TURÁN-TYPE REVERSE MARKOV INEQUALITIES FOR POLYNOMIALS WITH RESTRICTED ZEROS

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Abstract. Let $\mathcal{P}_n^c$ denote the set of all algebraic polynomials of degree at most n with complex coefficients. Let

$$D^+ := \{z \in \mathbb{C} : |z| \leq 1, \;\; \text{Im}(z) \geq 0\}\,.$$

For integers $0 \leq k \leq n$ let $\mathcal{F}_{n,k}^c$ be the set of all polynomials $P \in \mathcal{P}_n^c$ having at least $n-k$ zeros in D^+. Let

$$\|f\|_A := \sup_{z \in A} |f(z)|$$

for complex-valued functions defined on $A \subset \mathbb{C}$. We prove that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \left(\frac{n}{k+1} \right)^{1/2} \leq \inf_P \frac{\|P'\|_{[-1,1]}}{\|P\|_{[-1,1]}} \leq c_2 \left(\frac{n}{k+1} \right)^{1/2}$$

for all integers $0 \leq k \leq n$, where the infimum is taken for all $0 \not\equiv P \in \mathcal{F}_{n,k}^c$ having at least one zero in $[-1,1]$. This is an essentially sharp reverse Markov-type inequality for the classes $\mathcal{F}_{n,k}^c$ extending earlier results of Turán and Komarov from the case $k=0$ to the cases $0 \leq k \leq n$.

1. Introduction and Notation

Let $\mathcal{P}_n$ denote the set of all algebraic polynomials of degree at most n with real coefficients Let $\mathcal{P}_n^c$ denote the set of all algebraic polynomials of degree at most n with complex coefficients. Let

$$\|f\|_A := \sup_{z \in A} |f(z)|$$

for complex-valued functions defined on $A \subset \mathbb{C}$. Turán [32] proved that

$$\|P'\|_{[-1,1]} \geq \frac{\sqrt{n}}{6} \|P\|_{[-1,1]} \tag{1.1}$$

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for all $P \in \mathcal{P}_n^c$ of degree n having all their zeros in the interval $[-1,1]$. The examples $P(x) = (x^2-1)^m$ and $P(x) = (x^2-1)^m(x+1)$ show that Turán's reverse Markov-type inequality (1.1) is essentially sharp, even though the multiplicative constant 1/6 in (1.1) is not the best possible. Note that the best possible multiplicative constant $c = c_n$ in (1.1) had been found by Erőd [10], see also [11]. Another simple observation of Turán [32] is the inequality

$$\|P'\|_D \geq \frac{n}{2}\,\|P\|_D \tag{1.2}$$

for all $P \in \mathcal{P}_n^c$ of degree n having all their zeros in the closed unit disk $D \subset \mathbb{C}$. Malik [23] established an extension of (1.2) proving that

$$\|P'\|_D \geq \frac{n}{1+R}\,\|P\|_D$$

for all $P \in \mathcal{P}_n^c$ of degree n having all their zeros in the disk $D(0,R) \subset \mathbb{C}$ of radius $R \leq 1$ centered at 0, while Govil [16] showed that

$$\|P'\|_D \geq \frac{n}{1+R^n}\,\|P\|_D$$

for all $P \in \mathcal{P}_n^c$ of degree n having all its zeros in the disk $D(0,R) \subset \mathbb{C}$ of radius $R \geq 1$ centered at 0. See also [18, Section 4].

