Improved lower bound for the number of unimodular zeros of self-reciprocal polynomials with coefficients in a finite set

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Abstract

Let $n_1 < n_2 < \cdots < n_N$ be non-negative integers. In a private communication Brian Conrey asked how fast the number of real zeros of the trigonometric polynomials $T_N(\theta) = \sum_{j=1}^N \cos(n_j\theta)$ tends to ∞ as a function of N. Conrey's question in general does not appear to be easy. Let $\mathcal{P}_n(S)$ be the set of all algebraic polynomials of degree at most n with each of their coefficients in S. For a finite set $S \subset \mathbb{C}$ let $M = M(S) := \max\{|z| : z \in S\}$. It has been shown recently that if $S \subset \mathbb{R}$ is a finite set and (P_n) is a sequence of self-reciprocal polynomials $P_n \in \mathcal{P}_n(S)$ with $|P_n(1)|$ tending to ∞, then the number of zeros of P_n on the unit circle also tends to ∞. In this paper we show that if $S \subset \mathbb{Z}$ is a finite set, then every self-reciprocal polynomial $P \in \mathcal{P}_n(S)$ has at least

$$c(\log\log\log|P(1)|)^{1-\varepsilon} - 1$$

zeros on the unit circle of $\mathbb{C}$ with a constant $c > 0$ depending only on $\varepsilon > 0$ and $M = M(S)$. Our new result improves the exponent $1/2 - \varepsilon$ in a recent result by Sahasrabudhe to $1 - \varepsilon$. Sahasrabudhe's new idea [66] is combined with the approach used in [34] offering an essentially simplified way to achieve our improvement. We note that in both Sahasrabudhe's paper and our paper the assumption that the finite set S contains only integers is deeply exploited.

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1 Introduction and Notation.

Research on the distribution of the zeros of algebraic polynomials has a long and rich history. In fact, most of the papers [1]–[74] in our list of references are just some of the papers devoted to this topic. The study of the number of real zeros of trigonometric polynomials and the number of unimodular zeros (that is, zeros lying on the unit circle of the complex plane) of algebraic polynomials with various constraints on their coefficients is the subject of quite a few of these. We do not try to survey these in our introduction.

Let $S \subset \mathbb{C}$. Let $\mathcal{P}_n(S)$ be the set of all algebraic polynomials of degree at most n with each of their coefficients in S. An algebraic polynomial P of the form

$$(1.1) \qquad P(z) = \sum_{j=0}^{n} a_j z^j \,, \qquad a_j \in \mathbb{C}\,,$$

is called conjugate-reciprocal if

$$(1.2) \qquad \overline{a}_j = a_{n-j}\,, \qquad j = 0, 1, \ldots, n\,.$$

Functions T of the form

$$T(t) = \alpha_0 + \sum_{j=1}^{n} (\alpha_j \cos(jt) + \beta_j \sin(jt))\,, \qquad \alpha_j, \beta_j \in \mathbb{R}\,,$$

are called real trigonometric polynomials of degree at most n. It is easy to see that any real trigonometric polynomial T of degree at most n can be written as $T(t) = P(e^{it})e^{-int}$, where P is a conjugate-reciprocal algebraic polynomial of the form

$$(1.3) \qquad P(z) = \sum_{j=0}^{2n} a_j z^j \,, \qquad a_j \in \mathbb{C}\,.$$

Conversely, if P is conjugate-reciprocal algebraic polynomial of the form (1.3), then there are $\theta_j \in \mathbb{R}$, $j = 1, 2, \ldots n$, such that T defined by

$$T(t) := P(e^{it})e^{-int} = a_n + \sum_{j=1}^{n} 2|a_{j+n}| \cos(jt + \theta_j)$$

is a real trigonometric polynomial of degree at most n. A polynomial P of the form (1.1) is called self-reciprocal if

$$(1.4) \qquad a_j = a_{n-j}, \qquad j = 0, 1, \ldots, n\,.$$

If a conjugate-reciprocal algebraic polynomial P has only real coefficients, then it is obviously self-reciprocal. If the algebraic polynomial P of the form (1.3) is self-reciprocal, then

$$T(t) := P(e^{it})e^{-int} = a_n + \sum_{j=1}^{n} 2a_{j+n} \cos(jt)\,.$$

In this paper, whenever we write "$P \in \mathcal{P}_n(S)$ is conjugate-reciprocal" we mean that P is of the form (1.1) with each $a_j \in S$ satisfying (1.2). Similarly, whenever we write "$P \in \mathcal{P}_n(S)$ is self-reciprocal" we mean that P is of the form (1.1) with each $a_j \in S$ satisfying (1.4). This is going to be our understanding even if the degree of $P \in \mathcal{P}_n(S)$ is less than n. It is easy to see that $P \in \mathcal{P}_n(S)$ is self-reciprocal and n is odd, then $P(-1) = 0$. We call any subinterval $[a, a + 2\pi)$ of the real number line $\mathcal{R}$ a period. Associated with an algebraic polynomial P of the form (1.1) we introduce the numbers

$$\mathrm{NC}(P) := |\{j \in \{0, 1, \dots, n\} : a_j \neq 0\}|\,.$$

Here, and in what follows $|A|$ denotes the number of elements of a finite set A. Let $\mathrm{NZ}(P)$ denote the number of real zeros (by counting multiplicities) of an algebraic polynomial P on the unit circle. Associated with an even trigonometric polynomial (cosine polynomial) of the form

$$T(t) = \sum_{j=0}^{n} a_j \cos(jt)$$

we introduce the numbers

$$\mathrm{NC}(T) := |\{j \in \{0, 1, \dots, n\} : a_j \neq 0\}|\,.$$

Let $\mathrm{NZ}(T)$ denote the number of real zeros (by counting multiplicities) of a real trigonometric polynomial T in a period. Let $\mathrm{NZ}^*(T)$ denote the number of sign changes of a real trigonometric polynomial T in a period. The quotation below is from [8]. "Let $0 \leq n_1 < n_2 < \cdots < n_N$ be integers. A cosine polynomial of the form $T(\theta) = \sum_{j=1}^{N} \cos(n_j\theta)$ must have at least one real zero in a period. This is obvious if $n_1 \neq 0$, since then the integral of the sum on a period is 0. The above statement is less obvious if $n_1 = 0$, but for sufficiently large N it follows from Littlewood's Conjecture simply. Here we mean the Littlewood's Conjecture proved by Konyagin [45] and independently by McGehee, Pigno, and Smith [55] in 1981. See also pages 285-288 in [19] for a book proof. It is not difficult to prove the statement in general even in the case $n_1 = 0$ without using Littlewood's Conjecture. One possible way is to use the identity

$$\sum_{j=1}^{n_N} T\left(\frac{(2j-1)\pi}{n_N}\right) = 0\,.$$

See [46], for example. Another way is to use Theorem 2 of [56]. So there is certainly no shortage of possible approaches to prove the starting observation of this paper even in the case $n_1 = 0$.

