\,,\;\; s_1, s_2, \dots, s_k \in S\,,\;\; \text{implies}\;\; s_1 = s_2 = \cdots = s_k = 0\,,$$

that is, any sum of nonzero elements of S is different from 0. If $P_n \in \mathcal{P}_n(S)$ are self-reciprocal polynomials and

$$\lim_{n\to\infty} \mathrm{NC}(P_n) = \infty\,,$$

then

$$\lim_{n\to\infty} \mathrm{NZ}(P_n) = \infty\,.$$

J. Sahasrabudhe [S-19a] examined the case when $S \subset \mathbb{Z}$ is finite. Exploiting the assumption that the coefficients are integer he proved that for any finite set $S \subset \mathbb{Z}$ a self-reciprocal polynomial $P \in \mathcal{P}_{2n}(S)$ has at least

$$c\,\left(\log\log\log|P(1)|\right)^{1/2-\varepsilon} - 1$$

zeros on the unit circle of $\mathbb{C}$ with a constant $c > 0$ depending only on $M = M(S) := \max\{|z| : z \in S\}$ and $\varepsilon > 0$.

Let $\varphi(n)$ denote the Euler's totient function defined as the number of integers $1 \le k \le n$ that are relative prime to n. In an earlier version of his paper Sahasrabudhe [S-19a] used the trivial estimate $\varphi(n) \neq \sqrt{n}$ for $n \ge 3$ and he proved his result with the exponent $1/4-\varepsilon$ rather than $1/2-\varepsilon$. Using the nontrivial estimate $\varphi(n) \ge n/8(\log\log n)$ [R-62] for all $n > 3$ allowed him to prove his result with $1/2-\varepsilon$.

In the papers [BE-07], [E-16b], and [S-19a] the already mentioned Littlewood Conjecture, proved by Konyagin [K-81] and independently by McGehee, Pigno, and B. Smith [Mc-81], plays a key role, and we rely on it heavily in the proof of the main results of this paper as well. This states the following.

Theorem 12.6. *There is an absolute constant $c > 0$ such that*

$$\int_0^{2\pi} \Big| \sum_{j=1}^m a_j e^{i\lambda_j t} \Big|\, dt \ge c\delta \log m$$

whenever $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct integers and $a_1, a_2, \dots, a_m$ are complex numbers of modulus at least $\delta > 0$. Here $c = 1/30$ is a suitable choice.

This is an obvious consequence of the following result a book proof of which has been worked out by DeVore and Lorentz in [D-93, pages 285-288].

Theorem 12.7. *If $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ are integers and $a_1, a_2, \dots, a_m$ are complex numbers, then*

$$\int_0^{2\pi} \Big| \sum_{j=1}^m a_j e^{i\lambda_j t} \Big|\, dt \ge \frac{1}{30} \sum_{j=1}^m \frac{|a_j|}{j}\,.$$

In [E-19e] we proved the following results.

Theorem 12.8. *If $S \subset \mathbb{Z}$ is a finite set, $M = M(S) := \max\{|z| : z \in S\}$, $P \in \mathcal{P}_{2n}(S)$ is a self-reciprocal polynomial,*

$$T(t) := P(e^{it})e^{-int}\,,$$

then

$$\mathrm{NZ}^*(T_n) \ge \left(\frac{c}{1+\log M}\right) \frac{\log\log\log|P(1)|}{\log\log\log\log|P(1)|} - 1$$

with an absolute constant $c > 0$, whenever $|P(1)| \ge 16$.

Corollary 12.9. *If $S \subset \mathbb{Z}$ is a finite set, $M = M(S) := \max\{|z| : z \in S\}$, $P \in \mathcal{P}_n(S)$ is a self-reciprocal polynomial, then*

$$\mathrm{NZ}(P) \geq \left(\frac{c}{1+\log M}\right) \frac{\log\log\log|P(1)|}{\log\log\log\log|P(1)|} - 1$$

with an absolute constant $c > 0$, whenever $|P(1)| \geq 16$.

This improves the exponent $1/2 - \varepsilon$ to $1 - \varepsilon$ in a recent breakthrough result [S-19a] by Sahasrabudhe. We note that in both Sahasrabudhe's paper and this paper the assumption that the finite set S contains only integers is deeply exploited. Our next result is an obvious consequence of Corollary 12.9.