Let $\varepsilon \in [0,1]$ and let D_ε be the ellipse of the complex plane with large axis $[-1,1]$ and small axis $[-i\varepsilon, i\varepsilon]$. Let $\mathcal{P}_n^c(D_\varepsilon)$ denote the collection of all $P \in \mathcal{P}_n^c$ of degree n having all their zeros in D_ε. Extending Turán's reverse Markov-type inequality (1.1), Erőd [10, III. tétel] proved that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1(n\varepsilon + \sqrt{n}) \leq \inf_P \frac{\|P'\|_{D_\varepsilon}}{\|P\|_{D_\varepsilon}} \leq c_2(n\varepsilon + \sqrt{n})\,,$$

where the infimum is taken for all $P \in \mathcal{P}_n^c(D_\varepsilon)$. Levenberg and Poletsky [21] proved that

$$\frac{\sqrt{n}}{20\,\mathrm{diam} K} \leq \inf_P \frac{\|P'\|_K}{\|P\|_K}$$

for all compact convex set $K \subset \mathbb{C}$, where the infimum is taken for all $P \in \mathcal{P}_n^c$ of degree n having all their zeros in K.

Let $\varepsilon \in [0,1]$ and let S_ε be the diamond of the complex plane with diagonals $[-1,1]$ and $[-i\varepsilon, i\varepsilon]$. Let $\mathcal{P}_n^c(S_\varepsilon)$ denote the collection of all $P \in \mathcal{P}_n^c$ of degree n having all their zeros in S_ε. It has been proved in [5] that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1(n\varepsilon + \sqrt{n}) \leq \inf_P \frac{\|P'\|_{S_\varepsilon}}{\|P\|_{S_\varepsilon}} \leq c_2(n\varepsilon + \sqrt{n})\,,$$

where the infimum is taken for all $P \in \mathcal{P}_n^c(S_\varepsilon)$ with the property

$$|P(z)| = |P(-z)|\,, \qquad z \in \mathbb{C}\,,$$

or where the infimum is taken for all $P \in \mathcal{P}_n^c(S_\varepsilon)$ with real coefficients. It is an interesting question whether or not the lower bound in the above inequality holds for all $P \in \mathcal{P}_n^c(S_\varepsilon)$. Another result in [5] shows that this is the case at least when $\varepsilon = 1$, that is, there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 n \leq \inf_P \frac{\|P'\|_{S_1}}{\|P\|_{S_1}} \leq c_2 n\,,$$

where the infimum is taken for all (complex) $P \in \mathcal{P}_n^c(S_1)$. Motivated by the above results Révész [28] established the right order Turán-type reverse Markov inequalities on convex domains of the complex plane. His main theorem contains the above mentioned results in [5] as special cases. It states that

$$\frac{\|P'\|_K}{\|P\|_K} \geq c(K)n \qquad \text{with} \qquad c(K) = 0.0003\frac{w(K)}{d(K)^2}\,,$$

for all $P \in \mathcal{P}_n^c$ of degree n having all their zeros in a bounded convex set $K \subset \mathbb{C}$, where $d(K)$ is the diameter of K and

$$w(K) := \min_{\gamma \in [-\pi,\pi]} \left(\max_{z \in K} \mathrm{Re}(ze^{-i\gamma}) - \min_{z \in K} \mathrm{Re}(ze^{-i\gamma}) \right)$$

is the minimal width of K. The proof given by Révész is elementary, but rather subtle. Results on Turán and Erőd type reverses of Markov- and Bernstein-type inequalities include [25], [34], [9], [33], [21], [19], and [27]. The research on Turán and Erőd type reverses of Markov- and Bernstein-type inequalities got a new impulse suddenly in 2006 in large part by the work of Sz. Révész [28], see [5], [6], [8], [14], [15], and [29], for example.

G.G. Lorentz, M. von Golitschek, and Y. Makovoz devotes Chapter 3 of their book [22] to incomplete polynomials. E.B. Saff and R.S. Varga were among the researchers having contributed significantly to this topic. See [1], [30], and [31], for instance.