It seems likely that the number of zeros of the above sums in a period must tend to ∞ with N. In a private communication Conrey asked how fast the number of real zeros of the above sums in a period tends to ∞ as a function N. In [15] the authors observed that for an odd prime p the Fekete polynomial

$$f_p(z) = \sum_{k=0}^{p-1} \left(\frac{k}{p}\right) z^k$$

(the coefficients are Legendre symbols) has $\sim \kappa_0 p$ zeros on the unit circle, where $0.500813 > \kappa_0 > 0.500668$. Conrey's question in general does not appear to be easy.

Littlewood in his 1968 monograph 'Some Problems in Real and Complex Analysis [52] poses the following research problem (problem 22), which appears to still be open: 'If the n_m are integral and all different, what is the lower bound on the number of real zeros of $\sum_{m=1}^N \cos(n_m\theta)$? Possibly $N-1$, or not much less. Here real zeros are counted in a period. In fact no progress appears to have been made on this in the last half century. In a recent paper [8] we showed that this is false. There exist cosine polynomials $\sum_{m=1}^N \cos(n_m\theta)$ with the n_m integral and all different so that the number of its real zeros in a period is $O(N^{9/10}(\log N)^{1/5})$ (here the frequencies $n_m = n_m(N)$ may vary with N). However, there are reasons to believe that a cosine polynomial $\sum_{m=1}^N \cos(n_m\theta)$ always has many zeros in a period."

Let

$$\mathcal{L}_n := \left\{P:\ P(z) = \sum_{j=0}^n a_j z^j\,,\ \ a_j \in \{-1,1\}\right\}\,.$$

Elements of $\mathcal{L}_n$ are often called Littlewood polynomials of degree n. Let

$$\mathcal{K}_n := \left\{P:\ P(z) = \sum_{j=0}^n a_j z^j\,,\ \ a_j \in \mathbb{C},\ \ |a_0| = |a_n| = 1,\ \ |a_j| \le 1\right\}\,.$$

Observe that $\mathcal{L}_n \subset \mathcal{K}_n$. In [11] we proved that any polynomial $P \in \mathcal{K}_n$ has at least $8n^{1/2}\log n$ zeros in any open disk centered at a point on the unit circle with radius $33n^{-1/2}\log n$. Thus polynomials in $\mathcal{K}_n$ have quite a few zeros near the unit circle. One may naturally ask how many unimodular roots a polynomial in $\mathcal{K}_n$ can have. Mercer [56] proved that if a Littlewood polynomial $P \in \mathcal{L}_n$ of the form (1.1) is skew reciprocal, that is, $a_j = (-1)^j a_{n-j}$ for each $j = 0, 1, \ldots, n$, then it has no zeros on the unit circle. However, by using different elementary methods it was observed in both [27] and [56] that if a Littlewood polynomial P of the form (1.1) is self-reciprocal, that is, $a_j = a_{n-j}$ for each $j = 0, 1, \ldots, n$, $n \geq 1$, then it has at least one zero on the unit circle. Mukunda [58] improved this result by showing that every self-reciprocal Littlewood polynomial of odd degree has at least 3 zeros on the unit circle. Drungilas [21] proved that every self-reciprocal Littlewood polynomial of odd degree $n \geq 7$ has at least 5 zeros on the unit circle and every self-reciprocal Littlewood polynomial of even degree $n \geq 14$ has at least 4 zeros on the unit circle. In [4] two types of Littlewood polynomials are considered: Littlewood polynomials with one sign change in the sequence of coefficients and Littlewood polynomials with one negative coefficient, and the numbers of the zeros such Littlewood polynomials have on the unit circle and inside the unit disk, respectively, are investigated. Note that the Littlewood polynomials studied in [4] are very special. In [8] we proved that the average number of zeros of self-reciprocal Littlewood polynomials of degree n is at least $n/4$. However, it is much harder to give decent lower bounds for the quantities $\mathrm{NZ}_n := \min_P \mathrm{NZ}(P)$, where $\mathrm{NZ}(P)$ denotes the number of zeros of a polynomial P lying on the unit circle and the minimum is taken for all self-reciprocal Littlewood polynomials $P \in \mathcal{L}_n$. It has been conjectured for a long time that $\lim_{n\to\infty} \mathrm{NZ}_n = \infty$. In [34] we showed that $\lim_{n\to\infty} \mathrm{NZ}(P_n) = \infty$ whenever $P_n \in \mathcal{L}_n$ is self-reciprocal and $\lim_{n\to\infty} |P_n(1)| = \infty$. This follows as a consequence of a more general result, see Corollary 2.3 in [34], stated as Corollary 1.5 here, in which the coefficients of the self-reciprocal polynomials P_n of degree at most n belong to a fixed finite set of real numbers. This result also follows from the independent recent work of Sahasrabudhe [66]. In [7] we proved the following result.

Theorem 1.1. *If the set* $\{a_j : j \in \mathbb{N}\} \subset \mathbb{R}$ *is finite, the set* $\{j \in \mathbb{N} : a_j \neq 0\}$ *is infinite, the sequence* (a_j) *is not eventually periodic, and*

$$T_n(t) = \sum_{j=0}^{n} a_j \cos(jt)\,,$$

then $\lim_{n\to\infty} \mathrm{NZ}(T_n) = \infty\,.$

In [7] Theorem 1.1 is stated without the assumption that the sequence (a_j) is not eventually periodic. However, as the following example shows, Lemma 3.4 in [7], dealing with the case of eventually periodic sequences (a_j), is incorrect. Let

$$\begin{aligned} T_n(t) :=& \cos t + \cos((4n+1)t) + \sum_{k=0}^{n-1} (\cos((4k+1)t) - \cos((4k+3)t)) \\ =& \frac{1+\cos((4n+2)t)}{2\cos t} + \cos t \, . \end{aligned}$$

It is easy to see that $T_n(t) \neq 0$ on $[-\pi, \pi] \setminus \{-\pi/2, \pi/2\}$ and the zeros of T_n at $-\pi/2$ and $\pi/2$ are simple. Hence T_n has only two (simple) zeros in a period. So the conclusion of Theorem 1.1 above is false for the sequence (a_j) with $a_0 := 0$, $a_1 := 2$, $a_3 := -1$, $a_{2k} := 0$, $a_{4k+1} := 1$, $a_{4k+3} := -1$ for every $k = 1, 2, \ldots$. Nevertheless, Theorem 1.1 can be saved even in the case of eventually periodic sequences (a_j) if we assume that $a_j \neq 0$ for all sufficiently large j. See Lemma 3.11 in [34] where Theorem 1 in [7] is corrected as

Theorem 1.2. *If the set $\{a_j : j \in \mathbb{N}\} \subset \mathbb{R}$ is finite, $a_j \neq 0$ for all sufficiently large j, and*

$$T_n(t) = \sum_{j=0}^{n} a_j \cos(jt) \, ,$$

then $\lim_{n\to\infty} \mathrm{NZ}(T_n) = \infty$.

