ORTHOGONALITY AND IRRATIONALITY

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Abstract. The fairly recent proofs of the irrationality of $\log 2, \pi^2, \zeta(3)$ and various values of polylogarithms are re-examined by using a particular family of orthogonal forms. This allows us to give a uniform and arguably more natural treatment of these results.

1. Introduction

Apéry's astonishing proof of the irrationality of $\zeta(3)$ amounts to showing that

$$0 < |d_k^3 a_k \zeta(3) - b_k| \longrightarrow 0$$

where b_k is an integer,

$$a_k := \sum_{i=0}^{k} \binom{i+k}{k}^2 \binom{k}{i}^2$$

and

$$d_k := \operatorname{lcm}\{1, 2, \dots, k\}.$$

Here and throughout the paper lcm denotes the least common multiple. This is entertainingly proved in van der Poorten [15] and given a very elegant treatment by Beukers [4], [5]. In [5] Beukers puts the proof in the context of Padé approximants.

Many, maybe most, irrationality proofs may be based on approximation by Padé approximants and related orthogonal polynomials ([8], [9], [14]). Sometimes this is very natural, as with e, where the Padé approximants generate convergents of the simple continued fraction. This is, however, an exceptional case.

It is the intention of this paper to try to put the well known proofs of the irrationality of the constants $\log 2, \zeta(2)$, and $\zeta(3)$, as well as certain values of polylogarithms into the framework of orthogonality. This reproduces results of Apéry, Alladi and Robinson, and Chudnovsky in a unified fashion ([15], [2], [10]).

In [7], the authors considered orthogonalization of the system

$$\{x^{\lambda_0}, x^{\lambda_1}, x^{\lambda_2}, \dots\},$$

and from this point of view orthogonalization of the system

$$\{\underbrace{x^0, x^0, \dots, x^0}_{n \text{ times}},\ \underbrace{x^1, x^1, \dots, x^1}_{n \text{ times}},\ \dots\}$$

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is very natural to study (the repeated indices are taken to be in a limiting sense). This leads to orthogonal functions that generalize Legendre polynomials and are of the form

$$\sum_{j=0}^{n-1} A_{j,l}(x)\log^j x$$

where each $A_{j,l}(x)$ is a polynomial of degree l.

Legendre polynomials are closely tied to irrationality questions concerning logarithms ([2], [6, Chapter 11]) and higher order analogues prove to be the basis for dealing with the irrationality of the polylogarithm $\sum_n \frac{z^n}{n^m}$ ([12]).

We argue that the irrationality proofs flow quite naturally from this point of view. A nice feature of this approach is that the non-vanishing of the estimates, which is essential for the proofs, comes very easily from the orthogonality.

This paper is organized in the following way. First we produce a contour integral which generalizes the Legendre polynomials and consider the properties of this function. We then specialize this form for particular cases in order to prove the irrationality of $\log 2, \zeta(2), \zeta(3)$, and some values of the polylogarithm in general. The approach to $\zeta(2)$, $\zeta(3)$, and $\log(2)$ eventually reproduces the estimates of Apéry and Alladi and Robinson [2]. In the case of $\zeta(2)$ and $\zeta(3)$ we end up with the integrals of Beukers [4]. The results for irrationality of polylogarithms are very similar to those of Chudnovsky [10] but the estimates are different.

In all cases, the estimate allow for inequalities of the form

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{q^c}$$

for all integers p and $q > 0$ for some $c > 0$. So in all cases the stronger conclusion, that the numbers in question are not Liouville is possible.

2. The Contour Integral

The orthogonalization of repeated monomials can be expressed by contour integrals.

