THE MULTIPLICITY OF THE ZERO AT 1 OF POLYNOMIALS WITH CONSTRAINED COEFFICIENTS

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Abstract. For $n \in \mathbb{N}$, $L > 0$, and $p \geq 1$ let $\kappa_p(n,L)$ be the largest possible value of k for which there is a polynomial $P \neq 0$ of the form

$$P(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_0| \geq L\left(\sum_{j=1}^{n} |a_j|^p\right)^{1/p}, \quad a_j \in \mathbb{C}\,,$$

such that $(x-1)^k$ divides $P(x)$. For $n \in \mathbb{N}$ and $L > 0$ let $\kappa_\infty(n,L)$ be the largest possible value of k for which there is a polynomial $P \neq 0$ of the form

$$P(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_0| \geq L \max_{1 \leq j \leq n} |a_j|\,, \quad a_j \in \mathbb{C}\,,$$

such that $(x-1)^k$ divides $P(x)$.

We prove that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1\sqrt{n/L} - 1 \leq \kappa_\infty(n,L) \leq c_2\sqrt{n/L}$$

for every $L \geq 1$. The above result complements an earlier result of the authors showing that there is an absolute constant $c_3 > 0$ such that

$$\min\left\{\frac{1}{6}\sqrt{(n(1-\log L)} - 1\,, n\right\} \leq \kappa_\infty(n,L) \leq \min\left\{c_3\sqrt{n(1-\log L)}\,, n\right\}$$

for every $n \in \mathbb{N}$ and $L \in (0,1]$. Essentially sharp results on the size of $\kappa_2(n,L)$ are also proved.

1. Notation

For $n \in \mathbb{N}$, $L > 0$, and $p \geq 1$ we define the following numbers. Let $\kappa_p(n,L)$ be the largest possible value of k for which there is a polynomial $P \neq 0$ of the form

$$P(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_0| \geq L\left(\sum_{j=1}^{n} |a_j|^p\right)^{1/p}, \quad a_j \in \mathbb{C}\,,$$

2010 Mathematics Subject Classifications. 11C08, 41A17, 26C10, 30C15 such that $(x-1)^k$ divides $P(x)$. For $n \in \mathbb{N}$ and $L > 0$ let $\kappa_\infty(n,L)$ the largest possible value of k for which there is a polynomial $P \neq 0$ of the form

$$P(x) = \sum_{j=0}^{n} a_j x^j \,, \qquad |a_0| \geq L \max_{1 \leq j \leq n} |a_j| \,, \quad a_j \in \mathbb{C} \,,$$

such that $(x-1)^k$ divides $P(x)$. In [3] we proved that there is an absolute constant $c_3 > 0$ such that

$$\min\left\{\frac{1}{6}\sqrt{(n(1-\log L)} - 1 \,, n\right\} \leq \kappa_\infty(n,L) \leq \min\left\{c_3\sqrt{n(1-\log L)} \,, n\right\}$$

for every $n \in \mathbb{N}$ and $L \in (0,1]$. However, we were far from being able to establish the right result in the case of $L \geq 1$. It is our goal in this paper to prove the right order of magnitude of $\kappa_\infty(n,L)$ and $\kappa_2(n,L)$ in the case of $L \geq 1$. Our results in [3] have turned out to be related to a number of recent papers from a rather wide range of research areas. See [1,6-15,17-23], for example.

2. New Results

We extend some of our main results in [3] to the case $L \geq 1$. Our main result is the following.

Theorem 2.1. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1\sqrt{n/L} - 1 \leq \kappa_\infty(n,L) \leq c_2\sqrt{n/L}$$

for every $n \in \mathbb{N}$ and $L \geq 1/2$.

To prove the above theorem, its lower bound, in particular, requires some subtle new ideas. An interesting connection to number theory is explored. Namely, the fact that the density of square free integers is positive (in fact, it is $\pi^2/6$), appears in our proof in an elegant fashion.

While we consider Theorem 2.1 to be our main result in this paper we also prove the following.

Theorem 2.2. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1\sqrt{n}/L - 1 \leq \kappa_2(n,L) \leq c_2\sqrt{n}/L$$

for every $n \in \mathbb{N}$ and $L > 2^{-1/2}$, and

$$\min\left\{c_1\sqrt{n(-\log L)} - 1 \,, n\right\} \leq \kappa_2(n,L) \leq \min\left\{c_2\sqrt{n(-\log L)} \,, n\right\}$$

for every $n \in \mathbb{N}$ and $L \in (0, 2^{-1/2}]$.