It was expected that the conclusion of the above theorem remains true even if the coefficients of T_n do not come from the same sequence, that is,

$$T_n(t) = \sum_{j=0}^{n} a_{j,n} \cos(jt) \, ,$$

where the set

$$S := \{a_{j,n} : j \in \{0, 1, \ldots, n\}, \, n \in \mathbb{C}\} \subset \mathbb{R}$$

is finite and

$$\lim_{n\to\infty} |\{j \in \{0, 1, \ldots, n\}, \, a_{j,n} \neq 0\}| = \infty \, .$$

Associated with an algebraic polynomial

$$P(z) = \sum_{j=0}^{n} a_j z^j \, , \qquad a_j \in \mathbb{C} \, ,$$

let

$$\mathrm{NC}_k(P) := |\{u : 0 \leq u \leq n-k+1,\ \ a_u + a_{u+1} + \cdots + a_{u+k-1} \neq 0\}|\ .$$

In [34] we proved the following results.

Theorem 1.3. *If $S \subset \mathbb{R}$ is a finite set, $P_{2n} \in \mathcal{P}_{2n}(S)$ are self-reciprocal polynomials,*

$$T_n(t) := P_{2n}(e^{it})e^{-int}\,,$$

and

$$\lim_{n\to\infty} \mathrm{NC}_k(P_{2n}) = \infty$$

for every $k \in \mathbb{N}$, then

$$\lim_{n\to\infty} \mathrm{NZ}(P_{2n}) = \lim_{n\to\infty} \mathrm{NZ}(T_n) = \infty\,.$$

Some of the most important consequences of the above theorem obtained in [34] are stated below.

Corollary 1.4. *If $S \subset \mathbb{R}$ is a finite set, $P_n \in \mathcal{P}_n(S)$ are self-reciprocal polynomials, and*

$$\lim_{n\to\infty} |P_n(1)| = \infty\,,$$

then

$$\lim_{n\to\infty} \mathrm{NZ}(P_n) = \infty\,.$$

Corollary 1.5. *Suppose the finite set $S \subset \mathbb{R}$ has the property that*

$$s_1 + s_2 + \cdots + s_k = 0\,,\ \ s_1, s_2, \ldots, s_k \in S\,,\ \ \text{implies}\ \ s_1 = s_2 = \cdots = s_k = 0\,,$$

that is, any sum of nonzero elements of S is different from 0. If $P_n \in \mathcal{P}_n(S)$ are self-reciprocal polynomials and

$$\lim_{n\to\infty} \mathrm{NC}(P_n) = \infty\,,$$

then

$$\lim_{n\to\infty} \mathrm{NZ}(P_n) = \infty\,.$$

J. Sahasrabudhe [66] examined the case when $S \subset \mathbb{Z}$ is finite. Exploiting the assumption that the coefficients are integer he proved that for any finite set $S \subset \mathbb{Z}$ a self-reciprocal polynomial $P \in \mathcal{P}_{2n}(S)$ has at least

$$c\ (\log\log\log |P(1)|)^{1/2-\varepsilon} - 1$$

zeros on the unit circle of $\mathbb{C}$ with a constant $c > 0$ depending only on $M = M(S) := \max\{|z| : z \in S\}$ and $\varepsilon > 0$.

Let $\phi(n)$ denote the Euler's totient function defined as the number of integers $1 \leq k \leq n$ that are relative prime to n. In an earlier version of his paper Sahasrabudhe [66] used the trivial estimate $\phi(n) \geq \sqrt{n}$ for $n \geq 3$ and he proved his result with the exponent $1/4 - \varepsilon$ rather than $1/2 - \varepsilon$. Using the nontrivial estimate $\phi(n) \geq n/(8 \log\log n)$ in [65] for all $n > 3$ allowed him to prove his result with $1/2 - \varepsilon$.

In the papers [7], [34], and [66] the already mentioned Littlewood Conjecture, proved by Konyagin [45] and independently by McGehee, Pigno, and Smith [55], plays a key role, and we rely on it heavily in the proof of the main results of this paper as well. This states the following.

Theorem 1.6. *There is an absolute constant $c > 0$ such that*

$$\int_0^{2\pi} \Big| \sum_{j=1}^m a_j e^{i\lambda_j t} \Big| \, dt \geq c\gamma \log m$$

whenever $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct integers and $a_1, a_2, \ldots, a_m$ are complex numbers of modulus at least $\gamma > 0$. Here $c = 1/30$ is a suitable choice.

This is an obvious consequence of the following result a book proof of which has been worked out by Lorentz and DeVore, see pages 285–288 in [19].

Theorem 1.7. *If $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ are integers and $a_1, a_2, \ldots, a_m$ are complex numbers, then*

$$\int_0^{2\pi} \Big| \sum_{j=1}^m a_j e^{i\lambda_j t} \Big| \, dt \geq \frac{1}{30} \sum_{j=1}^m \frac{|a_j|}{j}\,.$$

Associated with a finite set $S \subset \mathbb{C}$ we will use the notation $M = M(S) := \max\{|z| : z \in S\}$ throughout the paper.

2 New Results.

The goal of this paper is to improve the exponent $1/2 - \varepsilon$ to $1 - \varepsilon$ in Sahasrabudhe's lower bound in [66] mentioned in Section 1. Sahasrabudhe's new idea is combined with the approach used in [34] offering an essentially simplified way to achieve our improvement.

Let, as before, $\mathrm{NZ}(T)$ denote the number of real zeros (by counting multiplicities) of a real trigonometric polynomial T in a period. Let $\mathrm{NZ}^*(T)$

denote the number of sign changes of a real trigonometric polynomial T in a period. Obviously $\mathrm{NZ}(T) \geq \mathrm{NZ}^*(T)$.

Theorem 2.1. *If $S \subset \mathbb{Z}$ is a finite set, $P \in \mathcal{P}_{2n}(S)$ is a self-reciprocal polynomial,*

$$T(t) := P(e^{it})e^{-int}\,,$$

then

$$\mathrm{NZ}^*(T) \geq \left(\frac{c}{1+\log M}\right) \frac{\log\log\log |P(1)|}{\log\log\log\log |P(1)|} - 1$$

with an absolute constant $c > 0$, whenever $|P(1)| \geq 16$.

Let, as before, $\mathrm{NZ}(P)$ denote the number of real zeros (by counting multiplicities) of an algebraic polynomial P on the unit circle.

Corollary 2.2. *If $S \subset \mathbb{Z}$ is a finite set, $P \in \mathcal{P}_n(S)$ is a self-reciprocal polynomial, then*

$$\mathrm{NZ}(P) \geq \left(\frac{c}{1+\log M}\right) \frac{\log\log\log |P(1)|}{\log\log\log\log |P(1)|} - 1$$

with an absolute constant $c > 0$, whenever $|P(1)| \geq 16$.

This improves the exponent $1/2 - \varepsilon$ to $1 - \varepsilon$ in a recent breakthrough result [66] by Sahasrabudhe. We note that in both Sahasrabudhe's paper and this paper the assumption that the finite set S contains only integers is deeply exploited. Our next result is an obvious consequence of Corollary 2.2.