Theorem 1. *Let k, l, m, and n be positive integers satisfying $(l+1)n - km \geq 1$. Define*

$$F(x;k,l;m,n) := \frac{(n-1)!}{2\pi i}\,\frac{(l!)^n}{(k!)^m}\oint_\gamma \frac{\prod_{j=1}^{k}(t+j)^m}{\prod_{j=0}^{l}(t-j)^n}x^t dt$$

where γ is any simple contour containing the poles at $t = 0, 1, \ldots, l$. Then

$$F(x;k,l;m,n) = (-1)^{nl}\sum_{j=0}^{n-1} A_{j,l}(x)\log^j x$$

where $A_{i,l} := A_{i,l,k,m,n}$ is a polynomial of degree l. So,

$$A_{n-1,l}(x) := \sum_{i=0}^{l}(-1)^{ni}\binom{i+k}{k}^m\binom{l}{i}^n x^i$$

and

$$A_{n-2,l}(x) := \sum_{i=0}^{l} (-1)^{ni} \binom{i+k}{k}^m \binom{l}{i}^n (n-1) \left(\sum_{j=1}^{k} \frac{m}{i+j} - \sum_{j=0, j\neq i}^{l} \frac{n}{i-j} \right) x^i$$

Furthermore, this function enjoys the orthogonality relations

$$\int_0^1 F(x;k,l;m,n) x^i \log^j x dx = 0,\ i = 0,1,\ldots,k-1,\ j = 0,1,\ldots,m-1.$$

The polynomial $A_{n-1,l}$ *has integer coefficients. If* $k = l$ *and* $n \geq 2$ *then the polynomial* $d_k A_{n-2,l}$ *also has integer coefficients* ($d_k := \mathrm{lcm}\{1,2,\ldots,k\}$).

Proof. The representation of F is just the evaluation of the integral at the poles $t = 0,\ 1,\ \ldots,\ l$ by the Residue Theorem. The orthogonality conditions follow by interchanging the order of integrations and observing the behaviour as the contour is allowed to become arbitrarily large.

The fact that $A_{n-1,l}$ has integer coefficients is obvious from the above forms. To show that $A_{n-2,l}$ has integer coefficients whenever $k = l$ and $n \geq 2$, we need to observe that if p is a prime and α is a positive integer so that $k \leq p^\alpha \leq k+i$ then p divides $\binom{i+k}{k}$. This is straightforward from Euler's formula for the largest power of a prime dividing a factorial. □

It is interesting to note that the orthogonality relation stated above encompasses an m-dimensional orthogonality conditions on the m-dimensional cube I^m. That is,

$$\int_{I^m} F(\pi_m(\boldsymbol{x}); k,l;m,n) \prod_{i=1}^{m} x_i^{s_i} d\boldsymbol{x} = 0, \qquad s_i \in \{0,1,\ldots,k-1\}$$

where $\pi_m(\boldsymbol{x}) := \prod_{i=1}^{m} x_i$ and where $\boldsymbol{x} := (x_1, x_2, \ldots, x_m)$. Here and throughout

$$\int_{I^m} g(\boldsymbol{x}) d\boldsymbol{x} = \int_0^1 \cdots \int_0^1 \int_0^1 g(\boldsymbol{x}) dx_1 dx_2 \cdots dx_m.$$

This can be deduced from the following theorem.

Theorem 2. *We have*

$$\int_{I^n} F(\pi_n(\boldsymbol{x})) d\boldsymbol{x} = \frac{1}{(n-1)!} \int_0^1 F(u) |\log u|^{n-1} du.$$

Proof. It is straightforward to verify by using the change of variables

$$u := xy, \quad v := \frac{1}{2}(x^2 - y^2)$$

that the double integral can be transformed as

$$\int_0^1 \int_0^1 F(xy) dx dy = -\int_0^1 F(u) \log u du.$$

In the general case, the proof follows recursively. □

Various specializations of the parameters give us the results we are looking for.

3. Irrationality of log 2 and $\zeta(3)$

For these cases we set $l = k$ and $n = m$ in Theorem 1, so

$$F(x; k, k; m, m) = \frac{(m-1)!}{2\pi i} \oint_\gamma \frac{\prod_{j=1}^{k}(t+j)^m}{\prod_{j=0}^{k}(t-j)^m} x^t dt$$

which are orthogonal in the sense that

$$\int_0^1 F(x; k, k; m, m) F(x; k', k'; m, m) dx = 0$$

for any two distinct positive integers k and k'. Also

$$\frac{d^j}{dx^j} F|_{x=1} \;=\; 0, \qquad j = 0, 1, \ldots, m-2$$

and

$$\frac{d^{m-1}}{dx^{m-1}} F|_{x=1} \;=\; (m-1)!.$$

The above equations can be easily verified by using the results of Theorem 1.