We think that the right result on the size of $\kappa_2(n,L)$ offered by Theorem 2.2 is also of some interest.

3. Lemmas

In this section we list our lemmas needed in the proofs of Theorems 2.1 and 2.2. These lemmas are proved in Section 4.

Lemma 3.1. *For any $L \geq 1$ there are polynomials P_n of the form*

$$P_n(x) = \sum_{j=0}^{n} a_{j,n} x^j \,, \qquad a_{j,n} \in \mathbb{R}\,, \quad a_{0,n} = \frac{6L}{\pi^2} + o(L)\,,$$

$$|a_{j,n}| \leq 2, \quad j = 1, 2, \dots, n\,, \quad n = 1, 2, \dots\,,$$

such that P_n has at least $\lfloor \sqrt{n/L} \rfloor$ zeros at 1.

Lemma 3.2. *For any $L > 0$ there are polynomials P_n of the form*

$$P_n(x) = \sum_{j=0}^{n} a_{j,n} x^j \,, \qquad a_{j,n} \in \mathbb{R}\,, \quad a_{0,n} = 1\,, \quad \sum_{j=1}^{n} a_{j,n}^2 \leq L^{-2}\,, \quad n = 1, 2, \dots\,,$$

such that
(a) P_n has at least $\lfloor \frac{1}{4}\sqrt{n}/L \rfloor$ zeros at 1 if $2^{-1/2} \leq L$.
(b) P_n has at least $\lfloor \frac{1}{4}\sqrt{n(-\log L)} \rfloor$ zeros at 1 if $4^{-n} \leq L \leq 2^{-1/2}$.
(c) P_n has at least n zeros at 1 if $0 < L \leq 4^{-n}$.

To prove Lemma 3.2 our tool is the next lemma due to Halász [24]. Let $\mathcal{P}_m$ denote the collection of all polynomials of degree at most m with real coefficients. Let $\mathcal{P}_m^c$ denote the collection of all polynomials of degree at most m with complex coefficients.

Lemma 3.3. *For every $m \in \mathbb{N}$, there exists a polynomial $Q_m \in \mathcal{P}_m$ such that*

$$Q_m(0) = 1\,, \qquad Q_m(1) = 0\,, \qquad |Q_m(z)| < e^{2/m}, \qquad |z| \leq 1\,.$$

The observation below is well known, easy to prove, and recorded in several papers. See [3], for example.

Lemma 3.4. *Let $P \neq 0$ be a polynomial of the form $P(x) = \sum_{j=0}^{n} a_j x^j$. Then $(x-1)^k$ divides P if and only if $\sum_{j=0}^{n} a_j Q(j) = 0$ for all polynomials $Q \in \mathcal{P}_{k-1}^c$.*

For $n \in \mathbb{N}$, $1 < q \leq \infty$, and $L > 0$ we define the following numbers. Let $\mu_q(n, L)$ be the smallest value of k for which there is a polynomial of degree k with complex coefficients such that

$$|Q(0)| > \frac{1}{L} \left(\sum_{j=1}^{n} |Q(j)|^q \right)^{1/q} .$$

Let $\mu_\infty(n, L)$ be the smallest value of k for which there is a polynomial of degree k with complex coefficients such that

$$|Q(0)| > \frac{1}{L} \max_{1 \leq j \leq n} |Q(j)|\,.$$

Our next lemma is a simple consequence of Hölder's inequality.

Lemma 3.5. *Let $1 \le p, q \le \infty$ and $1/p + 1/q = 1$. Then for every $n \in \mathbb{N}$ and $L > 0$, we have*

$$\kappa_p(n, L) \le \mu_q(n, L)\,.$$

The next lemma is stated as Lemma 3.4 and proved in [18].

Lemma 3.6. *For arbitrary real numbers $A, M > 0$, there exist a polynomial g such that $f = g^2$ is a polynomial of degree*

$$m < \sqrt{\pi}\sqrt{A}\sqrt[4]{M} + 2$$

with real coefficients such that $f(0) = M$ and

$$|f(x)| \le \min\left\{M, \frac{1}{x^2}\right\}\,, \qquad x \in (0, A]\,.$$

We also need Lemma 5.7 from [3] which may be stated as follows.

