HEREDITARY APPROXIMATION PROPERTY

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Abstract. If X is a Banach space such that the isomorphism constant to ℓ_2^n from n dimensional subspaces grows sufficiently slowly as $n \to \infty$, then X has the approximation property. A consequence of this is that there is a Banach space X with a symmetric basis but not isomorphic to ℓ_2 so that all subspaces of X have the approximation property. This answers a problem raised in 1980 [8]. An application of the main result is that there is a separable Banach space X that is not isomorphic to a Hilbert space, yet every subspace of X is isomorphic to a complemented subspace of X. This contrasts with the classical result of Lindenstrauss and Tzafriri [15] that a Banach space in which every closed subspace is complemented must be isomorphic to a Hilbert space.

Dedicated to the memory of Joram Lindenstrauss

1. Introduction

The first Banach space not isomorphic to a Hilbert space, all of whose subspaces have the approximation property, was constructed in [8]. We say that such a space has the *hereditary approximation property (HAP)* or is a HAPpy space. Later on Pisier [23], [24] developed the theory of spaces called weak Hilbert spaces that share many properties of Hilbert space and proved that they all have the HAP.

The spaces constructed in [8] as well as all weak Hilbert spaces are asymptotically Hilbertian. A space X is *asymptotically Hilbertian* provided there is a constant β such that for every n there is a finite codimensional subspace of X all of whose n dimensional subspaces are β-isomorphic to the n-dimensional Hilbert space ℓ_2^n. It was noted in [8] that an asymptotically Hilbertian space is superreflexive (= isomorphic to a uniformly convex space) and cannot have a symmetric or even subsymmetric basis unless it is isomorphic to ℓ_2. This induced the first named author to conclude [8] with two problems:

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(S) Is there a HAPpy space that has a symmetric basis but is not isomorphic to ℓ_2?

(R) Is every HAPpy space reflexive?

In this paper we give an affirmative answer to (S) by constructing a HAPpy Orlicz sequence space that is not isomorphic to ℓ_2. Problem (R) remains open.

Before stating in more detail the results herein, we recall some definitions and set our notation. "Space" means "infinite dimensional Banach space" unless specified otherwise. $L(X)$ denotes the space of bounded operators on the space X while $F(X)$ denotes the finite rank operators in $L(X)$. The identity operator on X is written Id_X. B_X denotes the unit ball of X.

Recall that a Banach space X is said to have the *approximation property* (AP) if for every compact set K in X and for every $\varepsilon > 0$, there is a $T \in F(X)$ such that $\|Tx - x\| \leq \varepsilon$ for every $x \in K$.

As was already mentioned, we say that a Banach space has the *hereditary approximation property* (HAP) if all of its subspaces have the AP. Results of Davie/Figiel and the second author combined with results of Krivine and Maurey and Pisier (cf. [16, Theorem 1.g.6]) imply that if X has the HAP then X is of type $2-\varepsilon$ and of cotype $2+\varepsilon$ for every $\varepsilon > 0$. This means that X has to be "very close" to a Hilbert space since a space that is both of type 2 and cotype 2 is isomorphic to a Hilbert space (this is a remarkable result of Kwapień from 1972 [3, Corollary 12.20]).

Both the AP and the HAP are in a natural way related to the trace formula. Let us recall here some main points (this topic is discussed in more detail in [24, Chap. 4]):

For $x^* \in X^*, x \in X$ let $x^* \otimes x \in F(X)$ be defined by

$$(x^* \otimes x)(y) = x^*(y)x.$$

A $T \in B(X)$ is called *nuclear* if $T = \sum_{i=1}^{\infty} x_i^* \otimes x_i$ with $\sum \|x_i^*\|\|x_i\| < \infty$. Let $N(T)$ denote the space of all nuclear operators on X. It is tempting to define the trace of a $T \in N(X)$ by

$$\text{tr } T = \sum_{i=1}^{\infty} x_i^*(x_i).$$

Grothendieck [5] (cf. [16, Theorem 1.a.4.]) discovered that tr T is well defined for every $T \in N(X)$ if and only if X has the approximation property, i.e. X does not have the AP iff there are $x_i^* \in X^*, x_i \in X$ so that

$$\sum \|x_i^*\| \|x_i\| < \infty, \sum x_i^*(x) x_i = 0 \quad \text{for every } x \in X, \tag{1}$$
$$\text{but } \sum x_i^*(x_i) \neq 0.$$

