# Upper Bounds for the Derivative of Exponential Sums

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**ABSTRACT.** The equality

$$\sup_p \frac{|\, p'(a)\, |}{\|p\|_{[a,b]}} = \frac{2n^2}{b-a}$$

is shown, where the supremum is taken for all exponential sums $p$ of the form

$$p(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}, \qquad a_j \in \mathbf{R},$$

with nonnegative exponents $\lambda_j$. The inequalities

$$\|p'\|_{[a+\delta,b-\delta]} \leq 4(n+2)^3 \delta^{-1} \|p\|_{[a,b]}$$

and

$$\|p'\|_{[a+\delta,b-\delta]} \leq 4\sqrt{2}(n+2)^3 \delta^{-3/2} \|p\|_{L_2[a,b]}$$

are also proved for all exponential sums of the above form with arbitrary real exponents. These results improve inequalities of Lorentz and Schmidt and partially answer a question of Lorentz.

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## 1. Introduction and Notation

Let $\Lambda_n := \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\}, \quad \lambda_j \neq 0, \quad j = 1, 2 \ldots, n,$

$$E(\Lambda_n) = \{f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}, \quad a_j \in \mathbf{R}\}$$

and

$$E_n := \bigcup_{\Lambda_n} E(\Lambda_n) = \{f : f(t) = a_0 + \sum_{i=1}^{n} a_j e^{\lambda_j t}, \;\; a_j, \lambda_j \in \mathbf{R}\}.$$

We will use the norms

$$\|f\|_{[a,b]} := \max_{x \in [a,b]} |f(x)|$$

and

$$\|f\|_{L_2[a,b]} := \left( \int_a^b \mid f(x) \mid^2 \, dx \right)^{1/2}$$

for functions $f \in C[a, b]$.

Schmidt [3] proved that there is a constant $c(n)$ depending only on $n$ so that

$$\|p'\|_{[a+\delta, b-\delta]} \leq c(n) \delta^{-1} \|p\|_{[a,b]}$$

for every $p \in E_n$ and $\delta \in (0, (b-a)/2)$. Lorentz [2] improved Schmidt's result by showing that for every $\alpha > \frac{1}{2}$ there is a constant $c(\alpha)$ depending only on $\alpha$ so that $c(n)$ in the above inequality can be replaced by $c(\alpha) n^{\alpha \log n}$, and he speculated that there may be an absolute constant $c$ so that Schmidt's inequality holds with $c(n) = cn$. Theorem 2 of this paper shows that Schmidt's inequality holds with $c(n) = 4(n+2)^3$. Our first theorem establishes the sharp inequality

$$\mid p'(a) \mid \leq \frac{2n^2}{b-a} \|p\|_{[a,b]}$$

for every $p \in E_n$ with nonnegative exponents $\lambda_j$.

## 2. New Results

**Theorem 1.** *We have*

$$\sup_p \frac{\mid p'(a) \mid}{\|p\|_{[a,b]}} = \frac{2n^2}{b-a}$$

*for every $a < b$, where the supremum is taken for all exponential sums $p \in E_n$ with nonnegative exponents. The equality*

$$\sup_p \frac{\mid p'(a) \mid}{\|p\|_{[a,b]}} = \frac{2n^2}{a(\log b - \log a)}$$

*also holds for every $0 < a < b$, where the supremum is taken for all Müntz polynomials of the form*

$$p(x) = a_0 + \sum_{j=1}^{n} a_j x^{\lambda_j}, \qquad a_j \in \mathbf{R}, \quad \lambda_j \geq 0.$$

**Theorem 2.** *The inequalities*

$$\|p'\|_{[a+\delta,b-\delta]} \leq 4(n+2)^3 \, \delta^{-1} \|p\|_{[a,b]}$$

*and*

$$\|p'\|_{[a+\delta,b-\delta]} \leq 4\sqrt{2}(n+2)^3 \, \delta^{-3/2} \|p\|_{L_2[a,b]}$$

*hold for every $p \in E_n$ and $\delta \in (0,(b-a)/2)$.*

## 3. Proofs

To prove Theorem 1 we need some notation. If $\Lambda_n := \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\}$ is a set of positive real numbers then the real span of

$$\{1, x^{\lambda_1}, x^{\lambda_2}, \cdots x^{\lambda_n}\}, \qquad x \geq 0,$$

will be denoted by $M(\Lambda_n)$. It is well-known that these are Chebyshev spaces (see [1] for instance), so $M(\Lambda_n)$ possesses a unique Chebyshev "polynomial" $T_{\Lambda_n}$ on $[a,b]$, $0 < a < b$, with the properties