Lemma 3.7. *Let n and R be a positive integers with $1 \le R \le \sqrt{n}$. Then there exists a polynomial $f \in \mathcal{P}_m$ with*

$$m \le 4\sqrt{n} + \tfrac{9}{7}R\sqrt{n} + R + 4 \le \tfrac{44}{7}R\sqrt{n} + 4$$

such that

$$f(1) = f(2) = \cdots = f(R^2) = 0$$

and

$$|f(0)| > \exp(R^2)\big(|f(R^2+1)| + |f(R^2+2)| + \cdots + |f(n)|\big) \ge \exp(R^2)\left(\sum_{j=1}^{n}|f(j)|^2\right)^{1/2}\,.$$

Lemmas 3.6 and 3.7 imply the following results needed in the proof of Theorems 2.1 and 2.2.

Lemma 3.8. *For every $n \in \mathbb{N}$ and $0 < K \le \exp(n - 2\sqrt{n})$, there exists a polynomial h of degree m with real coefficients satisfying*

$$|h(0)| > K\sum_{j=1}^{n}|h(j)|\,, \quad \textit{and} \quad m \le \begin{cases} c_4\sqrt{nK}\,, & K < 2\,,\\ c_4\sqrt{n\log K}\,, & K \ge 2\,,\end{cases}$$

with an absolute constant $c_4 > 0$.

Lemma 3.9. *For every $n \in \mathbb{N}$ and $0 < K \le \exp(n - 2\sqrt{n})$, there exists a polynomial H of degree m with real coefficients satisfying*

$$|H(0)| > \sqrt{K}\left(\sum_{j=1}^{n}|H(j)|^2\right)^{1/2}\,, \quad \textit{and} \quad m \le \begin{cases} c_5\sqrt{nK}\,, & K < 2\,,\\ c_5\sqrt{n\log K}\,, & K \ge 2\,,\end{cases}$$

with an absolute constant $c_5 > 0$.

4. Proofs

Proof of Lemma 3.1. Following page 138 of [2] we define

$$H_m(x) := \frac{(m!)^2}{2\pi i} \int_\Gamma \frac{x^t\, dt}{\prod_{k=0}^m (t-k^2)}\,, \qquad m = 0, 1, \dots\,, \qquad x \in (0,\infty)\,,$$

where the simple closed contour Γ surrounds the zeros of the denominator of the integrand. Then H_m is a polynomial of degree m^2 with a zero at 1 with multiplicity at least m. (This can be seen easily by repeated differentiation and then evaluation of the above contour integral by expanding the contour to infinity.) Also, by the residue theorem,

$$H_m(x) = 1 + \sum_{k=1}^m c_{k,m} x^{k^2}\,,$$

where

$$c_{k,m} = \frac{(-1)^m (m!)^2}{\prod_{j=0, j\neq k}^m (k^2 - j^2)} = \frac{(-1)^k 2 (m!)^2}{(m-k)!(m+k)!}\,.$$

It follows that each $c_{k,m}$ is real and

$$|c_{k,m}| \le 2\,, \qquad k = 1, 2, \dots, m\,.$$

Let S_L be the collection of all square free integers in $[1, L]$. Let $m := \lfloor \sqrt{n/L} \rfloor$. We define

$$P_n(x) := \sum_{j \in S_L} H_m(x^j)\,.$$

Then P_n is of the form

$$P_n(x) = \sum_{j=0}^n a_{j,n} x^j\,, \qquad a_{j,n} \in \mathbb{R}\,, \quad j = 0, 1, \dots, n\,.$$

Since $ju^2 \neq lv^2$ whenever $j, l \in S_L$, $j \neq l$, and $u, v \in \{1, 2, \dots, m\}$, we have

$$|a_{j,n}| \le 2, \quad j = 1, 2, \dots, n\,.$$

Also, $a_{0,n} = |S_L|$, where $|S_L|$ denotes the number of the elements in S_L, and it is well known that

$$|S_L| = \frac{6L}{\pi^2} + o(L)\,,$$

see [16, pp. 267-268], for example. Finally, observe that each term in P_n has a zero at 1 with multiplicity at least $m = \lfloor \sqrt{n/L} \rfloor$ zeros at 1, and hence so does P_n. $\square$