Corollary 2.3. *If the set $S \subset \mathbb{Z}$ is finite,*

$$T(t) = \sum_{j=0}^{n} a_j \cos(jt)\,, \qquad a_j \in S\,,$$

then

$$\mathrm{NZ}^*(T) \geq \left(\frac{c}{1+\log M}\right) \frac{\log\log\log |T(0)|}{\log\log\log\log |T(0)|} - 1$$

with an absolute constant $c > 0$, whenever $|T(0)| \geq 16$.

3 Lemmas.

Our first four lemmas are quite similar to some of the lemmas used in [34], but some modifications in the formulation of these lemmas and their proofs are needed.

Lemma 3.1. *If $S \subset \mathbb{C}$ is a finite set, $k \in \mathbb{N}$,*

$$S_k := \{s_1 + s_2 + \cdots + s_k : \ s_j \in S \cup \{0\}\} \,,$$

$$\gamma := \min_{z \in S_k \setminus \{0\}} |z| \,,$$

$P \in \mathcal{P}_{2n}(S)$, $H(z) := z^k - 1$, and

$$\mathrm{NC}(PH) \le \mu \,, \tag{3.1}$$

then

$$\int_{-\delta}^{\delta} |P(e^{it})| \, dt > \frac{\gamma}{30k} \, \log(\mathrm{NC}_k(P)) - \frac{\pi^2 \mu M}{\delta}$$

for every $\delta \in (0, \pi)$.

Proof of Lemma 3.1.

We define

$$G(z) := \sum_{j=0}^{k-1} z^j$$

so that $H(z) = G(z)(z-1)$. As $P \in \mathcal{P}_{2n}(S)$ and the set S is finite, the set S_k is also finite. By Theorem 1.6 there is an absolute constant $c > 0$ such that

$$\int_0^{2\pi} |(PG)(e^{it})| \, dt \ge \frac{\gamma}{30} \, \log(\mathrm{NC}(PG)) \ge \frac{\gamma}{30} \, \log(\mathrm{NC}_k(P)) \,. \tag{3.2}$$

We define

$$M_k := M(S_k) = \max\{|z| : z \in S_k\} \le k \max\{|z| : z \in S\} \le kM \,.$$

Combining this with (3.1) we have

$$\begin{aligned} |(PG)(e^{it})| =& \frac{1}{|e^{it} - 1|} \, |(PH)(e^{it})| \le \frac{\mu M_k}{|e^{it} - 1|} \\ =& \frac{\mu k M}{|2 \sin(t/2)|} \le \frac{\pi \mu k M}{|2t|} \,, \qquad t \in (-\pi, \pi) \,. \end{aligned}$$

It follows that

$$\int_{[-\pi,\pi] \setminus [-\delta,\delta]} |(PG)(e^{it})| \, dt \le 2\pi \frac{\pi \mu k M}{2\delta} = \frac{\pi^2 \mu k M}{\delta} \,. \tag{3.3}$$

Now (3.2) and (3.3) give

$$\begin{aligned}
\int_{-\delta}^{\delta} |P(e^{it})|\,dt &\geq \frac{1}{k}\int_{-\delta}^{\delta} |(PG)(e^{it})|\,dt \\
&= \frac{1}{k}\left(\int_0^{2\pi} |(PG(e^{it})|\,dt - \int_{[-\pi,\pi]\setminus[-\delta,\delta]} |(PG)(e^{it})|\,dt\right) \\
&\geq \frac{\gamma}{30k}\,\log(\mathrm{NC}_k(P)) - \frac{\pi^2\mu k M}{k\delta} \\
&\geq \frac{\gamma}{30k}\,\log(\mathrm{NC}_k(P)) - \frac{\pi^2\mu M}{\delta}\,.
\end{aligned}$$

□

Lemma 3.2. *If* $S \subset \mathbb{R}$ *is a finite set,* $P \in \mathcal{P}_{2n}(S)$ *is self-reciprocal,* $k \in \mathbb{N}$*,* $H(z) := z^k - 1$*, (3.1) holds,*

$$T(t) := P(e^{it})e^{-int}\,, \qquad R(x) := \int_0^x T(t)\,dt\,,$$

and $0 < \delta \leq (2k)^{-1}$*, then*

$$\max_{x\in[-\delta,\delta]} |R(x)| < 42k(\mu+1)M\,.$$

Proof of Lemma 3.2.

Let

$$P(z) = \sum_{j=0}^{2n} a_j z^j\,, \qquad a_j \in S\,,$$

be self-reciprocal. We have

$$T(t) = a_n + \sum_{j=1}^{n} 2a_{j+n}\cos(jt)\,, \qquad a_j \in S\,.$$

Observe that (3.1) implies that

$$|\{j : n+k \leq j \leq 2n,\ a_j \neq a_{j-k}\}| \leq \mu\,. \tag{3.4}$$

We have

$$R(x) = a_n x + \sum_{j=1}^{n} \frac{2a_{j+n}\sin(jx)}{j}\,.$$

Now (3.4) implies that

$$R(x) = a_0 x + \sum_{m=1}^{u} F_{m,k}(x)\,,$$

where

$$F_{m,k}(x) := \sum_{j=0}^{n_m-1} \frac{2A_{m,k}\sin((j_m+jk)x)}{j_m+jk}$$

with some $A_{m,k} \in S$, $m = 1, 2, \dots, u$, $j_m \in \mathbb{N}$, and $n_m \in \mathbb{N}$, where $u \leq k(\mu+1)$ (we do not know much about j_m and n_m). Since $S \subset [-M, M]$, it is sufficient to prove that

$$\max_{x\in[-\delta,\delta]} |F_{m,k}(x)| \leq \frac{41}{2}\,, \qquad m = 1, 2, \dots, u\,,$$

that is, it is sufficient to prove that if $j_0 \in \mathbb{N}$ and

$$F(x) := \sum_{j=0}^{\nu} \frac{\sin((j_0+jk)x)}{j_0+jk}\,,$$

then

$$\max_{x\in[-\delta,\delta]} |F(x)| = \max_{x\in[0,\delta]} |F(x)| \leq \frac{41}{2}\,. \tag{3.5}$$

Note that the equality in (3.5) holds as F is odd. To prove the inequality in (3.5) let $x \in (0, \delta]$, where $0 < \delta \leq (2k)^{-1}$. We break the sum as

$$F = R + S\,, \tag{3.6}$$

where

$$R(x) := \sum_{\substack{j=0 \\ j_0+jk\leq x^{-1}}}^{\nu} \frac{\sin((j_0+jk)x)}{j_0+jk}$$

and

$$S(x) := \sum_{\substack{j=0 \\ x^{-1}<j_0+jk}}^{\nu} \frac{\sin((j_0+jk)x)}{j_0+jk}\,.$$