Corollary 12.10. *If the set $S \subset \mathbb{Z}$ is finite, $M = M(S) := \max\{|z| : z \in S\}$,*

$$T(t) = \sum_{j=0}^{n} a_j \cos(jt)\,, \qquad a_j \in S\,,$$

then

$$\mathrm{NZ}^*(T) \geq \left(\frac{c}{1+\log M}\right) \frac{\log\log\log|T(0)|}{\log\log\log\log|T(0)|} - 1$$

with an absolute constant $c > 0$, whenever $|T(0)| \geq 16$.

13. Bourgain's L_1 problem and related results

For $n \geq 1$ let

$$\mathcal{A}_n := \left\{P : P(z) = \sum_{j=1}^{n} z^{k_j} : 0 \leq k_1 < k_2 < \cdots < k_n\,, k_j \in \mathbb{Z}\right\},$$

that is, $\mathcal{A}_n$ is the collection of all sums of n distinct monomials. For $p \geq 0$ we define

$$S_{n,p} := \sup_{Q \in \mathcal{A}_n} \frac{M_p(Q)}{\sqrt{n}} \qquad \text{and} \qquad S_p := \liminf_{n\to\infty} S_{n,p} \leq \Sigma_p := \limsup_{n\to\infty} S_{n,p}\,.$$

We also define

$$I_{n,p} := \inf_{Q \in \mathcal{A}_n} \frac{M_p(Q)}{\sqrt{n}} \qquad \text{and} \qquad I_p := \limsup_{n\to\infty} I_{n,p} \geq \Omega_p := \liminf_{n\in\to\infty} I_{n,p}\,.$$

Observe that Parseval's formula gives $\Omega_2 = \Sigma_2 = 1$. The problem of calculating Σ_1 appears in a paper of Bourgain [B-93]. Deciding whether $\Sigma_1 < 1$ or $\Sigma_1 = 1$ would be a major step toward confirming or disproving other important conjectures. Karatsuba [K-98] observed that $\Sigma_1 \geq 1/\sqrt{2} \geq 0.707$. Indeed, taking, for instance,

$$P_n(z) = \sum_{k=0}^{n-1} z^{2^k}\,, \qquad n = 1, 2, \ldots\,,$$

it is easy to see that

$$M_4(P_n)^4 = 2n(n-1) + n\,, \tag{13.1}$$

and as Hölder's inequality implies

$$n = M_2(P_n)^2 \leq M_1(P_n)^{2/3} M_4(P_n)^{4/3}\,,$$

we conclude

$$M_1(P_n) \geq \frac{n^{3/2}}{(2n(n-1)+n)^{1/2}} = \frac{n}{(2n-1)^{1/2}} \geq \frac{\sqrt{n}}{\sqrt{2}}\,. \tag{13.2}$$

Similarly, if $S_n := \{a_1 < a_2 < \cdots < a_n\}$ is a Sidon set (that is, S_n is a subset of integers such that no integer has two essentially distinct representations as the sum of two elements of S_n), then the polynomials

$$P_n(z) = \sum_{a \in S_n} z^a\,, \qquad n = 1, 2, \ldots\,,$$

satisfy (13.1) and (13.2). In fact, it was observed in [BC-08] that

$$\min_{P \in \mathcal{A}_n} M_4(P)^4 = 2n(n-1) + n\,,$$

and such minimal polynomials in $\mathcal{A}_n$ are precisely constructed by Sidon sets as above.

Improving Karatsuba's result, by using a probabilistic method Aistleitner [A-13] proved that $\Sigma_1 \geq \sqrt{\pi}/2 \geq 0.886$. We note that P. Borwein and Lockhart [BL-01] investigated the asymptotic behavior of the mean value of normalized L_p norms of Littlewood polynomials for arbitrary $p > 0$. Using the Lindeberg Central Limit Theorem and the Dominated Convergence Theorem, they proved that

$$\lim_{n \to \infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} \frac{(M_p(f))^p}{n^{p/2}} = \Gamma(1 + p/2)\,,$$

where $\mathcal{L}_n$ is, as before, the set of Littlewood polynomials of degree n. It follows simply from the case $p = 1$ of the result in [BL-01] quoted above that $\Sigma_1 \geq \sqrt{\pi/8} \geq 0.626$. Moreover, this can be achieved by taking the sum of approximately half of the monomials of $\{x^0, x^1, \ldots, x^{2n}\}$ and letting n tend to ∞.

Observe that Parseval's formula gives $\Omega_2 = \Sigma_2 = 1$. In [C-15b] we proved the following results.