Let $\mathcal{P}_{n,k}$ be the set of all algebraic polynomials, with real coefficients, of degree at most $n+k$ having at least $n+1$ zeros at 0. That is, every $P \in \mathcal{P}_{n,k}$ is of the form

$$P(x) = x^{n+1}R(x)\,, \qquad R \in \mathcal{P}_{k-1}\,.$$

Let

$$V_a^b(f) := \int_a^b |f'(x)|\,dx$$

denote the total variation of a continuously differentiable function f on an interval $[a,b]$. In [7] a question [12] asked by A. Eskenazis and P. Ivanisvili related to their paper [13] as well as to [26] is answered by proving that there are absolute constants $c_3 > 0$ and $c_4 > 0$ such that

$$c_3 \frac{n}{k} \leq \min_{0 \not\equiv P \in \mathcal{P}_{n,k}} \frac{\|P'\|_{[0,1]}}{V_0^1(P)} \leq \min_{0 \not\equiv P \in \mathcal{P}_{n,k}} \frac{\|P'\|_{[0,1]}}{|P(1)|} \leq c_4 \left(\frac{n}{k} + 1\right)$$

for all integers $n \geq 1$ and $k \geq 1$. Here $c_3 = 1/12$ is a suitable choice.

In [7] we also proved that there are absolute constants $c_3 > 0$ and $c_4 > 0$ such that

$$c_3 \left(\frac{n}{k}\right)^{1/2} \leq \min_{0 \not\equiv P \in \mathcal{P}_{n,k}} \frac{\|P'(x)\sqrt{1-x^2}\|_{[0,1]}}{V_0^1(P)} \leq \min_{0 \not\equiv P \in \mathcal{P}_{n,k}} \frac{\|P'(x)\sqrt{1-x^2}\|_{[0,1]}}{|P(1)|} \leq c_4 \left(\frac{n}{k}+1\right)^{1/2}$$

for all integers $n \geq 1$ and $k \geq 1$. Here $c_3 = 1/8$ is a suitable choice.

Let

$$D^+ := \{z \in \mathbb{C} : |z| \leq 1, \ \operatorname{Im}(z) \geq 0\}.$$

In [20] Komarov proved that

$$\|P'\|_{[-1,1]} \geq A\sqrt{n}\,\|P\|_{[-1,1]}, \qquad A = \frac{2}{3\sqrt{210e}} = 0.0279\ldots,$$

for all polynomials P of degree n having all their zeros in the closed upper half-disk D^+.

For integers $0 \leq k \leq n$ let $\mathcal{F}_{n,k}^c$ be the set of all polynomials $P \in \mathcal{P}_n^c$ having at least $n-k$ zeros in D^+. In this paper we prove an essentially sharp reverse Markov-type inequality for the classes $\mathcal{F}_{n,k}^c$ extending the above mentioned results of Turán and Komarov from the case $k = 0$ to the cases $0 \leq k \leq n$.

2. New Results

The lower bound of Theorem 2.1 below is quite a new result even in the case when the infimum is taken for polynomials $P \in \mathcal{P}_n^c$ having at least $n-k$ zeros only in $[-1,1]$ rather than D^+.

Theorem 2.1. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 \left(\frac{n}{k+1}\right)^{1/2} \leq \inf_P \frac{\|P'\|_{[-1,1]}}{\|P\|_{[-1,1]}} \leq c_2 \left(\frac{n}{k+1}\right)^{1/2}$$

for all integers $0 \leq k \leq n$, where the infimum is taken for all $0 \not\equiv P \in \mathcal{F}_{n,k}^c$ having at least one zero in $[-1,1]$. Here $c_1 = 1/636$ is a suitable choice. When $0 \leq k \leq n/100000$ the lower bound remains valid even if the infimum is taken for all $0 \not\equiv P \in \mathcal{F}_{n,k}^c$.

Theorem 2.1 follows from the results below.

Theorem 2.2. *Let $1 \leq k \leq n/100000$. We have*

$$\|P'\|_{[-1,1]} \geq \frac{1}{144e}\left(\frac{n-k}{2k}\right)^{1/2} \|P\|_{[-1,1]}$$

for all $P \in \mathcal{F}_{n,k}^c$.

Corollary 2.3. *Let $1 \le k \le n$. We have*

$$\|P'\|_{[-1,1]} \ge \max\left\{\frac{1}{2}, \frac{1}{448}\left(\frac{n-k}{2k}\right)^{1/2}\right\}\|P\|_{[-1,1]}$$

for all $P \in \mathcal{F}^c_{n,k}$ with at least one zero in $[-1,1]$.