(i) $T_{\Lambda_n} \in M(\Lambda_n)$,

(ii) $\|T_{\Lambda_n}\|_{[a,b]} = 1$

and

(iii) there are $a = x_0 < x_1 < \cdots < x_n = b$ so that

$$T_{\Lambda_n}(x_j) = (-1)^j, \qquad j = 0,1,\cdots,n.$$

It is routine to prove (see [1] again) that $T_{\Lambda_n}$ has exactly $n$ distinct zeros on $(a,b)$,

$$\max_{0 \not\equiv p \in M(\Lambda_n)} \frac{|\, p'(a) \,|}{\|p\|_{[a,b]}} = \frac{|\, T'_{\Lambda_n}(a) \,|}{\|T_{\Lambda_n}\|_{[a,b]}} = |\, T'_{\Lambda_n}(a) \,| \tag{1}$$

and

$$\max_{0 \not\equiv p \in M(\Lambda_n)} \frac{|\, p(0) \,|}{\|p\|_{[a,b]}} = \frac{|\, T_{\Lambda_n}(0) \,|}{\|T_{\Lambda_n}\|_{[a,b]}} = |\, T_{\Lambda_n}(0) \,| \,. \tag{2}$$

**Lemma 3.** *Let*

$$\Lambda_n := \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\} \qquad \textit{and} \qquad \Gamma_n := \{\gamma_1 < \gamma_2 < \cdots < \gamma_n\}$$

*be so that* $0 < \lambda_j \le \gamma_j$ *for each* $j = 1, 2, \cdots, n$. *Then*

$$| T'_{\Gamma_n}(a) | \le | T'_{\Lambda_n}(a) | . \tag{3}$$

**Proof.** Without loss of generality we may assume that there is an index $m, \quad 1 \le m \le n$, so that $\lambda_m < \gamma_m$ and $\lambda_j = \gamma_j$ if $j \ne m$, since repeated applications of the result in this situation give the lemma in the general case. First we show that

$$| T_{\Gamma_n}(0) | < | T_{\Lambda_n}(0) | . \tag{4}$$

Indeed, let $R_{\Gamma_n} \in M(\Gamma_n)$ interpolate $T_{\Lambda_n}$ at the zeros of $T_{\Lambda_n}$, and be normalized so that $R_{\Gamma_n}(0) = T_{\Lambda_n}(0)$. Then the Improvement Theorem of Pinkus and Smith [4, Theorem 2] yields

$$| R_{\Gamma_n}(x) | \le | T_{\Lambda_n}(x) | \le 1, \qquad x \in [a, b].$$

Hence, using (2) with $\Lambda_n$ replaced by $\Gamma_n$, we obtain

$$| T_{\Lambda_n}(0) | = | R_{\Gamma_n}(0) | \le | T_{\Gamma_n}(0) |,$$

which proves (4). Using the defining properties of $T_{\Lambda_n}$ and $T_{\Gamma_n}$, we deduce that $T_{\Lambda_n} - T_{\Gamma_n}$ has at least $n+1$ zeros in $[a, b]$ (we count every zero without sign change twice). Now assume that (3) does not hold, then

$$| T'_{\Lambda_n}(a) | > | T'_{\Gamma_n}(a) | .$$

This, together with (4), implies that $T_{\Lambda_n} - T_{\Gamma_n}$ has at least one zero in $(0, a)$. Hence $T_{\Lambda_n} - T_{\Gamma_n}$ has at least $n + 2$ zeros in $(0, b]$. This is a contradiction, since

$$T_{\Lambda_n} - T_{\Gamma_n} \in \text{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \cdots, x^{\lambda_n}, x^{\gamma_m}\},$$

and every function from the above span can have only at most $n + 1$ zeros in $(0, \infty)$ (see [3]). □

**Proof of Theorem 1.** It is sufficient to prove only the second statement of the theorem, the first one can be obtained by the change of variable $x = e^t$. We obtain from (1) and Lemma 3 that

$$\frac{| p'(a) |}{\|p\|_{[a,b]}} \le \lim_{\delta \to 0+} \frac{| T'_{\Lambda_{n,\delta}}(a) |}{\|T_{\Lambda_{n,\delta}}\|_{[a,b]}} = \lim_{\delta \to 0+} | T'_{\Lambda_{n,\delta}}(a) |$$

for every $p$ of the form

$$p(x) = a_0 + \sum_{j=1}^{n} a_j x^{\lambda_j}, \qquad a_j \in \mathbf{R}, \quad \lambda_j > 0,$$

where

$$\Lambda_{n,\delta} := \{\delta, 2\delta, 3\delta, \cdots, n\delta\}$$

and $T_{n,\delta}$ is the Chebyshev "polynomial" of $M(\Lambda_{n,\delta})$ on $[a,b]$. From the definition and uniqueness of $T_{\Lambda_{n,\delta}}$ it follows that

$$T_{\Lambda_{n,\delta}}(x) = T_n \left( \frac{2}{b^\delta - a^\delta} x^\delta - \frac{b^\delta + a^\delta}{b^\delta - a^\delta} \right),$$

where $T_n(y) := \cos(n \arccos y)$. Therefore

$$\begin{aligned} \mid T'_{\Lambda_{n,\delta}}(a) \mid &= \mid T'_n(-1) \mid \frac{2}{b^\delta - a^\delta} \delta a^{\delta-1} \\ &= \frac{2n^2}{\delta^{-1}(b^\delta - 1) - \delta^{-1}(a^\delta - 1)} a^{\delta-1} \underset{\delta \to 0+}{\longrightarrow} \frac{2n^2}{a(\log b - \log a)} \end{aligned}$$

and the theorem is proved. □

To prove Theorem 2 we need two lemmas.