Proof of Lemma 3.2. (a) Let $2^{-1/2} \le L$. We define $k := \lfloor \frac{1}{4}\sqrt{n}/L \rfloor$ and $m := \lfloor 4\sqrt{n}L \rfloor$. Observe that $m \ge 1$ holds. Let $P_n := Q_m^k \in \mathcal{P}_n$, where $Q_m \in \mathcal{P}_m$ is a polynomial satisfying the properties of Lemma 3.3. Then

$$P_n(x) = \sum_{j=0}^{n} a_{j,n} x^j \,, \qquad a_{j,n} \in \mathbb{R} \,, \quad j = 0, 1, \ldots, n \,,$$

has at least $k = \lfloor \frac{1}{4}\sqrt{n}/L \rfloor$ zeros at 1. Clearly, $a_{0,n} = P_n(0) = 1$, and since $k \le \frac{1}{4}\sqrt{n}/L$ and $m \ge 2\sqrt{n}L$, we have

$$|P_n(z)| < \exp(2k/m) \le \exp\left(\frac{1}{4L^2}\right) , \qquad |z| \le 1 \,.$$

Hence, it follows from the Parseval formula that

$$\sum_{j=1}^{n} a_{j,n}^2 = \left(\frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{it})|^2 \, dt\right) - 1 \le \exp\left(\frac{1}{2L^2}\right) - 1 \le \frac{1}{L^2} \,.$$

In the last step we used the inequality $e^x \le 1 + 2x$ valid for $x \in [0,1]$ with $x = \dfrac{1}{2L^2}$.

(b) Let $2^{-1/2} \le L \le 4^{-n}$. Let $k := \lfloor \frac{1}{2}\sqrt{n(-\log L)} \rfloor$ and $m := \lfloor 2\sqrt{n/(-\log L)} \rfloor$. Observe that $m \ge 1$ holds. Let $P_n := Q_m^k \in \mathcal{P}_n$, where $Q_m \in \mathcal{P}_m$ is a polynomial satisfying the properties of Lemma 3.3. Then

$$P_n(x) = \sum_{j=0}^{n} a_{j,n} x^j \,, \qquad a_{j,n} \in \mathbb{R} \,, \quad j = 0, 1, \ldots, n \,,$$

has at least $k = \lfloor \frac{1}{2}\sqrt{n(-\log L)} \rfloor$ zeros at 1. Clearly, $a_{0,n} = P_n(0) = 1$, and since $k \le \frac{1}{2}\sqrt{n(-\log L)}$ and $m \ge \sqrt{n/(-\log L)}$, we have

$$|P_n(z)| < \exp(2k/m) \le \exp(-\log L) = \frac{1}{L} \,, \qquad |z| \le 1 \,.$$

Hence, it follows from the Parseval formula that

$$\sum_{j=1}^{n} a_{j,n}^2 = \left(\frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{it})|^2 \, dt\right) - 1 \le \frac{1}{L^2} \,.$$

(c) Observe that the polynomial P_n defined by $P_n(z) = (z-1)^n$ has at least n zeros at 1, $P(0) = 1$, and

$$\sum_{j=1}^{n} a_{j,n}^2 = \sum_{j=0}^{n} \binom{n}{j}^2 \le 4^n \,.$$

□

Proof of Lemma 3.5. Let $m := \mu_q(n, L)$. Let Q be a polynomial of degree m with complex coefficients such that

$$|Q(0)| > \frac{1}{L} \left(\sum_{j=1}^{n} |Q(j)|^q \right)^{1/q} .$$

Now let P be a polynomial of the form

$$P(x) = \sum_{j=0}^{n} a_j x^j \,, \qquad |a_0| \geq L \left(\sum_{j=1}^{n} |a_j|^p \right)^{1/p} , \quad a_j \in \mathbb{C} \,.$$

It follows from Hölder's inequality that

$$\left| \sum_{j=1}^{n} a_j Q(j) \right| \leq \left(\sum_{j=1}^{n} |a_j|^p \right)^{1/p} \left(\sum_{j=1}^{n} |Q(j)|^q \right)^{1/q} < \frac{|a_0|}{L} L |Q(0)| = |a_0 Q(0)| \,.$$

Then $\sum_{j=0}^{n} a_j Q(j) \neq 0$, and hence Lemma 3.4 implies that $(x-1)^{m+1}$ does not divide P. We conclude that $\kappa_p(n, L) \leq m = \mu_q(n, L)$. $\square$