Theorem 2.4. *There is an absolute constant $c_2 > 0$ and there are polynomials $0 \not\equiv P = P_{n,k} \in \mathcal{F}^c_{2n,2k}$ of the form*

$$P(x) = (x^2-1)^{n-k}R(x)\,, \qquad R \in \mathcal{P}_{2k}\,,$$

such that

$$\frac{\|P'\|_{[-1,1]}}{\|P\|_{[-1,1]}} \le c_2\left(\frac{n}{k}\right)^{1/2}$$

for every $1 \le k \le n$.

We remark that the upper bound of Theorem 2.1 remains valid if we replace the closed upper half-disk D^+ with the closed unit disk D in the definition of $\mathcal{F}^c_{n,k}$, as then the infimum is taken for a larger class of polynomials. However, the lower bound of Theorem 2.1 does not remain valid if we replace the closed upper half-disk D^+ with the closed unit disk D in the definition of $\mathcal{F}^c_{n,k}$, not even in the case that $k = 0$. This can be seen by the example given in [20] (see also [21], where the case of star-shaped compact sets was considered). For completeness we present here a slight modification of the calculation made in [20] in a few lines. Given $\varepsilon > 0$, let m be the even integer for which $1/\varepsilon < m \le 1/\varepsilon + 2$. We claim that for every $\varepsilon > 0$ and for every integer $n \ge 1$ there is a $P_n \in \mathcal{P}^c_{mn}$ of degree mn having all its zeros on the unit circle ∂D such that

$$\|P_n'\|_{[-1,1]} \le (1/\varepsilon + 2)^{1-\varepsilon}(mn)^\varepsilon\|P_n\|_{[-1,1]}\,.$$

To see this let $P_n \in \mathcal{P}^c_{mn}$ be defined by $P_n(z) := (z^m - 1)^n$. Observe that $\|P_n\|_{[-1,1]} = 1$ (as m is even), and the function

$$|P_n'(x)| = mn(1-x^m)^{n-1}|x|^{m-1}$$

achieves its maximum on $[-1,1]$ at the point $a \in (0,1)$, where

$$a^m = \frac{m-1}{mn-1} \le \frac{1}{n}\,.$$

Hence

$$|P_n'(a)| \le mna^{m-1} \le mnn^{1/m-1} \le mn^\varepsilon \le m^{1-\varepsilon}(mn)^\varepsilon \le (1/\varepsilon+2)^{1-\varepsilon}(mn)^\varepsilon\,.$$

3. Lemmas

Our proof of Theorem 2.2 is based on the following two non-trivial results. Lemma 3.1 below is proved in [17].

Lemma 3.1. *If* $Q \in \mathcal{F}_{n,0}^c$ *and*

$$E_\delta := \left\{ x \in [-1,1] : \left| \frac{Q'(x)}{Q(x)} \right| \leq n\delta \right\}, \qquad \delta > 0,$$

then

$$m(E_\delta) < A\delta, \qquad \delta > 0,$$

where $A := 70e$ *is a suitable choice.*

Lemma 3.2 below was first proved in [24]. Its proof may also be found in [4, Section 7.2] with the larger constant $B = 8\sqrt{2}$.

Lemma 3.2. *If* $R \in \mathcal{P}_k^c$ *and*

$$F_\alpha := \left\{ x \in \mathbb{R} : \left| \frac{R'(x)}{R(x)} \right| \geq \alpha \right\}, \qquad \alpha > 0,$$

then

$$m(F_\alpha) \leq \frac{Bk}{\alpha}, \qquad \alpha > 0,$$

where $B := 2e$ *is a suitable choice.*

To prove Theorem 2.4 we need the following two lemmas. Lemma 3.3 below is stated and proved as Theorem 2.1 in [7] by using deep results from [2] and [3]. Recall that $\mathcal{P}_{n-k,k}$, $0 \leq k \leq n$, denotes the set of all algebraic polynomials with real coefficients, of degree at most n having at least $n-k+1$ zeros at 0.