When we set $m = 1$ and $m = 2$ these are in fact the polynomials needed in the proof of the irrationality of $\log 2$ and $\zeta(3)$ respectively. Both proofs are very similar and we outline them below.

Theorem 3. *We have*

$$0 < \left| \int_0^1 \frac{F(x; k, k; 1, 1)}{1+x} dx \right| = |A_{0,k}(-1) \log 2 + R_k| \le (\sqrt{2}-1)^{2k} \log 2$$

where $A_{0,k}(-1)$ is an integer, R_k is a rational number, and $d_k R_k$ is an integer ($d_k := \operatorname{lcm}\{1, 2, \ldots, k\}$).

Proof. We decompose the rational function as follows

$$\frac{F(x; k, k; 1, 1)}{1+x} = (-1)^k \left\{ \frac{A_{0,k}(-1)}{1+x} + \frac{A_{0,k}(x) - A_{0,k}(-1)}{1+x} \right\}.$$

The first term of the right-hand side integrates to $A_{0,k}(-1) \log 2$ while the second integrates to a rational number which can be integralized by the factor d_k introduced by the integration process. To establish the inequality we note that

$$A_{0,k}(x) = \sum_{i=0}^{k} (-1)^i \binom{i+k}{k} \binom{k}{i} x^i = \frac{d^k}{dx^k} \frac{x^k (1-x)^k}{k!}$$

which is the shifted Legendre polynomial of degree k. By making use of the above Rodrigues' formula for these polynomials, and integrating by parts, we find

$$0 < \left| \int_0^1 \frac{F(x; k, k; 1, 1)}{1+x} dx \right| = \left| \int_0^1 \frac{x^k (1-x)^k}{(1+x)^{k+1}} dx \right| \le \left| \int_0^1 \frac{dx}{1+x} \right| \cdot (\sqrt{2}-1)^{2k}$$

since

$$0 < \frac{x(1-x)}{1+x} \le (\sqrt{2}-1)^2$$

for every $x \in (0, 1)$. $\square$

The irrationality of $\log 2$ now simply follows by multiplying both sides of the inequality by d_k, which by the prime number theorem is of $O(e^{(1+\delta)k})$ for any $\delta > 0$.

For $\zeta(3)$ we proceed as follows.

Theorem 4. *We have*

$$0 < \left| \int_0^1 \int_0^1 \frac{F(xy; k, k; 2, 2)}{1 - xy} dxdy \right| = |2A_{1,k}(1)\zeta(3) + R_k| \le 2\zeta(3)(\sqrt{2} - 1)^{4k}$$

where $A_{1,k}(1)$ is an integer, R_k is a rational number, and $d_k^3 R_k$ is an integer.

Proof. As in the proof of Theorem 3 we write

$$\frac{F(xy; k, k; 2, 2)}{1 - xy} = A_{1,k}(1)\,\frac{\log(xy)}{1 - xy} + \frac{A_{1,k}(xy) - A_{1,k}(1)}{1 - xy} \log(xy) + \frac{A_{0,k}(xy)}{1 - xy}.$$

Note that $F(1; k, k; 2, 2) = 0$ implies that $A_{0,k}(1) = 0$. Hence the third term above is a polynomial in xy, while the second term is $\log(xy)$ times a polynomial in xy. (Both polynomials are of degree $k - 1$.) Now recall that

$$-\frac{1}{2} \int_0^1 \int_0^1 \frac{\log(xy)}{1 - xy} = \zeta(3)$$

since

$$\int_0^1 \int_0^1 (xy)^n \log(xy) dx\, dy = \frac{-2}{(n + 1)^3}.$$

This accounts for the form of the double integral above. The right-hand inequality and the non-vanishing follow from the next lemmas and the comment after them.