Theorem 13.1. *Let (k_j) be a strictly increasing sequence of nonnegative integers satisfying*

$$k_{j+1} > k_j \left(1 + \frac{c_j}{j^{1/2}}\right), \qquad j = 1, 2, \ldots\,,$$

where $\lim_{j\to\infty} c_j = \infty$. *Let*

$$P_n(z) = \sum_{j=1}^{n} z^{k_j}, \qquad n = 1, 2, \dots .$$

We have

$$\lim_{n\to\infty} \frac{M_p(P_n)}{\sqrt{n}} = \Gamma(1 + p/2)^{1/p}$$

for every $p \in (0, 2)$.

Theorem 13.2. *Let* (k_j) *be a strictly increasing sequence of nonnegative integers satisfying*

$$k_{j+1} > qk_j\,, \qquad j = 1, 2, \dots ,$$

where $q > 1$. *Let*

$$P_n(z) = \sum_{j=1}^{n} z^{k_j}, \qquad n = 1, 2, \dots .$$

We have

$$\lim_{n\to\infty} \frac{M_p(P_n)}{\sqrt{n}} = \Gamma(1 + p/2)^{1/p}$$

for every $p \in [1, \infty)$.

Corollary 13.3. *We have* $\Sigma_p \ge S_p \ge \Gamma(1 + p/2)^{1/p}$ *for all* $p \in (0, 2)$.

The special case $p = 1$ recaptures a recent result of Aistleitner [A-13], the best known lower bound for Σ_1.

Corollary 13.4. *We have* $\Sigma_1 \ge S_1 \ge \sqrt{\pi}/2$.

Corollary 13.5. *We have* $\Omega_p \le I_p \le \Gamma(1 + p/2)^{1/p}$ *for all* $p \in (2, \infty)$.

We remark here that the same results also hold for the polynomials $\sum_{j=1}^{n} a_j z^{k_j}$ with coefficients a_j if a general form of the Salem-Zygmund theorem is used (e.g. see (2) in [E-62]).

Our final result in [C-15b] shows that the upper bound $\Gamma(1 + p/2)^{1/p}$ in Corollary 13.5 is optimal at least for even integers.

Corollary 13.6. *For any even integer* $p = 2m \ge 2$, *we have*

$$\lim_{n\to\infty} \min_{P\in\mathcal{A}_n} \frac{M_p(P)}{\sqrt{n}} = \Gamma(1 + p/2)^{1/p}\,.$$

Observe that a standard way to prove a Nikolskii-type inequality for trigonometric polynomials [2, p. 394] applies to the classes $\mathcal{A}_n$. Indeed,

$$\begin{aligned} M_p(P)^p &= \frac{1}{2\pi}\int_0^{2\pi} \left|P(e^{it})\right|^p \,dt \le \left(\frac{1}{2\pi}\int_0^{2\pi} \left|P(e^{it})\right|^2 \,dt\right) \left(\max_{t\in[0,2\pi]} \left|P(e^{it})\right|\right)^{p-2} \\ &= nn^{p-2} = n^{p-1} \end{aligned}$$

for every $P \in \mathcal{A}_n$ and $p \ge 2$, and the Dirichlet kernel $D_n(z) := 1 + z + \cdots + z^n$ shows the sharpness of this upper bound up to a multiplicative factor constant $c > 0$. So if we study the original Bourgain problem in the case of $p > 2$, we should normalize by dividing by $n^{1-1/p}$ rather than $n^{1/2}$.

In [C-15c] we examined

$$S_{n,0}(I) := \sup_{Q \in \mathcal{A}_n} \frac{M_0(Q,I)}{\sqrt{n}} \qquad \text{and} \qquad S_0(I) := \liminf_{n \to \infty} S_{n,0}(I)$$

for intervals $I = [\alpha, \beta]$ with $0 < |I| := \beta - \alpha \le 2\pi$ and proved the following results.

Theorem 13.7. *There are polynomials $Q_n \in \mathcal{A}_n \cap \mathcal{P}_N$ with $N = 2n + o(n)$ such that*

$$M_0(Q_n) \ge \left(\frac{1}{2\sqrt{2}} + o(1) \right) \sqrt{n}\,, \qquad n = 1, 2, \dots\,,$$

and hence $S_0 \ge \dfrac{1}{2\sqrt{2}}$.