Here

$$\begin{aligned} |R(x)| \leq & \sum_{\substack{j=0 \\ j_0+jk\leq x^{-1}}}^{\nu} \left|\frac{\sin((j_0+jk)x)}{j_0+jk}\right| \leq (x^{-1}+1)|x| \leq 1+|x| \\ & \leq 1+\delta = 1+(2k)^{-1} \leq \frac{3}{2}\,, \end{aligned} \tag{3.7}$$

where each term in the sum in the middle is estimated by

$$\left|\frac{\sin((j_0+jk)x)}{j_0+jk}\right| \leq \left|\frac{(j_0+jk)x)}{j_0+jk}\right| = |x|\,,$$

and the number of terms in the sum in the middle is clearly at most $x^{-1}+1$. Further, using Abel rearrangement, we have

$$S(x) = -\frac{B_v(x)}{j_0+vk} + \frac{B_u(x)}{j_0+uk} + \sum_{\substack{j=0 \\ x^{-1}<j_0+jk}}^{\nu} B_j(x)\left(\frac{1}{j_0+jk} - \frac{1}{j_0+(j+1)k}\right)$$

with

$$B_j(x) := B_{j,k}(x) := \sum_{h=0}^{j} \sin((j_0+hk)x)$$

and with some $u, v \in \mathbb{N}_0$ for which $x^{-1} < j_0+(u+1)k$ and $x^{-1} < j_0+(v+1)k$. Hence,

$$\begin{aligned} |S(x)| \le & \left|\frac{B_v(x)}{j_0+vk}\right| + \left|\frac{B_u(x)}{j_0+uk}\right| \\ & + \sum_{\substack{j=0 \\ x^{-1}<j_0+jk}}^{\nu} |B_j(x)|\left(\frac{1}{j_0+jk} - \frac{1}{j_0+(j+1)k}\right). \end{aligned} \tag{3.8}$$

Note that $x \in (0,\delta]$, $0 < \delta \le (2k)^{-1}$, $x^{-1} < j_0+(w+1)k$, and $w \in \mathbb{N}_0$ imply

$$x^{-1} < j_0+(w+1)k < 2(j_0+wk) \qquad \text{if } \ w \ge 1\,,$$

and

$$2k \le \delta^{-1} \le x^{-1} < j_0+k \qquad \text{if } \ w = 0\,,$$

and hence

$$\frac{1}{j_0+wk} \le 2x\,. \tag{3.9}$$

Observe also that $x \in (0,\delta]$ and $0 < \delta \le (2k)^{-1}$ imply that $0 < x < \pi k^{-1}$. Hence, with $z = e^{ix}$ we have

$$\begin{aligned} |B_j(x)| &= \left|\mathrm{Im}\left(\sum_{h=0}^{j} z^{j_0+hk}\right)\right| \le \left|\sum_{h=0}^{j} z^{j_0+hk}\right| = \left|\sum_{h=0}^{j} z^{hk}\right| \\ &= \left|\frac{1-z^{(j+1)k}}{1-z^k}\right| \le |1-z^{(j+1)k}|\,\frac{1}{|1-z^k|} \le \frac{2}{|1-z^k|} \\ &\le \frac{2}{2\sin(kx/2)} \le \frac{\pi}{kx}\,. \end{aligned} \tag{3.10}$$

Combining (3.8), (3.9), and (3.10), we conclude

$$|S(x)| \le \frac{\pi}{kx}\,2x + \frac{\pi}{kx}\,2x + \frac{\pi}{kx}\,2x \le \frac{6\pi}{k}\,. \tag{3.11}$$

Now (3.6), (3.7), and (3.11) give the inequality in (3.5) as $3/2 + 6\pi/k \le 41/2$. $\square$

Our next lemma was used in [34] in the same form. To prove it by contradiction is a simple exercise.

Lemma 3.3. *If R is a continuously differentiable real-valued function on the interval $[-\delta, \delta]$, $\delta > 0$,*

$$L := \int_{-\delta}^{\delta} |R'(x)|\, dx \qquad \text{and} \qquad N := \max_{x \in [-\delta,\delta]} |R(x)|\,,$$

then there is an $\eta \in [-N, N]$ such that $R - \eta$ has at least $L(2N)^{-1}$ distinct zeros in $[-\delta, \delta]$.

Lemma 3.4. *If $S \subset \mathbb{R}$ is a finite set, $k \in \mathbb{N}$,*

$$S_k := \{s_1 + s_2 + \cdots + s_k : \; s_j \in S \cup \{0\}\}\,,$$

$$\gamma := \min_{z \in S_k \setminus \{0\}} |z|\,,$$

$P \in \mathcal{P}_{2n}(S)$ is self-reciprocal, $T(t) := P(e^{it})e^{-int}$, $H(z) := z^k - 1$, and (3.1) holds, that is,

$$\mathrm{NC}(PH) \le \mu\,,$$

then

$$\mathrm{NZ}^*(T) \ge \left(\frac{\gamma}{30} \log(\mathrm{NC}_k(P)) - 2k\pi^2 \mu M\right) (84k(\mu+1)M)^{-1}\,.$$

Proof of Lemma 3.4.
Let $0 < \delta := (2k)^{-1}$. Let R be defined by

$$R(x) := \int_0^x T(t)\, dt\,.$$

Observe that $|T(x)| = |P(e^{ix})|$ for all $x \in \mathbb{R}$, and hence Lemma 3.1 yields that

$$\begin{aligned} \int_{-\delta}^{\delta} |R'(x)|\, dx \; &= \int_{-\delta}^{\delta} |T(x)|\, dx = \int_{-\delta}^{\delta} |P(e^{ix})|\, dx \\ &> \frac{\gamma}{30k} \log(\mathrm{NC}_k(P)) - \frac{\pi^2 \mu M}{\delta} \\ &= \frac{\gamma}{30k} \log(\mathrm{NC}_k(P)) - 2k\pi^2 \mu M\,, \end{aligned}$$

while by Lemma 3.2 we have

$$\max_{x \in [-\delta,\delta]} |R(x)| < 42k(\mu+1)M\,.$$

Therefore, by Lemma 3.3 there is an $\eta \in \mathbb{R}$ such that $R-\eta$ has at least

$$\left(\frac{\gamma}{30k}\,\log(\mathrm{NC}_k(P)) - 2k\pi^2\mu M\right)(84k(\mu+1)M)^{-1}\,.$$

distinct zeros in $[-\delta,\delta]$. However, $T(x) = (R-\eta)'(x)$ for all $x \in \mathbb{R}$, and hence

$$\mathrm{NZ}^*(T) \geq \left(\frac{\gamma}{30}\,\log(\mathrm{NC}_k(P)) - 2k\pi^2\mu M\right)(84k(\mu+1)M)^{-1}$$

follows by Rolle's Theorem. □

The following lemma, in which the assumption $S \subset \mathbb{Z}$ is crucial, is simple to prove. It is stated as Lemma 9 in [66]. Its straightforward proof given in [66] is reduced to the fact that a determinant of integer entries is an integer, and hence if it is not 0, then its modulus is at least 1.