Lemma 3.3. *There are absolute constants* $c_3 > 0$ *and* $c_4 > 0$ *such that*

$$c_3 \frac{n-k}{k} \leq \min_{0 \not\equiv P \in \mathcal{P}_{n-k,k}} \frac{\|P'\|_{[0,1]}}{V_0^1(P)} \leq \min_{0 \not\equiv P \in \mathcal{P}_{n-k,k}} \frac{\|P'\|_{[0,1]}}{|P(1)|} \leq c_4 \frac{n}{k}$$

for all integers $1 \leq k \leq n-1$. *Here* $c_3 = 1/12$ *is a suitable choice.*

Lemma 3.4 below follows directly from Lemma 3.2 in [7].

Lemma 3.4. *Let* $1 \leq k \leq n/11$ *and let* $S(x) := x^{n-k}R(x)$ *with* $R \in \mathcal{P}_k$. *We have*

$$|S(x)| < \|S\|_{[0,1]}, \qquad x \in \left[0, 1 - \frac{10k}{n-k}\right].$$

Lemma 3.5 below follows simply from Lemma 3.4.

Lemma 3.5. *Let $1 \le k \le (n-10)/20$ and let $W(x) := (1-x)^{n-k}V(x)$ with $0 \not\equiv V \in \mathcal{P}_k$. We have*

$$|y^{1/2}W(y)| < \|u^{1/2}W(u)\|_{[0,1]}\,, \qquad y \in \left[\frac{10(2k+1)}{n}, 1\right]\,.$$

Proof of Lemma 3.5. Replacing n by $2n+1$ and k by $2k+1$ in Lemma 3.4 we obtain that

$$(3.1) \qquad |S(x)| < \|S\|_{[0,1]}\,, \qquad x \in \left[0, 1 - \frac{10(2k+1)}{n}\right] \subset \left[0, 1 - \frac{10(2k+1)}{2n-2k}\right]\,,$$

whenever $1 \le k \le (n-10)/20 \le n/2$ and $S(x) := x^{2n-2k}R(x)$ with $R \in \mathcal{P}_{2k+1}$. Replacing the variable x by $1-x$ in (3.1) yields that

$$(3.2) \qquad |S(x)| < \|S\|_{[0,1]}\,, \qquad x \in \left[\frac{10(2k+1)}{n}, 1\right]\,,$$

whenever $1 \le k \le (n-10)/20$ and $S(x) := (1-x)^{2n-2k}R(x)$ with $R \in \mathcal{P}_{2k+1}$. Now let $1 \le k \le (n-10)/20$ and let $W(x) := (1-x)^{n-k}V(x)$ with $0 \not\equiv V \in \mathcal{P}_k$. Applying (3.2) to S defined by

$$S(x) = xW(x)^2 = (1-x)^{2n-2k}(xV(x)^2)\,, \qquad V \in \mathcal{P}_k\,,$$

we get the conclusion of the lemma. □

4. Proof of the Theorems

Proof of Theorem 2.2. Let $0 \not\equiv P \in \mathcal{F}^c_{n,k}$, that is, $P = QR$, where $Q \in \mathcal{F}^c_{n-k,0}$ and $R \in \mathcal{P}^c_k$. We have

$$(4.1) \qquad \frac{P'}{P} = \frac{Q'}{Q} + \frac{R'}{R}\,.$$

By Lemma 3.1 we have

$$(4.2) \qquad m(E_\delta) < A\delta, \qquad \delta > 0, \qquad A := 70e\,,$$

where

$$(4.3) \qquad E_\delta := \left\{x \in [-1,1] : \left|\frac{Q'(x)}{Q(x)}\right| \le (n-k)\delta\right\}\,, \qquad \delta > 0\,.$$