Suppose now that X is a complex Banach space with the AP. It is natural to ask whether the trace formula holds for every $T \in N(X)$. More precisely, let $T \in N(X)$ be such that $\sum |\lambda_j(T)| < \infty$, where $\lambda_1(T), \lambda_2(T), \dots$ are all the eigenvalues of T, with their multiplicities (this assumption is necessary, because for every X not isomorphic to a Hilbert space there is a $T \in N(X)$ such that $\sum |\lambda_j(T)| = \infty$, by a result of [9]). We ask whether then

$$\text{tr } T = \sum \lambda_j(T). \tag{2}$$

If the trace formula (2) holds for every $T \in N(Y)$ with summable eigenvalues, then Y is a HAPpy space. Indeed, suppose Y fails the HAP, and let $X \subset Y$ be a subspace without the AP. By the Grothendieck result quoted above, there are $x_i^* \in X_i^*, x_i \in X$ so that (1) holds. Let $y_i^* \in Y^*$ be a Hahn-Banach extension of x_i^* and let $T \in N(Y)$ be defined by $T = \sum y_i^* \otimes x_i$. Then tr $T = \sum y_i^*(x_i) = \sum x_i^*(x_i) \neq 0$. On the other hand, $Tx = 0$ for every $x \in X$, $TY \subset X$, therefore $T^2 = 0$, hence 0 is the only eigenvalue of T. Therefore (2) does not hold.

To put it tersely, the AP is necessary (and sufficient) for the formula (2) to make sense, while the HAP is necessary for the formula (2) to be true. We do not know if it is sufficient.

The information about the class $\mathcal{L}$ of Banach spaces satisfying (2) is still very scarce. Lidskii proved in [14] that Hilbert spaces belong to $\mathcal{L}$. We do not know whether the non weak Hilbert HAPpy spaces constructed in [8] belong to $\mathcal{L}$. Pisier [23], [24] proved that the weak Hilbert spaces belong to $\mathcal{L}$, and, for the time being, there are no other examples, although it might be true that every HAPpy space is in $\mathcal{L}$.

Rather more is known about the class of spaces that satisfy the HAP. Unfortunately, this class is very difficult to work with, partly because the HAP is not very stable. For example, there are two HAPpy spaces whose direct sum fails the HAP [1]. In fact, all the known examples of HAPpy spaces come from verifying that some hereditary (i.e. which passes to subspaces) property implies the AP and constructing spaces that satisfy the property. Examples of such properties are several conditions that are equivalent to the weak Hilbert property [24] and properties of being sufficiently asymptotically Hilbertian [8]. To be more precise, it is enough that X satisfies the condition that for some β and infinitely many n, there is a $\log n$ codimensional subspace all of whose 4^n dimensional subspaces are β-isomorphic to $\ell_2^{4^n}$ [8]. A space that satisfies this need not be a weak Hilbert space. On the other hand, it is open whether every weak Hilbert space satisfies the condition, although it is true for a weak Hilbert space that has an unconditional basis [21].

In this paper we give another hereditary condition that implies the AP but does not imply the asymptotically Hilbertian property. The condition is that $d_n(X)$ goes to infinity sufficiently slowly , where $d_n(X)$ is the supremum over the n-dimensional subspaces E of X of the isomorphism constant from E to ℓ_2^n (cf. (7)). Hence, in order to get an affirmative answer to problem (S), it is enough, given any sequence $\delta_k \to \infty$, to produce a Banach space with symmetric basis X non isomorphic to ℓ_2 such that $d_k(X) \le \delta_k$ for every k. It is more or less obvious that this can be done, but we were unable to find such constructions in the literature. The simplest ones we know are of modified Tsirelson/Schlumprecht type, presented in section 3 as example (A). We also show in example (B) that there are Orlicz spaces other than ℓ_2 that have this property. This looks rather obvious but is tedious to verify.

2. Basic Theorem

Let X be a Banach space. Let n be a natural number. For $m \ge n$ let

$$(3)\quad f(n,m) = f_X(n,m) = \sup_{E\subset X, \dim E = n} \inf\{\|T\| : T_{|E} = Id_E \text{ and rk } T \le m\}.$$

Observe that if $\dim E = n$, then $f(n,n)$ is the minimal norm of a projection of X onto E, thus

$$f(n,n) = \lambda_n(X),$$

where, as usual, $\lambda_n(X)$ is the supremum over all n dimensional subspaces E of the relative projection constant of E in X.