Lemma 3.5. *For $b \in \mathbb{N}$ let A be a $b \times b$ invertible matrix with entries from $\widetilde{S} \subset \mathbb{Z}$. If $A\boldsymbol{x} = \boldsymbol{y}$ with*

$$\boldsymbol{x} = (x(1), x(2), \ldots, x(d)) \in \mathbb{C}^b \quad \text{and} \quad \boldsymbol{y} = (y(1), y(2), \ldots, y(d)) \in \mathbb{C}^b\,,$$

then

$$\max\{|x(1)|, |x(2)|, \ldots, |x(d)|\} \leq \widetilde{M}^{d-1} d^{d/2} \max\{|y(1)|, |y(2)|, \ldots, |y(b)|\}\,,$$

where $\widetilde{M} := M(\widetilde{S}) := \max\{|z| : z \in \widetilde{S}\}$.

For integers $1 \leq b \leq N$ we call

$$(x(1+r), x(2+r), \ldots, x(b+r)) \in \mathbb{C}^N\,, \qquad r = 0, 1, \ldots, N-b\,,$$

the b-tuples of

$$(x(1), x(2), \ldots, x(N)) \in \mathbb{C}^N\,.$$

The following lemma is Lemma 10 in [66].

Lemma 3.6. *For $u, v, b, t \in \mathbb{N}$, let $S \subset \mathbb{Z}$ be a finite set such that $v - u > |S|^b + 3b$, and let*

$$(x(u+1), x(u+2), \ldots, x(v)) \in S^{v-u}\,.$$

Let V denote the linear space spanned by the b-tuples

$$(x(r+1), x(r+2), \ldots, x(r+b)) \in S^b\,, \qquad r = u, u+1, \ldots, v-b\,,$$

over $\mathbb{R}$. If $\dim(V) = t < b$, then there are

$$(x_j(u+b), x_j(u+b+1), \ldots, x_j(v-b)) \in \mathbb{C}^{v-u-2b+1}\,, \qquad j = 1, 2, \ldots, t\,,$$

such that

$$x(r) = x_1(r) + x_2(r) + \cdots + x_t(r)\,, \qquad r \in [u+b, v-b]\,,$$

where

$$(x_j(u+b), x_j(u+b+1), \ldots, x_j(v-b)) \in \mathbb{C}^{v-u-2b+1}$$

are periodic with period $\alpha_j \leq 16t \log\log(t+3)$ *for each* $j = 1, 2, \ldots, t$.

Let f be a continuous, even, real-valued function on $K := \mathbb{R}$ (mod 2π) which changes sign on $(0, \pi)$ exactly at $t_1 < t_2 < \cdots < t_d$, $d \geq 1$. We define the *companion polynomial* Q of f by

$$Q(e^{it})e^{-idt} := (-1)^p 2^d \prod_{j=1}^{d} (\cos t - \cos t_j)\,,$$

where $p \in \{0, 1\}$ is chosen so that $f(t)e^{-idt}Q(e^{it}) \geq 0$ for all $t \in (-\pi, \pi)$. Observe that

$$Q(z) := (-1)^p \prod_{j=1}^{d} (z - e^{it_j})(z - e^{-it_j}) =: \sum_{j=0}^{2d} b_j z^j$$

is a monic self-reciprocal algebraic polynomial of degree $2d$ with real coefficients and with constant term 1. Observe that

$$|b_j| = \left| \frac{1}{2\pi} \int_0^{2\pi} Q(e^{it})e^{-ijt}\,dt \right| \leq \max_{t \in [0, 2\pi} |Q(e^{it})| \leq 4^d\,, \qquad j = 0, 1, \ldots, 2d\,. \tag{3.12}$$

Associated with $m \in \mathbb{N}$ let $d_m := \mathrm{LCM}(1, 2, \ldots, m)$. It is shown in [65] that $d_m < 3^m$ holds for all $m \in \mathbb{N}$, and this upper bound will be useful for us later in this paper. We remark though that $\lim_{m \to \infty} d_m/e^m = 1$ holds and it is equivalent to the Prime Number Theorem, see [73].

Lemma 3.7. *Suppose* $S \subset \mathbb{Z}$ *is a finite set,* $P \in \mathcal{P}_{2n}(S)$ *is self-reciprocal, and* $T(t) := P(e^{it})e^{-int}$ *has exactly* $2d \geq 2$ *sign changes in* $(-\pi, \pi)$. *Let* Q *be the companion polynomial of* T *(so the degree of the monic self-reciprocal algebraic polynomial* Q *is* $2d$*) and let*

$$F(z) := P(z)(z^{d_m} - 1)^2 Q(z) = \sum_{j=0}^{2n+2d_m+2d} a_j z^j\,, \tag{3.13}$$

where $m := \lfloor 32d \log\log(2d+3) \rfloor$*, and let*

$$\{j \in [0, 2n + 2d_m + 2d] : a_j \neq 0\} = \{j_1 < j_2 < \cdots < j_q\}\,.$$

If $1 \le r \le s \le q$ *are integers and*

$$|a_{j_k}| < (4M)^{-2d}(2d+1)^{-d-1/2}\,, \qquad k \in [r,s]\,, \tag{3.14}$$

then

$$s - r < (|S|+2)^{4m+2} + 6d + 3\,,$$

where $|S|$ *denotes the number of elements in the set* S.

Proof of Lemma 3.7.
Let

$$S^* := \{s_1 - s_2 :\ s_1, s_2 \in S \cup \{0\}\} \subset \mathbb{Z}$$

and

$$S^{**} := \{s_1 - s_2 :\ s_1, s_2 \in S^*\} \subset \mathbb{Z}\,.$$

Obviously

$$|S^*| \le (|S|+1)^2, \qquad |S^{**}| \le |S^*|^2 \le (|S|+1)^4\,, \tag{3.15}$$

and

$$M(S^{**}) \le 2M(S^*) \le 4M(S) = 4M\,. \tag{3.16}$$

Put $b := 2d+1$. Suppose to the contrary that

$$s - r \ge (|S|+2)^{4m+2} + 3b\,. \tag{3.17}$$

Let

$$G_1(z) := P(z)(z^{d_m} - 1) := \sum_{j=0}^{2n+d_m} \alpha_j z^j\,, \qquad \alpha_j \in S^*\,,$$

and

$$G_2(z) := G_1(z)(z^{d_m} - 1) = P(z)(z^{d_m} - 1)^2 = \sum_{j=0}^{2n+2d_m} \beta_j z^j\,, \tag{3.18}$$

where

$$\beta_j := \alpha_{j-d_m} - \alpha_j \in S^{**}, \qquad j = 0, 1, \dots, 2n+2d_m\,, \tag{3.19}$$

with

$$\alpha_{j-d_m} := 0, \quad \alpha_{2n+d_m+j+1} := 0\,, \qquad j = 0, 1, \dots, d_m - 1\,.$$