By Lemma 3.2 we have

$$(4.4) \qquad m(F_\delta) \le B\delta, \qquad \delta > 0, \quad B := 2e\,,$$

where

$$(4.5) \qquad F_\delta := \left\{x \in [-1,1] : \left|\frac{R'(x)}{R(x)}\right| \ge \frac{k}{\delta}\right\}\,, \qquad \delta > 0\,.$$

Now we choose $\delta > 0$ such that

$$\frac{k}{\delta} = \frac{1}{2}(n-k)\delta\,, \tag{4.6}$$

that is,

$$\delta := \left(\frac{2k}{n-k}\right)^{1/2}. \tag{4.7}$$

Then, combining (4.1)–(4.7), we can deduce that

$$\left|\frac{P'(x)}{P(x)}\right| \geq \left|\frac{Q'(x)}{Q(x)}\right| - \left|\frac{R'(x)}{R(x)}\right| \geq (n-k)\delta - \frac{k}{\delta} = \left(\frac{(n-k)k}{2}\right)^{1/2}, \quad x \in [-1,1] \setminus H_\delta\,, \tag{4.8}$$

where $H_\delta := E_\delta \cup F_\delta$ with

$$m(H_\delta) < (A+B)\delta = 72e\delta\,. \tag{4.9}$$

Note that

$$1 \leq k \leq \frac{n}{100000}$$

implies that

$$72e\delta = 72e\left(\frac{2k}{n-k}\right)^{1/2} \leq 72e\left(\frac{2}{99999}\right)^{1/2} < 1\,. \tag{4.10}$$

Choose an $x_0 \in [-1,1]$ such that $|P(x_0)| := \|P\|_{[-1,1]}$. It follows from (4.10) that the length of the interval $[x_0 - 72e\delta, x_0 + 72e\delta] \cap [-1,1]$ is at least $72e\delta$, and hence (4.9) implies that there is a

$$y \in [x_0 - 72e\delta, x_0 + 72e\delta] \cap [-1,1] \tag{4.11}$$

such that

$$y \in [-1,1] \setminus H_\delta\,. \tag{4.12}$$

If

$$|P(y)| \geq \frac{1}{2}\|P\|_{[-1,1]}\,, \tag{4.13}$$

then combining (4.12), (4.8) and (4.13), we obtain

$$\begin{aligned}\|P'\|_{[-1,1]} \geq& |P'(y)| \geq \left(\frac{1}{2}(n-k)k\right)^{1/2}|P(y)| \\ &\geq \left(\frac{1}{2}(n-k)k\right)^{1/2}\frac{1}{2}\|P\|_{[-1,1]} \geq \frac{1}{144e}\left(\frac{n-k}{2k}\right)^{1/2}\|P\|_{[-1,1]}\,,\end{aligned}$$

and the theorem follows. If (4.13) does not hold, that is, $|P(y)| < \frac{1}{2}\|P\|_{[-1,1]}$, then it follows from the Mean Value Theorem and (4.11) that there is a value ξ in the open interval between y and x_0 such that

$$\begin{aligned}\|P'\|_{[-1,1]} \geq & |P'(\xi)| \geq \left|\frac{P(y)-P(x_0)}{y-x_0}\right| \geq \frac{1}{2}\|P\|_{[-1,1]}|y-x_0|^{-1} \\ \geq & (144e\delta)^{-1}\|P\|_{[-1,1]} = \frac{1}{144e}\left(\frac{n-k}{2k}\right)^{1/2}\|P\|_{[-1,1]}\,,\end{aligned}$$

and the theorem follows. □

Proof of Corollary 2.3. Let $1 \leq k \leq n$. Suppose $0 \not\equiv P \in \mathcal{F}^c_{n,k}$ has at least one zero in $[-1,1]$. Choose $a, b \in [-1,1]$ such that $P(a) = 0$, and $|P(b)| = \|P\|_{[-1,1]}$. By the Mean Value Theorem there is a $\xi \in (-1,1)$ between a and b such that

$$\|P'\|_{[-1,1]} \geq |P'(\xi)| \geq \left|\frac{P(b)-P(a)}{b-a}\right| \geq \frac{1}{2}\|P\|_{[-1,1]}\,. \tag{4.14}$$