Also, by taking a weak cluster point of an appropriate sequence of T's, we see that the infimum on the right side of (3) is a minimum provided X is reflexive.

With this notation the space X is said to have the λ-*uniform approximation property* (λ-UAP) if for every $n \in \mathbb{N}$ there is $j(n)$ so that $f(n, j(n)) \le \lambda$. In this case $j(n)$ is called a λ-*uniformity function* of X.

We say that the space X has the *uniform approximation property* (UAP) if it has the λ-uniform approximation property for some $\lambda < \infty$.

A HAPpy space X will be said to have the *hereditary UAP* (denoted HUAP)- or to be uniformly HAPpy if all of its subspaces have the UAP.

There are two basic ingredients in the proof of Theorem 2.1. The first is the averaging argument of Lindenstrauss and Tzafriri [18] to prove that a uniformly convex space with the UAP actually has the $(1+\varepsilon)$-UAP for every $\varepsilon > 0$. An important difference is that in [18] uniformly bounded operators were averaged to produce an operator with norm close to one. Here we get the same conclusion but average operators whose norms grow slowly.

The second ingredient in the proof of Theorem 2.1 is the argument in [8] that spaces that are sufficiently Hilbertian must have many finite rank projections with controlled norms.

We begin with a simple lemma that is a variation of one in [18].

Lemma 1. *Assume that X is a Banach space and $\delta > 0$, $\varepsilon > 0$ satisfy*

$$x, y \in B_X, \|x - y\| > \varepsilon \Rightarrow \|\tfrac{x+y}{2}\| < 1 - 2\delta. \tag{4}$$

Let $A = \frac{3}{\delta}$. If $T \in F(X)$ with $\text{rk}\, T = k$ *and*

$$K = \{x \in B_X : \|Tx\| \geq (1-\delta)\|T\|\},$$

then K can be covered by $[A^k]$ sets of diameter ε.

Proof. By a standard volumetric estimate, TB_X can be covered by $[A^k]$ balls of diameter $2\|T\|\delta$, centered at a maximal $\|T\|\delta$-separated subset of TB_X; say $TB_X \subset \bigcup_{i=1}^{[A^k]} B_i, \text{diam } B_i \leq 2\|T\|\delta$. Let $K_i = (T^{-1}B_i) \cap K$. We claim that diam $K_i \leq \varepsilon$.

For assume that there are $x, y \in K_i$ with $\|x - y\| > \varepsilon$. By (4), $\|\frac{x+y}{2}\| < 1 - 2\delta$, hence $\|\frac{Tx+Ty}{2}\| < (1-2\delta)\|T\|$. Since $Tx, Ty \in B_i$, we have $\|\frac{Tx-Ty}{2}\| \leq \frac{1}{2}\text{diam } B_i \leq \|T\|\delta$. Summing these inequalities, we have by the triangle inequality that $\|Tx\| < (1-\delta)\|T\|$, hence $x \notin K$, a contradiction. ∎

Notice that if X is uniformly convex and $1 > \varepsilon > 0$, condition (4) is satisfied for all sufficiently small $\delta > 0$.

Lemma 1 is used to prove that, under certain (extremal) conditions, for a fixed n we can shrink $f_X(n, j)$ (see (3)) by a constant factor by changing j to a suitable larger integer.

To formulate the next lemma about a uniformly convex space X, we make the following technical assumptions:

$\delta > 0$ and $0 < \varepsilon < 3/4$ satisfy (4),

δ is so small that we have

$$\tfrac{5}{8} + \tfrac{1}{2}(1 + \tfrac{\delta}{2})\varepsilon \leq 1 - \tfrac{\delta}{4}. \tag{5}$$

Then we denote

$A = \frac{3}{\delta}$, $\alpha = (1 + \frac{\delta}{2})^{-1}$, $\beta = 1 - \frac{\delta}{4}$.

Lemma 2. ***(The main lemma.)*** *Let X be a uniformly convex space, let A, α, β be as above. If for some n, j with $n \le j \le N$ we have $f_X(n, j) = f(n, j) \ge \alpha f([A^j] + n, N)$, then*

$$f(n, N + j) \le \max(4, \beta f(n, j)).$$

Proof. Let $E \subset X, \dim E = n$. We shall find U such that $U_{|E} = Id_E, \mathrm{rk}\, U \le N + j$ and $\|U\| \le \beta f(n, j)$.