The next detail is that $d_3^k R_k$ is an integer. For this we use the explicit form of $A_{1,k}$ and $A_{0,k}$ in Theorem 1 to see that $A_{1,k}$ and $d_k A_{0,k}$ have integer coefficients and observe that each integration introduces a d_k.

The final detail is to verify the inequalities of the theorem. This can be easily done by Lemma 3 below. This yields

$$\begin{aligned} 0 < \left| \int_0^1 \int_0^1 \frac{F(xy; k, k; 2, 2)}{1 - xy} dxdy \right| &\le \left| \int_0^1 \int_0^1 \int_0^1 \frac{dxdydv}{(1 - (1 - xy)v)} \right| \cdot (\sqrt{2} - 1)^{4k} \\ &= 2\zeta(3) \cdot (\sqrt{2} - 1)^{4k} \end{aligned}$$

since

$$0 < \frac{xyv(1 - x)(1 - y)(1 - v)}{1 - (1 - xy)v} \le (\sqrt{2} - 1)^4$$

for every $x, y, v \in (0, 1)$.

The irrationality of $\zeta(3)$ now follows directly by multiplying by d_k^3 and invoking the estimate $d_k = O(e^{(1+\delta)k})$ for every $\delta > 0$. $\square$

Lemma 1. *(Padé Approximation.) For each n there exist polynomials p_n and q_n of degree n so that*

$$\begin{aligned} \frac{(n!)^2}{2\pi i} \oint_\gamma \frac{x^t dt}{\prod_{j=0}^n (t - j)^2} &= (x - 1)^{2n+1} \int_0^1 \frac{v^n(1 - v)^n dv}{(1 - (1 - x)v)^{n+1}} \\ &= p_n(x) \log x + q_n(x) = O((x - 1)^{2n+1}) \end{aligned}$$

where γ is a simple contour containing all the poles. (In fact p_n/q_n is the (n, n) Padé approximant to $\log x$ at $x = 1$.)

Proof. An expansion of both integrals about the point $x = 1$, and a comparison of coefficients verifies the identity. $\square$

Lemma 2. *(Rodrigues-Type Formula.) With γ as in Lemma 1,*

$$\begin{aligned} F(xy;k,k;2,2) &= \frac{d^k}{dy^k}\frac{d^k}{dx^k}\frac{x^k y^k}{2\pi i}\oint_\gamma \frac{(xy)^t dt}{\prod_{j=0}^k (t-j)^2} \\ &= \frac{1}{(k!)^2}\frac{d^k}{dy^k}\frac{d^k}{dx^k}\int_0^1 \frac{(xy)^k(xy-1)^{2k+1}v^k(1-v)^k}{(1-(1-xy)v)^{k+1}}dv. \end{aligned}$$

Proof. This follows directly from the definition of $F(x;k,k;2,2)$ and differentiation in Lemma 1. $\square$

Lemma 3. *We have*

$$\int_0^1\int_0^1 \frac{F(xy;k,k;2,2)}{1-xy}dxdy = -\int_0^1\int_0^1\int_0^1 \frac{(xyv(1-x)(1-y)(1-v))^k}{(1-(1-xy)v)^{k+1}}dxdydv.$$

Proof. For i, k nonnegative integers

$$\begin{aligned} &\int_0^1\int_0^1 (xy)^{k+i}[(1-x)(1-y)]^k dxdy \\ &= \frac{-1}{(k!)^2}\int_0^1\int_0^1 \frac{1}{1-xy}\frac{d^k}{dy^k}\frac{d^k}{dx^k}(xy)^{k+i}(xy-1)^{2k+1}dxdy. \end{aligned}$$

(Both sides are equal to

$$\left[\frac{k!(k+i)!}{(2k+i+1)!}\right]^2$$

though this is not completely transparent. One can verify this by induction.) So

$$\begin{aligned} &\frac{1}{(k!)^2}\int_0^1\int_0^1 \frac{1}{1-xy}\frac{d^k}{dy^k}\frac{d^k}{dx^k}(xy)^k(xy-1)^{2k+1+i}dxdy \\ &= -\int_0^1\int_0^1 (xy(1-x)(1-y))^k(xy-1)^i dxdy. \end{aligned}$$