Observe that $G_2 \in \mathcal{P}_{2n+2d_m}(S^{**})$. Let V denote the linear space spanned by the b-tuples

$$(\beta_j, \beta_{j+1}, \dots, \beta_{j+b-1})\,, \qquad j \in [j_r, j_s - b + 1]\,,$$

over $\mathbb{R}$. Using Lemma 3.5 with $\widetilde{S} := S^{**}$, (3.14), (3.16), and the fact that the polynomial Q of degree $2d$ is monic, we can deduce that $\dim(V) = t < b$. It follows from (3.15) and (3.17) that

$$j_s - j_r \ge s - r \ge (|S|+2)^{4m+2} + 3b > |S^{**}|^m + 3b\,.$$

As $t < b = 2d+1$, we have $\lfloor 16t \log\log(t+3) \rfloor \le \lfloor 32d \log\log(2d+3) \rfloor = m$. Applying Lemma 3.6 we obtain that

$$(\beta_{j_r+b}, \beta_{j_r+b+1}, \ldots, \beta_{j_s-b+1})$$

is periodic with period d_m, that is,

$$\beta_{r+d_m} = \beta_r\,, \qquad r \in [j_r + b, j_s - d_m - b + 1]\,. \tag{3.20}$$

We claim that

$$\beta_u = 0, \qquad u \in [j_r + b, j_s - b - |S^*|d_m]\,. \tag{3.21}$$

Indeed, if $\beta_u \ne 0$ for some $u \in [j_r+b, j_s-b-|S^*|d_m]$, then (3.19) and (3.20) give

$$\begin{aligned} \alpha_{u+hd_m} &= \alpha_u + \sum_{j=1}^{h} (\alpha_{u+jd_m} - \alpha_{u+(j-1)d_m}) = \alpha_u - \sum_{j=1}^{h} \beta_{u+jd_m} \\ &= \alpha_u - h\beta_u \in S^*, \qquad h = 0, 1, \ldots, |S^*|\,, \end{aligned}$$

exhibiting $|S^*|+1$ distinct elements of S^*, which is impossible. It follows from (3.13), (3.18), and (3.21) that

$$a_j = 0, \qquad j \in [j_r + 2b, j_s - b - |S^*|d_m]\,,$$

hence, recalling $d_m < 3^m$ and (3.15), we obtain

$$\begin{aligned} s - r &\le |S^*|d_m + 3b < |S^*|3^m + 3b < (|S|+1)^2(|S|+2)^{4m} + 3b \\ &\le (|S|+2)^{4m+2} + 3b\,, \end{aligned}$$

which contradicts (3.17). In conclusion

$$s - r < (|S|+2)^{4m+2} + 3b = (|S|+2)^{4m+2} + 6d + 3\,.$$

$\square$

Lemma 3.8. *Under the assumptions of Lemma 3.7 we have*

$$\log q \le 60\pi(8M)^{2d+1}(2d+1)^{d+3/2}(|S|+2)^{4m+2} + 6d + 3)\,.$$

Proof of Lemma 3.8.
Let $L := (|S|+2)^{4m+2} + 6d + 3$, and $r := \lfloor q/L \rfloor$. Observe that

$$P(e^{it})e^{-int}(e^{id_m t}-1)^2 e^{-id_m t} Q(e^{it}) e^{-idt}$$

is real and nonnegative for all $t \in \mathbb{R}$. Combining this with Theorem 1.7 and Lemma 3.7 we obtain

$$\begin{aligned}
&\int_0^{2\pi} P(e^{it})e^{-int}(e^{id_m t}-1)^2 e^{-id_m t} Q(e^{it}) e^{-idt}\,dt \\
&= \int_0^{2\pi} |P(e^{it})(e^{id_m t}-1)^2 Q(e^{it})|\,dt \\
&\geq \frac{1}{30}\sum_{k=1}^{q}\frac{|a_{j_k}|}{k} \geq \frac{1}{30}\sum_{j=1}^{r}\sum_{k=(j-1)L+1}^{jL}\frac{|a_{j_k}|}{k} \qquad (3.22)\\
&\geq \frac{1}{30}(4M)^{-2d}(2d+1)^{-d-1/2}\sum_{j=1}^{r}\frac{1}{jL} \\
&\geq \frac{1}{30}(4M)^{-2d}(2d+1)^{-d-1/2}L^{-1}\log(r+1)\,.
\end{aligned}$$

On the other hand, using orthogonality, (3.18), $\beta_n + d_m - j \in S^{**}$, and (3.12) we have

$$\begin{aligned}
\int_0^{2\pi} P(e^{it})e^{-int}(e^{id_m t}-1)^2 e^{-id_m t} Q(e^{it}) e^{-idt}\,dt &= 2\pi \sum_{j=-d}^{d}\beta_{n+d_m-j}b_j \qquad (3.23)\\
&\leq 2\pi(2d+1)(4M)4^d\,.
\end{aligned}$$

Combining (3.22) and (3.23) we conclude

$$\frac{1}{30}(4M)^{-2d}(2d+1)^{-d-1/2}L^{-1}\log(r+1) \leq 2\pi(2d+1)(4M)4^d\,,$$

and hence

$$\begin{aligned}
\log q \leq \log(r+1) + \log L &\leq 60\pi(8M)^{2d+1}(2d+1)^{d+3/2}L \\
&\leq 60\pi(8M)^{2d+1}(2d+1)^{d+3/2}((|S|+2)^{4m+2}+6d+3)\,.
\end{aligned}$$

□

Our final lemma follows easily from Lemma 3.6.

Lemma 3.9. *If $S \subset \mathbb{Z}$ is a finite set, $P \in \mathcal{P}_{2n}(S)$, $0 \not\equiv R$ is a polynomial of degree at most u with real coefficients,*

$$\mathrm{NC}(PR) \leq q\,,$$

$v := \lfloor 16u \log\log(u+3) \rfloor$, $k := d_v = \mathrm{LCM}(1,2,\dots,v)$, *and* $H(z) = z^k - 1$, *then*

$$\mathrm{NC}(PH) \leq \mu := (q+1)(k + |S|^{u+1} + 3(u+1) + 2)\,.$$

4 Proof of the New Results.

Proof of Theorem 2.1.