Theorem 13.8. *There are polynomials $Q_n \in \mathcal{A}_n \cap \mathcal{P}_N$ with $N = 2n + o(n)$, an absolute constant $c_1 > 0$, and a constant $c_2(\varepsilon) > 0$ depending only on $\varepsilon > 0$ such that*

$$M_0(Q_n, I) \ge c_1 \sqrt{n}\,, \qquad n = 1, 2, \dots\,,$$

for every interval $I := [\alpha, \beta] \subset \mathbb{R}$ such that

$$\frac{4\pi}{n} \le \frac{(\log n)^{3/2}}{n^{1/2}} \le \beta - \alpha \le 2\pi\,, \tag{13.3}$$

while

$$M_1(Q_n, I) \le c_2(\varepsilon) \sqrt{n}\,, \qquad n = 1, 2, \dots\,,$$

for every interval $I := [\alpha, \beta] \subset \mathbb{R}$ such that

$$(n/2)^{-1/2+\varepsilon} \le \beta - \alpha \le 2\pi\,. \tag{13.4}$$

Note that Theorem 13.8 implies that there is an absolute constant $c_1 > 0$ such that $S_0(I) \ge c_1$ for all intervals $I := [\alpha, \beta] \subset \mathbb{R}$ satisfying (13.3).

Theorem 13.9. *There are polynomials $Q_n \in \mathcal{L}_n$ such that*

$$M_0(Q_n) \ge \left(\frac{1}{2} + o(1) \right) \sqrt{n}\,, \qquad n = 1, 2, \dots\,.$$

Theorem 13.10. *There are polynomials $Q_n \in \mathcal{L}_n$, an absolute constant $c_1 > 0$, and a constant $c_2(\varepsilon) > 0$ depending only on $\varepsilon > 0$ such that*

$$M_0(Q_n, I) \ge c_1 \sqrt{n}\,, \qquad n = 1, 2, \dots\,,$$

for every interval $I := [\alpha, \beta] \subset \mathbb{R}$ satisfying (13.3), while

$$M_1(Q_n, I) \le c_2(\varepsilon) \sqrt{n}\,, \qquad n = 1, 2 \dots\,,$$

for every interval $I := [\alpha, \beta] \subset \mathbb{R}$ satisfying (13.4).

References

A-13. C. Aistleitner, *On a problem of Bourgain concerning the L_1-norm of exponential sums*, Math. Z. **275** (2013), 681–688.

A-19. J.-P. Allouche, K.-K. S. Choi, A. Denise, T. Erdélyi, and B. Saffari, *Bounds on autocorrelation coefficients of Rudin-Shapiro polynomials*, Anal. Math. (2019) (to appear).

B-90. R.C. Baker and H.L. Montgomery, *Oscillations of Quadratic L-functions*, in Analytic Number Theory (ed. B.C. Berndt et. al.), Birkhuser, Boston pp. 23–40, 1990.

B-75. P.T. Bateman, G.B. Purdy, and S.S Wagstaff, Jr., *Some numerical results on Fekete polynomials*, Math. Comp. (1975), 7–23.

B-91. J. Beck, *"Flat" polynomials on the unit circle – note on a problem of Littlewood*, Bull. London Math. Soc. (1991), 269–277.

B-17. J. Bell and I. Shparlinski, *Power series approximations to Fekete polynomials*, J. Approx. Theory **222** (2017), 132142.

B-73. E. Beller and D.J. Newman,, *An extremal problem for the geometric mean of polynomials*, Proc. Amer. Math. Soc. **39** (1973), 313–317.

BP-32. A. Bloch and G. Pólya,, *On the roots of certain algebraic equations*, Proc. London Math. Soc. **33** (1932), 102–114.

BB-09. E. Bombieri and J. Bourgain, *On Kahane's ultraflat polynomials*, J. Eur. Math. Soc. **11** (2009), no. 3, 627–703.

B-02. P. Borwein, *Computational Excursions in Analysis and Number Theory*, Springer, New York, 2002.

BC-01a. P. Borwein and K.-K. S. Choi, *Merit factors of polynomials formed by Jacobi symbols*, Canad. J. Math. **53** (2001), no. 1, 33–50.

BC-02. P. Borwein and K.-K. S. Choi, *Explicit merit factor formulae for Fekete and Turyn polynomials*, Trans. Amer. Math. Soc. **354** (2002), no. 1, 219–234.

BC-15. P. Borwein, K.-K. S. Choi, R. Ferguson, and J. Jankauskas, *On Littlewood polynomials with prescribed number of zeros inside the unit disk*, Canad. J. Math. **67** (2015), 507–526.