**Lemma 4.** *For every set $\Lambda_n := \{\lambda_1 < \lambda_2 < \ldots \lambda_n\}$ of nonzero real numbers there is a point $y \in [-1,1]$ depending only on $\Lambda_n$ so that*

$$\mid p'(y) \mid \leq 2(n+2)^3 \|p\|_{L_2[-1,1]}$$

*for every $p \in E(\Lambda_n)$.*

**Proof.** Take the orthonormal set $\{p_k\}_{k=0}^n$ on $[-1,1]$ defined by

(i) $p_k \in \operatorname{span}\{1, e^{\lambda_1 t}, e^{\lambda_2 t}, \cdots, e^{\lambda_k t}\}, \qquad k = 0, 1, \ldots, n,$

(ii) $\int_{-1}^{1} p_i p_j = \delta_{i,j}, \qquad 0 \leq i \leq j \leq n.$

Writing $p \in E(\Lambda_n)$ as a linear combination of the functions $p_k$, $k = 0, 1, \cdots, n$, and using the Cauchy-Schwartz inequality and the orthonormality of $\{p_k\}_{k=0}^n$ on $[-1,1]$, we obtain in a standard fashion that

$$\max_{p \in E(\Lambda_n)} \frac{\mid p'(t_0) \mid}{\|p\|_{L_2[-1,1]}} = \left( \sum_{k=0}^{n} p'_k(t_0)^2 \right)^{1/2}, \qquad t_0 \in \mathbf{R}.$$

Let

$$A_k := \{t \in [-1,1] \ : \mid p_k(t) \mid \geq (n+1)^{1/2}\}, \qquad k = 0, 1, \cdots, n$$

and

$$B_k := \{t \in [-1,1] \backslash A_k \ : \mid p_k'(t) \mid \geq 2(n+2)^{5/2}\}, \qquad k = 0, 1, \cdots, n.$$

Since $\int\limits_{-1}^{1} p_k^2 = 1$, we have

$$m(A_k) \leq (n+1)^{-1}, \qquad k = 0, 1, \cdots, n.$$

Since $\text{span}\{1, e^{\lambda_1 t}, e^{\lambda_2 t}, \cdots, e^{\lambda_k t}\}$ is a Chebyshev system, each $\tilde{A}_k := [-1,1] \backslash A_k$ comprises of at most $k+1$ intervals, and each $B_k$ comprises of at most $2(k+1)$ intervals. Therefore

$$2(n+2)^{5/2} \ m(B_k) \leq \int_{B_k} \mid p_k'(t) \mid dt \leq 4(k+1)\sqrt{n+1},$$

whence

$$\sum_{k=0}^{n} m(B_k) \leq \frac{2\sqrt{n+1}}{(n+2)^{5/2}} \ \frac{(n+1)(n+2)}{2} < 1.$$

Now let

$$A := [-1,1] \backslash \bigcup_{k=0}^{n} (A_k \cup B_k).$$

Then

$$\begin{aligned} m(A) \ &\geq \ 2 - \sum_{k=0}^{n} m(A_k) - \sum_{k=0}^{n} m(B_k) \\ &> \ 2 - (n+1)(n+1)^{-1} - 1 > 0, \end{aligned}$$

so there is a point $y \in A \subset [-1,1]$, where

$$\mid p'(y) \mid \leq 2(n+1)^{5/2}, \qquad k = 0, 1, \cdots, n,$$

hence

$$\left( \sum_{k=0}^{n} p_k'(y)^2 \right)^{1/2} \leq 2(n+2)^3,$$

and the lemma is proved. □

**Lemma 5.** *We have*

$$\mid p'(0) \mid \leq 2(n+2)^3 \|p\|_{L_2[-2,2]} \leq 2(n+2)^3 \|p\|_{[-2,2]}$$

*for every* $p \in E_n$.

**Proof.** Let $\Lambda_n := \{\lambda_1 < \lambda_2 < \cdots, \lambda_n\}$ be a fixed set of nonzero real numbers, and let $y \in [-1, 1]$ be chosen by Lemma 4. Let $0 \not\equiv p \in E(\Lambda_n)$. Then

$$q(t) := p(t - y) \in E(\Lambda_n),$$

therefore, applying Lemma 4 to $q$, we obtain

$$\frac{|\, p'(0) \,|}{\|p\|_{L_2[-2,2]}} \leq \frac{|\, p'(0) \,|}{\|p\|_{L_2[-1-y,1-y]}} = \frac{|\, q'(y) \,|}{\|q\|_{L_2[-1,1]}} \leq 2(n+2)^3,$$

and the lemma is proved. □

**Proof of Theorem 2.** Let $t_0 \in [a + \delta, b - \delta]$. Applying Lemma 5 to $q(t) := p(\delta t/2 + t_0)$, we get the theorem. □

## References


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