Hence

$$\begin{aligned} &\frac{1}{(k!)^2}\int_0^1\int_0^1 \frac{1}{1-xy}\frac{d^k}{dy^k}\frac{d^k}{dx^k}\frac{(xy)^k(xy-1)^{2k+1}}{(1-(1-xy)v)^{k+1}}dxdy \\ &= -\int_0^1\int_0^1 \frac{(xy(1-x)(1-y))^k}{(1-(1-xy)v)^{k+1}}dxdy \end{aligned}$$

which together with Lemma 2 completes the proof. $\square$

4. Irrationality of $\zeta(2)$

In this section the values of the parameters are $l = k$, $m = 1$, and $n = 2$. Then

$$F(x;k,k';1,2) = \frac{k!}{2\pi i}\oint_\gamma \frac{\prod_{j=1}^k (t+j)}{\prod_{j=0}^k (t-j)^2}x^t dt$$

and the orthogonality relations

$$\int_0^1 F(x;k,k;1,2)F(x;k',k';1,2,)dx = 0$$

hold for any two distinct positive integers k and k'.

Theorem 5. *We have*

$$0 < \left| \int_0^1 \frac{F(x;k,k;1,2)}{1-x} dx \right| = |A_{1,k}(1)\zeta(2) + R_k| \leq C \left(\frac{\sqrt{5}-1}{2} \right)^{5k}$$

where $A_{1,k-1}(1)$ is an integer, R_k is a rational number, $d_k^2 R_k$ is an integer, and C is an absolute constant.

Proof. The proof proceeds in an entirely analogous manner as the two previous ones. The inequality of the theorem is derived by making use of Lemma 1 again. Here, the Rodrigues-type formula has the form

$$F(x;k,k;1,2) = -\frac{1}{k!}\frac{d^k}{dx^k}\left[x^k(1-x)^{2k+1} \int_0^1 \frac{v^k(1-v)^k}{(1-(1-x)v)^{k+1}} dv. \right].$$

This yields

$$0 < \left| \int_0^1 \frac{F(x;k,k;1,2)}{1-x} dx \right| = \int_0^1 \int_0^1 \frac{x^k(1-x)^k v^k(1-v)^k}{(1-(1-x)v)^{k+1}} dv dx$$

and the inequality of the theorem follows by estimating the the maximum of the integrand on the unit square. □

The irrationality of $\zeta(2)$ now follows easily as it did for $\zeta(3)$.

5. Irrationality of Polylogarithms

The parameters we use in this case are $m = N$ and $n = 1$ where N is an arbitrary positive integer N. We also set $k = \lceil l/N \rceil$ throughout this section, that is k is the smallest integer not less than l/N. Using the results of Theorem 1, we now write the contour integral as

$$\begin{aligned} F_l(x) := F(x;k,l;N,1) &= \frac{1}{2\pi i} \frac{l!}{(k!)^N} \oint_\gamma \frac{\prod_{i=1}^k (t+i)^N}{\prod_{i=0}^l (t-i)} x^t dt \\ &= (-1)^l \sum_{i=0}^l (-1)^i \binom{i+k}{k}^N \binom{l}{i} x^i. \end{aligned}$$

We use this form to prove the irrationality of some values of the polylogarithm function [13] defined as

$$\mathrm{Li}_N(z) := \sum_{j=1}^\infty \frac{z^j}{j^N}, \qquad |z| < 1.$$

Theorem 6. *Let N and α be fixed positive integers. With $\pi_N(\boldsymbol{x}) := x_1 x_2 \cdots x_N$ and $k := \lceil l/N \rceil$, we have*

$$\begin{aligned} \left| \int_{I^N} \frac{F_l(\pi_N(\boldsymbol{x}))}{\pi_N(\boldsymbol{x}) - \alpha} d\boldsymbol{x} \right| &= \left| \frac{1}{\alpha} F_l(\alpha) \,\mathrm{Li}_N\left(\frac{1}{\alpha}\right) + R_{N,k} \right| \\ &\leq \frac{1}{2}\left(\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\alpha-1}} \right)^2 \left(\frac{2^{N(N+2)}}{(\sqrt{\alpha}+\sqrt{\alpha-1})^2} \right)^k. \end{aligned}$$

where $R_{N,k}$ is a rational number and $(d_l)^N R_{N,k}$ is an integer. The integral does not vanish for infinitely many positive integers l. Note that $F_l(\alpha)$ is an integer.