BC-04. P. Borwein, K.-K. S. Choi, and J. Jedwab, *Binary sequences with merit factor greater than 6.34*, IEEE Trans. Inform. Theory **50** (2004), no. 12, 3234–3249.

BC-08. P. Borwein, K.-K. S. Choi, and I. Mercer, *Expected Norms of Zero-One Polynomials*, Canad. Math. Bull. **51** (2008), 297–507.

BC-01b. P. Borwein, K.-K. S. Choi, and S. Yazdani, *An extremal property of Fekete polynomials*, Proc. Amer. Math. Soc. **129** (2001), no. 1, 19–27.

BE-95. P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer, New York, 1995.

BE-97. P. Borwein and T. Erdélyi, *On the zeros of polynomials with restricted coefficients*, Illinois J. Math. **41** (1997), no. 4, 667–675.

BE-01. P. Borwein and T. Erdélyi, *Trigonometric polynomials with many real zeros and a Littlewood-type problem*, Proc. Amer. Math. Soc. **129** (no. 3, 2001), 725–730.

BE-07. P. Borwein and T. Erdélyi, *Lower bounds for the number of zeros of cosine polynomials in the period: a problem of Littlewood*, Acta Arith. **128** (2007), no. 4, 377–384.

BE-08a. P. Borwein, T. Erdélyi, R. Ferguson, and R. Lockhart, *On the zeros of cosine polynomials: solution to a problem of Littlewood*, Ann. of Math. (2) **167** (2008), no. 3, 1109–1117.

BE-99. P. Borwein, T. Erdélyi, and G. Kós, *Littlewood-type problems on* $[0,1]$, Proc. London Math. Soc. **79** (1999), 22–46.

BE-13. P. Borwein, T. Erdélyi, and G. Kós, *The multiplicity of the zero at* 1 *of polynomials with constrained coefficients*, Acta Arith. **159** (2013), no. 4, 387–395.

BE-08b. P. Borwein, T. Erdélyi, and F. Littmann, *Zeros of polynomials with finitely many different coefficients*, Trans. Amer. Math. Soc. **360** (2008), 5145–5154.

BL-01. P. Borwein and R. Lockhart, *The expected* L_p *norm of random polynomials*, Proc. Amer. Math. Soc. **129** (2001), 1463–1472.

BM-00. P. Borwein and M.J. Mossinghoff, *Rudin-Shapiro like polynomials in* L_4, Math. Comp. **69** (2000), 1157–1166.

B-93. J. Bourgain, *On the sprectal type of Ornstein's class one transformations*, Israel J. Math. **84** (1993), 53–63.

B-97. D. Boyd, *On a problem of Byrnes concerning polynomials with restricted coefficients*, Math. Comput. **66** (1997), 1697–1703.

B-73. J. Brillhart, *On the Rudin-Shapiro polynomials*, Duke Math. J. **40** (1973), no. 2, 335–353.

BL-76. J. Brillhart, J.S. Lemont, and P. Morton, *Cyclotomic properties of the Rudin-Shapiro polynomials*, J. Reine Angew. Math. (Crelle's J.) **288** (1976), 37–65.

C-19. K.-K. S. Choi, *Bounds on autocorrelation coefficients of Rudin-Shapiro polynomials II*, J. Approx. Theory (2019) (to appear).

C-15a. K.-K. S. Choi and T. Erdélyi, *On the average Mahler measures on Littlewood polynomials*, Proc. Amer. Math. Soc. Ser. B **1** (2015), 105–120.

C-15b. K.-K. S. Choi and T. Erdélyi, *On a problem of Bourgain concerning the* L_p *norms of exponential sums*, Math. Zeit. **279** (2015), 577–584.

C-15c. K.-K. S. Choi and T. Erdélyi, *Sums of monomials with large Mahler measure*, J. Approx. Theory **197** (2015), 49–61.

C-11. K.-K. S. Choi and M.J. Mossinghoff, *Average Mahler's measure and* L_p *norms of unimodular polynomials*, Pacific J. Math. **252** (2011), no. 1, 31–50.

C-35. S. Chowla, *Note on Dirichlet's L-functions*, Acta Arith. **1** (1935), 113–114.

C-00. B. Conrey, A. Granville, B. Poonen, and K. Soundararajan, *Zeros of Fekete polynomials*, Ann. Inst. Fourier (Grenoble) **50** (2000), 865–884.