Let T be such that $T_{|E} = Id_E, \mathrm{rk}\, T \le j$ and $\|T\| \le f(n, j)$ (remember that X is reflexive). Without loss of generality we can assume that $\|T\| \ge \alpha f([A^j] + n, N)$, and also assume that $\|T\| \ge 4$ since otherwise we are done.

Let $K = \{x \in B_X : \|Tx\| \ge (1-\delta)\|T\|\}$ and get $K_1, \dots, K_{[A^j]} \subset B_X$ from Lemma 1 so that $K \subset \bigcup_{i=1}^{[A^j]} K_i$ and diam $K_i \le \varepsilon$. For $i = 1, \dots, [A^j]$ pick any $y_i \in K_i$. Let S be an operator of rank at most N so that $S_{|E} = Id_E, Sy_i = y_i$ for $i = 1, \dots, [A^j]$ and $\|S\| \le f([A^j]+n, N)$. Thus

$$\|S\| \le \alpha^{-1}\|T\| = (1 + \tfrac{\delta}{2})\|T\|. \tag{6}$$

Set now $U = \frac{1}{2}(T + S)$. Evidently $U_{|E} = Id_E$ and $\mathrm{rk}\, U \le N + j$.

Let $x \in B_X$.

If $x \in K_i$, then, by (6) and (5),

$$\begin{aligned}\|Ux\| &\le \tfrac{1}{2}(\|Tx\| + \|Sy_i\| + \|S(x - y_i)\|) \le \tfrac{1}{2}(\|T\| + 1 + (1 + \tfrac{\delta}{2})\varepsilon\|T\|)\\ &\le \tfrac{1}{2}\|T\|(1 + \tfrac{1}{4} + (1 + \tfrac{\delta}{2})\varepsilon) \le \beta\|T\|.\end{aligned}$$

If $x \in B_X \setminus \bigcup_{i=1}^{[A^j]} K_i$, then, by (6),

$$\|Ux\| \le \tfrac{1}{2}[(1 - \delta)\|T\| + (1 + \tfrac{\delta}{2})\|T\|].$$

In both cases we obtain $\|Ux\| \le \beta\|T\| \le \beta f(n, j)$. ∎

Denote $d_n(X) = \sup\, \{d(E, l_2^n) : E \subset X, \dim E = n\}$. Here $d(E, F)$ is the isomorphism constant from E to F, i.e. the infimum of $\|T\| \cdot \|T^{-1}\|$ as T ranges over all isomorphisms from E onto F. In the sequel we are concerned with spaces X for which $d_n(X) \to \infty$ very slowly. Pisier [22, p. 348] proved that such a space is superreflexive and hence has an equivalent uniformly convex norm, under which the space satisfies (4) and (5) for some ε and δ.

In the proof of Theorem 2.1 we also need the concept of projection constant. For a subspace E of X, recall that $\lambda(E) = \lambda(E; X)$ is the infimum of $\|P\|$ as P ranges over all projections from X onto E. The parameter $\lambda_n(X)$ is the supremum of $\lambda(E; X)$ as E ranges over the n

dimensional subspaces of X. Already in [8] the relation between $\lambda_n(X)$ and $d_m(X)$ played an important role.

Here we use the fact that $\lambda_n(X) \leq Cd_{4^n}(X)$, although the weaker estimate proved in [8] would serve equally well. The improved estimate follows from the following lemma proved by Vitali Milman a couple of years after the results in [8] were obtained.

Lemma 3. *If E embeds isometrically into ℓ_∞^N and $E \subset X$, then $\lambda(E;X)$ is the infimum of $\lambda(E;F)$ as F ranges over the N dimensional subspaces of X that contain E.*

Proof. The number $\lambda(E;X)$ is, by duality, the supremum of $|\mathrm{tr}\ (T)|$ as T ranges over operators from E to X that have nuclear norm less than one and map E into E (or, by a small perturbation argument, map E onto E). Given such a T with $TE = E$ and regarding E as a subspace of ℓ_∞^N, we can extend T to an operator $\tilde{T} : \ell_\infty^N \to X$ that also has nuclear norm less than one. The nuclear norm of $\tilde{T}$ is $\sum_{k=1}^N \|\tilde{T}(e_k)\|$, where (e_k) is the unit vector basis of ℓ_∞^N. But $\tilde{T}$ can be written as $\sum_{k=1}^N e_k^* \otimes \tilde{T}(e_k)$ and $\sum_{k=1}^N \|e_k^*\| \cdot \|\tilde{T}(e_k)\| < 1$, so T has nuclear norm less than one when considered as an operator into the (at most N dimensional) subspace span $(\tilde{T}e_k)_{k=1}^N$. ∎