Proof. Using our standard decomposition, we write

$$\frac{F_l(\pi_N(\boldsymbol{x}))}{\pi_N(\boldsymbol{x})-\alpha} = \frac{F_l(\alpha)}{\pi_N(\boldsymbol{x})-\alpha} + \frac{F_l(\pi_N(\boldsymbol{x}))-F_l(\alpha)}{\pi_N(\boldsymbol{x})-\alpha}.$$

Integration of this form shows the structure of the terms as stated in the equality of the theorem.

To prove the inequality of the theorem, note that the orthogonality conditions satisfied by F and Theorem 2 imply that

$$\int_{I^N} \frac{F_l(\pi_N(\boldsymbol{x}))}{\pi_N(\boldsymbol{x})-\alpha} d\boldsymbol{x} = \int_{I^N} \left(\frac{1}{\pi_N(\boldsymbol{x})-\alpha} - P_{k-1}(\pi_N(\boldsymbol{x}))\right) F_l(\pi_N(\boldsymbol{x})) d\boldsymbol{x}$$

where $P_{k-1}(x)$ is any polynomial of degree $k-1$. Proceeding on, we obtain

$$\begin{aligned}
\left|\int_{I^N} \frac{F_l(\pi_N(\boldsymbol{x}))}{\pi_N(\boldsymbol{x})-\alpha} d\boldsymbol{x}\right| &\leq \max_{0\leq x\leq 1}\left|\frac{1}{x-\alpha} - P_{k-1}(x)\right| \max_{\boldsymbol{x}\in I^N} |F_l(\pi_N(\boldsymbol{x}))| \\
&\leq 2^{(l+k)N+l} \max_{0\leq x\leq 1}\left|\frac{1}{x-\alpha} - P_{k-1}(x)\right| \\
&\leq 2^{N(N+2)k} \max_{0\leq x\leq 1}\left|\frac{1}{x-\alpha} - P_{k-1}(x)\right|
\end{aligned}$$

where the upper bound of the second maximum is derived from the explicite form of the coefficients of the polynomial F_l.

To have the best possible upper bound for the remaining maximum in the above estimate, we wish to find a polynomial of degree $k-1$ such that the deviation of $\frac{1}{x-\alpha}$ from P_{k-1} is minimized. This problem was solved by Tchebysheff and is discussed by Akhieser [1] as in the next lemma from which the right hand-side inequality of the theorem follows.

The non-vanishing of the integral follows from the orthogonality conditions. Each F_l is a polynomial of exact degree l. So, with $g(x) := 1/(x-\alpha)$,

$$\int_{I^N} F_l(\pi_N(\boldsymbol{x}))g(\pi_N(\boldsymbol{x}))dx = 0$$

implies

$$\int_0^1 F_l(x)g(x)(\log x)^{N-1}dx = 0.$$

So vanishing at all large l implies that $g(x)(\log x)^{N-1}$ is a polynomial, which is a contradiction. □

Lemma 4. *The $(k-1)$-th degree polynomial*

$$P_{k-1}(x) = \frac{1}{x-\alpha} - \frac{M}{2}\left[v^{k-1}\frac{a-v}{1-av} + v^{-(k-1)}\frac{1-av}{a-v}\right]$$

where

$$x := \frac{1}{2}\left(v+\frac{1}{v}\right), \quad \alpha := \frac{1}{2}\left(a+\frac{1}{a}\right), \quad |a|<1, \quad M := \frac{4a^{k+1}}{(1-a^2)^2}$$

and

$$\max_{|x|\leq 1}\left|\frac{1}{x-\alpha} - P_{k-1}(x)\right| = M,$$

deviates the least from $\frac{1}{x-\alpha}$ on the interval $[-1,1]$ in the uniform norm.