D-93. R.A. DeVore and G.G. Lorentz, *Constructive Approximation*, Springer-Verlag, Berlin, 1993.

D-05. Ch. Doche, *Even moments of generalized Rudin-Shapiro polynomials*, Math. Comp. **74** (2005), no. 252, 1923–1935.

DH-04. Ch. Doche and L. Habsieger, *Moments of the Rudin-Shapiro polynomials*, J. Fourier Anal. Appl. **10** (2004), no. 5, 497–505.

D-08. P. Drungilas, *Unimodular roots of reciprocal Littlewood polynomials*, J. Korean Math. Soc. **45** (2008), no. 3, 835–840.

D-99. A. Dubickas, *On the order of vanishing at* 1 *of a polynomial*, Lithuanian Math. J. **39** (1999), 365–370.

D-14. A. Dubickas, *Polynomials with multiple roots at* 1, Int. J. Number Theory **10** (2014), no. 2, 391–400.

EZ-17. S.B. Ekhad and D. Zeilberger, *Integrals involving Rudin-Shapiro polynomials and sketch of a proof of Saffari's conjecture*, in Analytic number theory, modular forms and q-hypergeometric series **221** (2017), Springer Proc. Math. Stat., Springer, Cham, 253–265.

E-00a. T. Erdélyi, *The phase problem of ultraflat unimodular polynomials: the resolution of the conjecture of Saffari*, Math. Ann. **300** (2000), 39–60.

E-00b. T. Erdélyi, *The resolution of Saffari's Phase Problem*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), 803–808.

E-01a. T. Erdélyi, *How far is a sequence of ultraflat unimodular polynomials from being conjugate reciprocal*, Michigan Math. J. **49** (2001), 259–264.

E-01b. T. Erdélyi, *Proof of Saffari's near-orthogonality conjecture for ultraflat sequences of unimodular polynomials*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), 623–628.

E-01c. T. Erdélyi, *On the zeros of polynomials with Littlewood-type coefficient constraints*, Michigan Math. J. **49** (2001), 97–111.

E-02. T. Erdélyi, *Polynomials with Littlewood-type coefficient constraints*, in Approximation Theory X: Abstract and Classical Analysis, Charles K. Chui, Larry L. Schumaker, and Joachim Stöckler (Eds.) (2002), Vanderbilt University Press, Nashville, TN, 153–196.

E-03. T. Erdélyi, *On the real part of ultraflat sequences of unimodular polynomials*, Math. Ann. **326** (2003), 489–498.

E-08a. T. Erdélyi, *An improvement of the Erdős-Turán theorem on the distribution of zeros of polynomials*, C. R. Acad. Sci. Paris, Ser. I **346** (2008), no. 5, 267–270.

E-08b. T. Erdélyi, *Extensions of the Bloch-Pólya theorem on the number of real zeros of polynomials*, J. Théor. Nombres Bordeaux **20** (2008), no. 2, 281-287.

E-11. T. Erdélyi, *Sieve-type lower bounds for the Mahler measure of polynomials on subarcs*, Computational Methods and Function Theory **11** (2011), 213–228.

E-12. T. Erdélyi, *Upper bounds for the L_q norm of Fekete polynomials on subarcs*, Acta Arith. **153** (2012), no. 1, 81–91.

E-15. T. Erdélyi, *On the flatness of conjugate reciprocal unimodular polynomials*, J. Math. Anal. Appl. **432** (2015), no. 2, 699–714.

E-16a. T. Erdélyi, *Coppersmith-Rivlin type inequalities and the order of vanishing of polynomials at* 1, Acta Arith. **172** (2016), no. 3, 271–284.

E-16b. T. Erdélyi, *On the number of unimodular zeros of self-reciprocal polynomials with coefficients from a finite set*, Acta Arith. **176** (2016), no. 2, 177–200.

E-16c. T. Erdélyi, *The Mahler measure of the Rudin-Shapiro polynomials*, Constr. Approx. **43** (2016), no. 3, 357–369.

E-18. T. Erdélyi, *Improved lower bound for the Mahler measure of the Fekete polynomials*, Constr. Approx. **48** (2018), no. 2, 383–399.

E-19a. T. Erdélyi, *The asymptotic value of the Mahler measure of the Rudin-Shapiro polynomials*, J. Anal. Math. (to appear).

E-19b. T. Erdélyi, *On the oscillation of the modulus of Rudin-Shapiro polynomials on the unit circle*, Mathematika (2019) (to appear).