Corollary 1. *For every space X, $\lambda_n(X) \leq 2d_{4^n}(X)$.*

Proof. Let E be any n dimensional subspace of X. Then E is less than 2-isomorphic to a subspace of $\ell_\infty^{4^n}$, so Lemma 3 gives a 4^n dimensional subspace F of X that contains E and so that $\lambda(E;X) < 2\lambda(E;F)$. But clearly $\lambda(E;F) \leq d(F, \ell_2^{4^n}) \leq d_{4^n}(X)$. ∎

Let us notice that for type 2 spaces a much better estimate is valid, namely $\lambda_n(X) \leq Cd_n(X)$, where C is the type 2 constant of X (cf. [27]).

Given a map $D : \mathbb{N} \to \mathbb{N}$ and a natural number k, by $D^{\langle k \rangle}$ let us denote the k iterate of D, i.e. $D^{\langle k \rangle} = D \circ D \circ \cdots \circ D$, k times. Let $D(j) = 3[A^j]$ and let $\gamma(j)$ be the $3^j + 1$ iterate of D of 1, i.e. $\gamma(j) = D^{\langle 3^j+1 \rangle}(1)$.

Theorem 2.1. *Let X be a Banach space, let $0 < \alpha \leq \beta < 1$ and $A > 1$ be as in Lemma 2. If*

$$d_{\gamma(j)}(X) = o(\beta^{-j}), \tag{7}$$

then X has the 4-UAP . Consequently, X has the HUAP and there exists a function $j(n)$ that is a 4-uniformity function for all $Y \subset X$.

Proof. For a fixed $n \in \mathbb{N}$, we define by recursion some numbers $\kappa_j(n)$, which play the role of N in Lemma 2. Formally, we define:

$$\kappa_0(n) = n, \kappa_{j+1}(n) = \kappa_j(n) + \kappa_j(2A(\kappa_j(n))).$$

For $M \in \mathbb{N}$, let us denote

$$s_j(M) = \max\{f(n, \kappa_j(n)) : \kappa_j(n) \leq M\}.$$

We claim that

$$s_{j+1}(M) \leq \max(4, \beta s_j(M)). \tag{8}$$

Indeed, let n be such that $\kappa_{j+1}(n) \leq M$. We shall show that

$$f(n, \kappa_{j+1}(n)) \leq \beta s_j(M).$$

First, since $f(n, \kappa_{j+1}(n)) \leq f(n, \kappa_j(n))$, without loss of generality we can assume that $f(n, \kappa_j(n)) \geq \beta s_j(M)$.

In particular, since $\kappa_j(2A(\kappa_j(n)) \leq \kappa_{j+1}(n) \leq M$, we get

$$f(n, \kappa_j(n)) \geq \beta f(B(\kappa_j(n)), \kappa_j(B(\kappa_j(n)))).$$

Also $\beta \geq \alpha$; therefore, by Lemma 2,

$$f(n, \kappa_j(n) + \kappa_j(B(\kappa_j(n)))) \leq \max(4, \beta s_j(M))$$

which is (8).

Let $n, j \in \mathbb{N}$, let $M = \kappa_j(n)$. Since $s_0(M) \geq s_1(M) \geq \ldots$, by induction we get from (8) that

$$s_j(\kappa_j(n)) \leq \max(4, \beta^j s_0(\kappa_j(n))).$$

By Corollary 1 , we have

$$s_0(M) = f(M, M) = \lambda_M(X) \leq 2d_{4^M}(X),$$

thus

$$s_j(\kappa_j(n)) \leq \max(4, 2\beta^j d_{4^{\kappa_j(n)}}(X)),$$

in particular

$$f(n, \kappa_j(n)) \leq \max(4, 2\beta^j d_{4^{\kappa_j(n)}}(X)). \tag{9}$$

Let us now estimate $\kappa_j(n)$. By induction we obtain that $\kappa_j(n) \leq D^{\langle 3^j \rangle}(n)$. For a given n, let J be such that $n \leq D^{\langle 3^J \rangle}(1)$, thus $\kappa_j(n) \leq D^{\langle 3^{(j+J)} \rangle}(1)$. Observe that $4^k \leq D(k)$, thus

$$4^{\kappa_j(n)} \leq D(\kappa_j(n)) \leq D^{\langle 3^{(j+J)}+1 \rangle}(1) = \gamma(j+J).$$