Proof. See [1]. □

¿From these we obtain, on using $k=\lceil l/N\rceil$, and transforming the above Lemma to $[0,1]$, that

$$\left|\int_{I^N}\frac{F_l(\pi_N(\boldsymbol{x}))}{\pi_N(\boldsymbol{x})-\alpha}d\boldsymbol{x}\right|\le\frac{1}{2}\left(\frac{1}{\sqrt{\alpha}}+\frac{1}{\sqrt{\alpha-1}}\right)^2\left(\frac{2^{N(N+2)}}{(\sqrt{\alpha}+\sqrt{\alpha-1})^2}\right)^k.$$

Theorem 7. *For positive integers α the polylogarithm function values $\mathrm{Li}_N(1/\alpha)$ are irrational provided*

$$\alpha>\left(\frac{1}{2}\left(\sqrt{\beta}+\frac{1}{\sqrt{\beta}}\right)\right)^2$$

where $\beta:=2^{N(N+2)}e^{(1.0388\ldots)N^2}$.

Proof. We multiply the form in Theorem 6 by $(d_l)^N$ and demand for large k that

$$\begin{aligned}\left(\frac{2^{N(N+2)}}{(\sqrt{\alpha}+\sqrt{\alpha-1})^2}\right)^k(d_l)^N &\le \left(\frac{2^{N(N+2)}}{(\sqrt{\alpha}+\sqrt{\alpha-1})^2}\right)^k e^{((1.0388\ldots)lN/k)k}\\ &\le \left(\frac{2^{N(N+2)}e^{(1.0388\ldots)N^2}}{(\sqrt{\alpha}+\sqrt{\alpha-1})^2}\right)^k<1.\end{aligned}$$

For this is suffices that

$$\frac{2^{N(N+2)}e^{(1.0388\ldots)N^2}}{(\sqrt{\alpha}+\sqrt{\alpha-1})^2}<1$$

where we have used the sharp global estimate $d_l\le e^{(1.0388\ldots)l}$ (see [15]). The theorem now follows by solving the above inequality for α □

We can extend our results in Theorem 6 to show the irrationality of the function

$$\Phi(z,N,u):=\sum_{j=0}^{\infty}\frac{z^j}{(u+j)^N},\qquad |z|<1,\ \ u\ne 0,-1,-2,\ldots$$

(several properties of Φ may be found in [11]) whenever N is an integer, u is a rational number, and $z=1/\alpha$, where α is an integer greater than a constant depending only on N and u. This follows from

Theorem 8. *Let N and α be fixed positive integers. Let $u=q/p$ be a rational mumber where p and q are nonzero integers. Then there exists a constant $C_{N,u}$ depending only on N and u so that*

$$\begin{aligned}\left|\int_{I^N}\frac{F_l(\pi_N(\boldsymbol{x}))^{p/q}dx}{(\pi_N(\boldsymbol{x}))^{p/q}-\alpha}\right| &= \left|\frac{1}{\alpha}\left(\frac{q}{p}\right)^N F_l(\alpha)\Phi\left(\frac{1}{\alpha},N,\frac{q}{p}\right)+R_{N,k}\right|\\ &\le (C_{N,u})^k\max_{0\le x\le 1}\left|\frac{1}{x-\alpha}-P_{k-1}(x)\right|\end{aligned}$$

where $R_{N,k}$ is a rational number, $d_{pl+q}^N R_{N,k}$ is an integer, and $P_{k-1}(x)$ is an arbitrary polynomial of degree $k-1$. The integral is not zero for infinitely many positive integers l. Note that $F_l(\alpha)$ is an integer.

Proof. This follows analogously to the proof of Theorem 6 with the substitution

$$\pi_N(\boldsymbol{x})\to(\pi_N(\boldsymbol{x}))^{p/q}.$$

□

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