E-19c. T. Erdélyi, *Improved results on the oscillation of the modulus of Rudin-Shapiro polynomials on the unit circle*, Proc. Amer. Math. Soc. (2019) (to appear).

E-19d. T. Erdélyi, *The asymptotic distance between an ultraflat unimodular polynomial and its conjugate reciprocal*, https://arxiv.org/abs/1810.04287.

E-19e. T. Erdélyi, *Improved lower bound for the number of unimodular zeros of self-reciprocal polynomials with coefficients from a finite set*, https://arxiv.org/pdf/1702.05823.pdf, Acta Arith. (2019).

EL-07. T. Erdélyi and D. Lubinsky, *Large sieve inequalities via subharmonic methods and the Mahler measure of Fekete polynomials*, Canad. J. Math. **59** (2007), 730–741.

EN-16. T. Erdélyi and P. Nevai, *On the derivatives of unimodular polynomials (Russian)*, Mat. Sbornik **207** (2016), no. 4, 123-142, translation in Sbornik Math. 207 (2016), no. 3–4, 590–609.

E-57. P. Erdős, *Some unsolved problems*, Michigan Math. J. **4** (1957), 291–300.

E-62. P. Erdős, *On trigonometric sums with gaps*, Publ. Math. Inst. Hung. Acad. Sci., Ser. A **7** (1962), 37–42.

E-50. P. Erdős and Turán, *On the distribution of roots of polynomials*, Ann. of Math. **51** (1950), 105–119.

F-12. M. Fekete and G. Pólya, *Über ein Problem von Laguerre*, Rend. Circ. Mat. Palermo **34** (1912), 89–120.

G-51. M.J. Golay, *Static multislit spectrometry and its application to the panoramic display of infrared spectra,*, J. Opt. Soc. America **41** (1951), 468–472.

G-16. C. Günther and K.-U. Schmidt, *L_q norms of Fekete and related polynomials*, Canad. J. Math. **69** (2017), no. 4, 807–825.

J-13a. J. Jedwab, D.J. Katz, and K.-U. Schmidt, *Advances in the merit factor problem for binary sequences.*, J. Combin. Theory Ser. A **120** (2013), no. 4, 882–906.

J-13b. J. Jedwab, D.J. Katz, and K.-U. Schmidt, *Littlewood polynomials with small L_4 norm*, Adv. Math. **241** (2013), 127–136.

HL-52. G.H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge Univ. Press, London, 1952.

H-37. H. Heilbronn, *On real characters*, Acta Arith. **2** (1937), 212–213.

J-16. J. Jung and S.W. Shin, *On the sparsity of positive-definite automorphic forms within a family*, J. Anal. Math. **129** (2016), no. 1, 105–138.

K-80a. J.P. Kahane, *Sur les polynomes a coefficient unimodulaires*, Bull. London Math. Soc. **12** (1980), 321–342.

K-98. A.A. Karatsuba, *An estimate for the L_1-norm of an exponential sum*, Mat. Zametki **64** (1998), 465–468.

K-81. S.V. Konyagin, *On a problem of Littlewood*, Mathematics of the USSR, Izvestia **18** (1981), 205–225.

K-04. S.V. Konyagin and V.F. Lev, *Character sums in complex half planes*, J. Theor. Nombres Bordeaux **16** (2004), no. 3, 587–606.

K-80b. T. Körner, *On a polynomial of J.S. Byrnes*, Bull. London Math. Soc. **12** (1980), 219–224.

L-61. J.E. Littlewood, *On the mean values of certain trigonometrical polynomials*, J. London Math. Soc. **36** (1961), 307–334.

L-64. J.E. Littlewood, *On the real roots of real trigonometrical polynomials (II)*, J. London Math. Soc. **39** (1964), 511–552.

L-66a. J.E. Littlewood, *The real zeros and value distributions of real trigonometrical polynomials*, J. London Math. Soc. **41** (1966), 336–342.

L-66b. J.E. Littlewood, *On polynomials* $\sum \pm z^m, \sum \exp(\alpha_m i) z^m, z = e^{i\theta}$, J. London Math. Soc. **41**, 367–376, yr 1966.

L-68. J.E. Littlewood, *Some Problems in Real and Complex Analysis*, Heath Mathematical Monographs, Lexington, Massachusetts, 1968.

Mc-81. O.C. McGehee, L. Pigno, and B. Smith, *Hardy's inequality and the* L_1 *norm of exponential sums*, Ann. Math. **113** (1981), 613–618.