If $1 \leq k \leq \dfrac{n}{100000}$, the result follows from Theorem 2.2 and (4.14) as $1/448 \leq (144e)^{-1}$. If $\dfrac{n}{100000} < k \leq n$, then

$$\frac{1}{448}\left(\frac{n-k}{2k}\right)^{1/2} \leq \frac{1}{448}\left(\frac{99999}{2}\right)^{1/2} < \frac{1}{2}\,,$$

and the result follows simply from (4.14). □

Proof of Theorem 2.4. For $k = n$ the polynomials $P = P_{n,n} \in \mathcal{F}^c_{2n,2n}$ defined by $P(x) := x$ show the theorem with $c_2 = 1$. Let $1 \leq k \leq n-1$. By the upper bound of Lemma 3.3 there is an absolute constant $c_4 > 0$ and there are polynomials

$$0 \not\equiv Q = Q_{n,k} \in \mathcal{P}_{n-k,k}$$

such that

$$\frac{\|Q'\|_{[0,1]}}{\|Q\|_{[0,1]}} \leq c_4\frac{n}{k}\,. \tag{4.15}$$

Let

$$0 \not\equiv R(x) = R_{n,k}(x) = Q(1-x)\,. \tag{4.16}$$

Obviously R is of the form

$$R(x) = (1-x)^{n-k+1}U(x)\,, \qquad U \in \mathcal{P}_{k-1}\,,$$

and R' is of the form

$$R'(x) = (1-x)^{n-k} V(x)\,, \qquad V \in \mathcal{P}_{k-1}\,, \tag{4.17}$$

Let $0 \not\equiv P = P_{n,k}$ be defined by $P(x) := R(x^2)$. Observe that P is of the form

$$P(x) = (1-x^2)^{n-k+1} U(x)\,, \qquad U \in \mathcal{P}^c_{2k-2}\,,$$

hence $P \in \mathcal{F}^c_{2n,2k}$. Observe that $P(x) := R(x^2)$ and (4.16) imply that

$$\|P\|_{[-1,1]} = \|R\|_{[0,1]} = \|Q\|_{[0,1]} \tag{4.18}$$

and

$$P'(x) = 2xR'(x^2)\,. \tag{4.19}$$

First assume that $1 \leq k \leq (n-10)/20$. Let $y := x^2$. Using (4.19), (4.17), $R' \not\equiv 0$, and Lemma 3.5, we obtain

$$|P'(x)| = |2xR'(x^2)| = |2y^{1/2}R'(y)| < \|2u^{1/2}R'(u)\|_{[0,1]} = \|P'\|_{[-1,1]}$$

for every $y = x^2 \in [10(2k+1)/n, 1]$, and hence there is an

$$a \in \left[0, \left(\frac{10(2k+1)}{n}\right)^{1/2}\right] \subset [0,1] \tag{4.20}$$

such that

$$|P'(a)| = \|P'\|_{[0,1]}\,. \tag{4.21}$$

Note that $1 \leq k \leq (n-10)/20$ implies that $a \in [0,1]$. Using (4.19), (4.21), (4.19) again, (4.20), (4.15), and (4.18), we obtain

$$\begin{aligned}
\|P'\|_{[-1,1]} &= \|P'\|_{[0,1]} = |P'(a)| = |2aR'(a^2)| \\
&\leq 2\left(\frac{10(2k+1)}{n}\right)^{1/2} \|R'\|_{[0,1]} = 2\left(\frac{10(2k+1)}{n}\right)^{1/2} \|Q'\|_{[0,1]} \\
&\leq 2\left(\frac{10(2k+1)}{n}\right)^{1/2} c_4 \frac{n}{k} \|Q\|_{[0,1]} \\
&\leq c_2 \left(\frac{n}{k}\right)^{1/2} \|Q\|_{[0,1]} = c_2 \left(\frac{n}{k}\right)^{1/2} \|P\|_{[-1,1]}
\end{aligned}$$

with the absolute constant $c_2 = 12c_4 > 0$.