Suppose $S \subset \mathbb{Z}$ is a finite set, $P \in \mathcal{P}_{2n}(S)$ is self-reciprocal, and $T(t) := P(e^{it})e^{-int}$ has exactly $2d$ sign changes in $(-\pi, \pi)$. Without loss of generality we may assume that $d \geq 2$ otherwise we study the self-reciprocal polynomial $\widetilde{P} \in \mathcal{P}_{2n}(\widetilde{S}$ defined by $\widetilde{P}(z) := (z^2+1)P(z)$, where $M(\widetilde{S}) = 2M(S) = 2M$ and $\widetilde{P}(1) := 2P(1)$. Let Q be the companion polynomial of T. Let

$$F(z) := P(z)(z^{d_m} - 1)^2 Q(z) = \sum_{j=0}^{2n+2d_m+2d} a_j z^j \,,$$

where $m := \lfloor 32d \log\log(2d+3) \rfloor$. Let

$$\{j \in [0, 2n+2d_m+2d] : a_j \neq 0\} =: \{j_1 < j_2 < \cdots < j_q\} \,.$$

Lemma 3.8 together with $|S| + 2 \leq 2M(S) + 3 = 2M + 3 \leq 5M$ and $4m + 2d + 3 \leq 5m$ implies

$$\log q \leq 60\pi (8M)^{2d+1} (5M)^{4m+2} (2d+1)^{d+3/2} + 180\pi (8M)^{2d+1} (2d+1)^{d+5/2} \,,$$

and hence

$$\log q \leq 240\pi (8M)^{5m} (2d+1)^{d+5/2} \,. \tag{4.1}$$

Applying Lemma 3.9 with $u := 2d_m + 2d$, we have

$$\mathrm{NC}(PH) \leq \mu := (q+1)(d_v + |S|^{u+1} + 3(u+1) + 2) \tag{4.2}$$

with $v := \lfloor 16u \log\log(u+3) \rfloor$, $k := d_v$, and $H(z) = z^k - 1$. Observe that if $S \subset \mathbb{Z}$ and

$$S_k := \{s_1 + s_2 + \cdots + s_k : \ s_j \in S \cup \{0\}\} \,,$$

then

$$1 \leq \gamma := \min\{|z| : z \in S_k \setminus \{0\}\} \,.$$

Lemma 3.4 gives

$$\log(\mathrm{NC}_k(P)) \leq 2520 k^2 (\mu+1) k M(2d) + 60 k^2 \pi^2 \mu M \,. \tag{4.3}$$

Using $m := \lfloor 32d \log\log(2d+3) \rfloor$, $u := 2d_m + 2d$, $v := \lfloor 16u \log\log(u+3) \rfloor$, $k = d_v < 3^v$, $d_m < 3^m$, (4.1), (4.2), $|S| \leq 2M(S) + 1 = 2M + 1$, and the inequality $a + b \le ab$ valid for all $a \ge 1$ and $b \ge 1$, we obtain

$$\begin{aligned}
&\log(2520k(\mu+1)M(2d) + 60k^2\pi^2\mu M) \le \log(2520k^2(\mu+1)M(2d+1))\\
&\le 8 + 2\log k + \log(\mu+1) + \log M + \log(2d+1)\\
&\le 8 + 2\log k + \log(q+1) + \log d_v + \log(|S|^{u+1}) + \log(3u+6)\\
&\quad + \log M + \log(2d+1)\\
&\le 8 + 2(\log 3)v + \log 2 + 240\pi(8M)^{8m}(2d+1)^{d+5/2} + (\log 3)v\\
&\quad + (u+1)\log(2M+1) + \log(3u+6) + \log M + \log(2d+1)\\
&\le 9 + 2(\log 3)16u\log\log(u+3) + 240\pi(8M)^{5m}(2d+1)^{d+5/2}\\
&\quad + \ (2\cdot 3^m + 2d + 1)\log(2M+1)\\
&\quad + \ (\log 2 + 3(\log 3)m)\log(2d+1)\log(2M+1) + \log M + \log(2d+1)\,,
\end{aligned}$$

and hence

$$\begin{aligned}
&\log(2520k(\mu+1)M(2d) + 60k^2\pi^2\mu M)\\
\le& 9 + 3(\log 3)16(2\cdot 3^m + 2d)\log\log(2\cdot 3^m + 2d)\\
&+ 240\pi(8M)^{5m}(2d+1)^{d+5/2} + (2\cdot 3^m + 2d + 1)\log(2M+1)\\
&+ (\log 2 + (\log 3)m)\log(2d+1)\log(2M+1) + \log M + \log(2d+1)\,.
\end{aligned}$$

Combining this with (4.3) and $m := \lfloor 32d\log\log(2d+3)\rfloor$ gives that there is an absolute constant $c_1 > 0$ such that

$$\log\log\log \mathrm{NC}_k(P) \le c_1(d\log(d+1) + d\log\log(2d+1)(1+\log M))\,. \tag{4.4}$$

It is easy to see that

$$\mathrm{NC}_k(P) \ge \frac{k|P(1)| - k^2M}{kM} = \frac{|P(1)|}{M} - k\,.$$

Therefore if $|P(1)| \ge 2kM$, then $\mathrm{NC}_k(P) \ge \frac{1}{2}|P(1)|$, and the theorem follows from (4.4) after a straightforward calculus. If $|P(1)| < 2kM$, then it follows from $k := d_v < 3^v$, $v := \lfloor 16u\log\log(u+3)\rfloor$, $u := 2d_m + 2d < 2\cdot 3^m + 2d$, and $m := \lfloor 32d\log\log(2d+3)\rfloor$ that

$$\begin{aligned}
\log\log\log|P(1)| < \log\log\log(2kM) &\le c_2(1+\log M)\log(d+1)\\
&\le c_2(1+\log M)(2d+1)\,,
\end{aligned}$$

with an absolute constant $c_2 > 0$, and the theorem follows. □

Proof of Corollary 2.2.

Let $S \subset \mathbb{Z}$ be a finite set. If $P \in \mathcal{P}_{2\nu}(S)$ is self-reciprocal, then the corollary follows from Theorem 2.1. If $P \in \mathcal{P}_{2\nu+1}(S)$ is self-reciprocal, then $\widetilde{P} \in \mathcal{P}_{2\nu+2}(S^*)$ defined by

$$\widetilde{P}(z) := (z+1)P(z) \in \mathcal{P}_{2\nu+2}(S^*)$$

is also self-reciprocal, where the fact that $S \subset \mathbb{Z}$ is finite implies that the set

$$S^* := \{s_1 + s_2 : \; s_1, s_2 \in S \cup \{0\}\} \subset \mathbb{Z}\,,$$

is also finite. Observe also that

$$M(S^*) = \max\{|z| : z \in S^*\} = 2\max\{|z| : z \in S\} = 2M(S) = 2M$$

and

$$\widetilde{P}(1) = 2P(1)\,.$$

Hence applying Theorem 2.1 to $\widetilde{P} \in \mathcal{P}_{2\nu+2}(S^*)$, we obtain the statement of the corollary for $P \in \mathcal{P}_{2\nu+1}(S)$ from Theorem 2.1 again. □

Proof of Corollary 2.3.
The corollary follows from Theorem 2.1 and the fact that for every trigonometric polynomial T of the form

$$T(t) := a_0 + \sum_{j=1}^{n} a_j \cos(jt)\,, \qquad a_j \in \mathbb{Z}\,,$$

there is a self-reciprocal algebraic polynomial P of the form

$$P(z) = 2a_0 z^n + \sum_{j=1}^{n} a_j (z^{n+j} + z^{n-j})\,, \qquad a_j \in \mathbb{Z}\,,$$

such that

$$2T(t) := P(e^{it})e^{-int}\,.$$

□

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