Proof of Lemma 3.8. Note that $h \equiv 1$ is a trivial choice in the case of $K > 1/n$. First we consider the case of $2 \leq K \leq \exp(n - 2\sqrt{n})$. Let $R := \lfloor \sqrt{\log K} \rfloor + 1$, and let $h := f$, where f is the polynomial given in Lemma 3.7 with this R. Then

$$|h(0)| > K \sum_{j=1}^{n} |h(j)| \,,$$

and the degree m of h satisfies

$$m \leq \tfrac{44}{7} R \sqrt{n} + 4 \leq c_6 \sqrt{n \log K}$$

with an absolute constants $c_6 > 0$.

Now let $1/n \leq K < 2$. Let f be the polynomial given in Lemma 3.6 with $A := n$ and $M := 9K^2$. Let $h := f$. Then

$$\begin{aligned} \sum_{j=1}^{n} |h(j)| = \sum_{j=1}^{n} |f(j)| &\leq \sum_{j \leq (3K)^{-1}} M + \sum_{j > (3K)^{-1}} \frac{1}{j^2} \\ &< \frac{1}{3K} 9K^2 + 2(3K) = 9K \,, \end{aligned}$$

$$|h(0)| = |f(0)| = M = 9K^2 \,,$$

and the degree m of h satisfies

$$m < \pi \sqrt{n} \sqrt[4]{M} + 2 < c_7 \sqrt{nK}$$

with an absolute constants $c_7 > 0$. $\square$

Proof of Lemma 3.9. Note that $H \equiv 1$ is a trivial choice again in the case of $K < 1/n$. In the case of $2 \le K \le \exp(n - 2\sqrt{n})$ let $H := h$, where the polynomial h is the same as in Lemma 3.8. Then

$$|H(0)| > K\left(\sum_{j=1}^{n} |H(j)|\right) \ge K\left(\sum_{j=1}^{n} |H(j)|^2\right)^{1/2} \ge \sqrt{K}\left(\sum_{j=1}^{n} |H(j)|^2\right)^{1/2},$$

and the degree m of H satisfies

$$m \le c_6\sqrt{n\log K}\,,$$

where $c_6 > 0$ is the same absolute constant as in the proof of Lemma 3.8.

Now let $1/n \le K < 2$. Let g be the polynomial such that $f = g^2$ is the polynomial given in Lemma 3.6 with $A := n$ and $M := 9K^2$. Let $H := g$. Then $h = H^2$, where the polynomial h is the same as in Lemma 3.8. Then

$$\left(\sum_{j=1}^{n} |H(j)|^2\right)^{1/2} = \left(\sum_{j=1}^{n} |h(j)|\right)^{1/2} < 3\sqrt{K}\,, \qquad |H(0)| = \sqrt{h(0)} = 3K\,,$$

and the degree m of H satisfies $m < \frac{1}{2}c_7\sqrt{nK}$, where $c_7 > 0$ is the same absolute constant as in the proof of Lemma 3.8. $\square$

Proof of Theorem 2.1. The upper bound follows from Lemmas 3.5 and 3.8. Indeed, Lemma 3.5 implies $\kappa_\infty(n,L) \le \mu_1(n,L)$, and it follows from Lemma 3.8 with $K = L^{-1}$ that $\mu_1(n,L) \le c_4\sqrt{n/L}$. The lower bound of the theorem follows directly from Lemma 3.1. $\square$

Proof of Theorem 2.2. The upper bound follows from Lemmas 3.5 and 3.9. Indeed, Lemma 3.5 implies $\kappa_2(n,L) \le \mu_2(n,L)$ and it follows from Lemma 3.9 with $K = L^{-2}$ that

$$\mu_2(n,L) \le \begin{cases} c_5\sqrt{n}/L, & L > 2^{-1/2}\,, \\ c_5\sqrt{n(-2\log L)}\,, & \exp(-n/2 + \sqrt{n}) \le L \le 2^{-1/2}\,. \end{cases}$$

Combining this with $\kappa_2(n,L) \le \mu_2(n,L)$ and the trivial estimate $\kappa_2(n,L) \le n$, the upper bound of the theorem follows. The lower bound of the theorem is a direct consequence of Lemma 3.2. $\square$

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