Now assume that in addition to $1 \le k \le n-1$ we have $(n-10)/20 \le k \le n-1$. Hence $k \ge n/30$ also holds. Choose an $a \in [0,1]$ such that (4.21) holds. Using (4.19), (4.21), (4.19) again, (4.15), $k \ge n/30$, (4.18), and $1 \le k \le n$, we obtain

$$\begin{aligned}\|P'\|_{[-1,1]} &= \|P'\|_{[0,1]} = |P'(a)| = |2aR'(a^2)| \le 2\|R'\|_{[0,1]} = 2\|Q'\|_{[0,1]} \\ &\le 2c_4 \frac{n}{k}\, \|Q\|_{[0,1]} = 60c_4\|Q\|_{[0,1]} = 60c_4\|P\|_{[-1,1]} \le c_2 \left(\frac{n}{k}\right)^{1/2} \|P\|_{[-1,1]}\end{aligned}$$

with the absolute constant $c_2 = 60c_4 > 0$. □

Proof of Theorem 2.1. The case that $k = 0$ is the result of Komarov [20] mentioned in the Introduction, so we may assume that $1 \le k \le n$, in which cases the lower bound of the theorem follows immediately from Corollary 2.3. To see that $c_1 := 1/636$ can be chosen in the lower bound of the theorem we distinguish three cases. If $k = 0$, then Komarov's result mentioned in the Introduction gives the lower bound of the theorem with $c_1 := 1/636$ as

$$\frac{1}{636} < \frac{2}{3\sqrt{210e}}\,.$$

If $1 \le k \le n/318$, then Corollary 2.3 gives the lower bound of the theorem with $c_1 := 1/636$ as

$$\begin{aligned}\frac{1}{636}\left(\frac{n}{k+1}\right)^{1/2} &\le \frac{1}{636}\left(\frac{n}{k}\right)^{1/2} = \frac{1}{636}\left(\frac{2n}{n-k}\right)^{1/2}\left(\frac{n-k}{2k}\right)^{1/2} \\ &= \frac{\sqrt{2}}{636}\left(1+\frac{k}{n-k}\right)^{1/2}\left(\frac{n-k}{2k}\right)^{1/2} \\ &\le \frac{1}{449}\left(1+\frac{1}{317}\right)^{1/2}\left(\frac{n-k}{2k}\right)^{1/2} \\ &\le \frac{1}{448}\left(\frac{n-k}{2k}\right)^{1/2}.\end{aligned}$$

If $n/318 \le k \le n$, then $n/k \le 318$, and hence Corollary 2.3 gives the lower bound of the theorem with $c_1 := 1/636$ again as

$$\frac{1}{636}\left(\frac{n}{k+1}\right)^{1/2} \le \frac{1}{636}\left(\frac{n}{k}\right)^{1/2} \le \frac{1}{636}\sqrt{318} \le \frac{1}{2}\,.$$

To see the upper bound of the theorem let $f(n,k)$ defined by

$$f(n,k) := \min_{0 \not\equiv P \in \mathcal{F}^c_{n,k}} \frac{\|P'\|_{[-1,1]}}{\|P\|_{[-1,1]}}\,.$$

When $k = 0$ and $n = 2\nu$ is even the polynomial P defined by $P(x) = (x^2-1)^\nu$ shows the upper bound of the theorem. Observe that for a fixed positive integer n the function $f(n,k)$ is decreasing on the set of integers $0 \le k \le n$, and for a fixed integer $1 \le k \le n$ we have $f(n,k) \le f(n-1,k-1)$. So it is sufficient to show the upper bound of the theorem only for even numbers $n = 2\nu$ and $k = 2\kappa$ satisfying $1 \le \kappa \le \nu$ in which cases the upper bound of the theorem follows from Theorem 2.4. □

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