M-06a. I.D. Mercer, *Unimodular roots of special Littlewood polynomials*, Canad. Math. Bull. **49** (2006), no. 3, 438–447.

M-80. H.L. Montgomery, *An exponential polynomial formed with the Legendre symbol*, Acta Arith. **37** (1980), 375–380.

M-17. H.L. Montgomery, *Littlewood polynomials*, in Analytic Number Theory, Modular Forms and q-Hypergeometric Series, George Andrews and Frank Garvan (Eds.) (2017), Springer Proc. Math. Stat., vol. 221, Springer, Cham, 533–553.

M-06b. K. Mukunda, *Littlewood Pisot numbers*, J. Number Theory **117** (2006), no. 1, 106–121.

N-16. H. Nguyen, O. Nguyen, and V. Vu, *On the number of real roots of random polynomials*, 1550052, Commun. Contemp. Math. **18** (2016).

O-18. A. Odlyzko, *Search for ultraflat polynomials with plus and minus one coefficients*, in Connections in Discrete Mathematics, Steve Butler, Joshua Cooper, Glenn Hurlbert (Eds.) (2018), Cambridge Univ. Press, Cambridge, 39–55.

O-93. A.M. Odlyzko and B. Poonen, *Zeros of polynomials with* $0, 1$ *coefficients*, L'Enseign. Math. **39** (1993), 317–348.

P-99. C. Pinner, *Double roots of* $[-1, 1]$ *power series and related matters*, Math. Comp. **68** (1999), 1149–1178.

P-19. G. Pólya, *Verschiedene Bemerkungen zur Zahlentheorie*, Jahresber. Dtsch. Math. Ver. **28** (1919), 31–40.

P-14. I.E. Pritsker and A.A. Sola, *Expected discrepancy for zeros of random algebraic polynomials*, Proc. Amer. Math. Soc. **142** (2014), 4251–4263.

QS-95. H. Queffelec and B. Saffari, *Unimodular polynomials and Bernstein's inequalities*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995, 3), 313–318.

QS-96. H. Queffelec and B. Saffari, *On Bernstein's inequality and Kahane's ultraflat polynomials*, J. Fourier Anal. Appl. **2** (1996, 6), 519–582.

R-16. B. Rodgers, *On the distribution of Rudin-Shapiro polynomials and lacunary walks on* $SU(2)$, to appear in Adv. Math., arxiv.org/abs/1606.01637.

R-62. J.B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94.

S-92. B. Saffari, *The phase behavior of ultraflat unimodular polynomials*, in Probabilistic and Stochastic Methods in Analysis, with Applications (1992), Kluwer Academic Publishers, Dordrecht, 555–572.

S-01. B. Saffari, *Some polynomial extremal problems which emerged in the twentieth century*, in Twentieth Century Harmonic Analysis – A Celebration, James S. Byrnes (Ed.) (2001), Kluwer Academic Publishers, Dordrecht, 201–233.

S-19a. J. Sahasrabudhe, *Counting zeros of cosine polynomials: on a problem of Littlewood*, Adv. Math. **343** (2019), 495–521.

Sch-32. E. Schmidt, *Über algebraische Gleichungen vom Pólya-Bloch-Typos*, Sitz. Preuss. Akad. Wiss., Phys.-Math. Kl. (1932), 321.

Sch-33. I. Schur, *Untersuchungen über algebraische Gleichungen*, Sitz. Preuss. Akad. Wiss., Phys.-Math. Kl. (1933), 403–428.

S-51. H.S. Shapiro, *Extremal problems for polynomials and power series*, Master thesis, MIT, 1951.

S-19b. K. Soundararajan, *Equidistribution of zeros of polynomials*, Amer. Math. Monthly (to appear).

Sz-34. G. Szegő, *Bemerkungen zu einem Satz von E. Schmidt uber algebraische Gleichungen*, Sitz. Preuss. Akad. Wiss., Phys.-Math. Kl. (1934), 86–98.

T-15. T. Tao and V. Vu, *Local universality of zeros of random polynomials*, Int. Math. Res. Notices **13** (2015), 5053–5139.

T-93. V. Totik, *Distribution of simple zeros of polynomials*, Acta Math. **170** (1993), 1–28.

T-07. V. Totik and P. Varjú, *Polynomials with prescribed zeros and small norm*, Acta Sci. Math. (Szeged) **73** (2007), 593–612.

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