Hence

$$\beta^j d_{4^{\kappa_j(n)}}(X) \leq \beta^j d_{\gamma(j+J)}(X) = 2\beta^{-J} \beta^{j+J} d_{\gamma(j+J)}(X)^{1/2}$$

and, by (7), this tends to 0. Therefore, by (9), $f(n, \kappa_j(n)) \leq 4$ for sufficiently large j. ∎

3. Main Application

As was mentioned in the introduction, it has been an open question for 30+ years whether there exists a HAPpy space with symmetric basis that is not isomorphic to a Hilbert space. We prove the existence of such spaces. By Theorem 2.1, it is just enough, given any sequence $\delta_k \to \infty$, to produce a Banach space with symmetric basis X non isomorphic to ℓ_2 such that $d_k(X) \leq \delta_k$ for every k. We give two such examples:

(A) We build a space $X^{(2)}$ of modified Tsirelson/Schlumprecht type ([8]), [26]) that has a symmetric basis and so that $d_n(X^{(2)})$ tends to infinity as slowly as we wish.

(B) We show that there are Orlicz sequence spaces that have the same property; the arguments in this case are more involved than in (A).

Let us mention that if we are just looking for a nonhilbertian space X with symmetric basis so that $d_n(X)$ satisfy the estimate of Theorem 2.1, such a space has already appeared in the literature: the space $S(T^2)$, constructed by Cassazza and Nielsen in [2] satisfies this estimate, as follows from Proposition 3.9. in [2].

(A) Given any positive sequence $b_n \downarrow 0$, let $\mathbf{a} = a_n \downarrow 0$ be a strictly decreasing sequence such that $a_n > \delta b_n$ for some positive δ, $a_n = 1$, $a_n a_m \leq a_{nm}$, and na_n is concave. We build a space of sequences $X = X(\mathbf{a})$ so that the unit vector basis is a 1-symmetric basis for X, $X \neq \ell_1$ even up to an equivalent renorming, and so that for any choice (x_k) of n disjoint vectors in X we have $\| \sum_{k=1}^n x_k\| \geq a_n \sum_{k=1}^n \|x_k\|$. Then any collection of n disjointly supported unit vectors in the 2-convexification $X^{(2)}$ of X is a_n^{-n} equivalent to an orthonormal basis (see ([4] or [17, Section 1.d]) for a discussion of p-convexification). As was explained already in [8], this does the job (the reason being that an n dimensional subspace of a Banach lattice is a small perturbation of a subspace of the span of some set of n^n disjoint vectors).

The space X is the completion of c_{00} under the unique norm that satisfies the implicit equation

$$\|x\| = \|x\|_{c_0} \vee \sup \{a_n \sum_{k=1}^{n} \|A_k x\| : n = 1, 2, 3, \ldots ; (A_k) \text{ disjoint}\}. \tag{10}$$

(Multiplication of x by the indicator function of A is denoted by Ax.) The now standard argument for the existence of the norm $\| \cdot \|$ goes back to [4]. Define two sequences of norms on c_{00} by recursion. Set

$\|x\|_1 = \|x\|'_1 = \|x\|_{c_0}$ and

$$\|x\|_{m+1} = \|x\|_m \vee \sup \{a_n \sum_{k=1}^{n} \|A_k x\|_m : n = 2, 3, \ldots ; (A_k) \text{ disjoint}\}$$

$$\|x\|'_{m+1} = \|x\|'_{c_0} \vee \sup \{a_n \sum_{k=1}^{n} \|A_k x\|'_m : n = 2, 3, \ldots ; (A_k) \text{ disjoint}\}.$$

An easy induction argument shows that $\|x\|_n = \|x\|'_n$ from which it follows that $\|\cdot\|_n$ converges to a norm that satisfies (10).

It is obvious that the unit vector basis is a normalized 1-symmetric basis for X and that for any choice of (x_k) of n disjoint vectors in X we have $\|\sum_{k=1}^{n} x_k\| \geq a_n \sum_{k=1}^{n} \|x_k\|$. Just as in [26, Lemma 4], the submultiplicativity of a_n and the concavity of na_n easily implies that $\|\sum_{i=1}^{n} e_i\| = na_n$, so the constructed space is not ℓ_1 under an equivalent norm.

(B) Perhaps this theorem is known but we were unable to find it in the literature:

Theorem 3.1. *Let $1 < \delta_k \to \infty$. There exists an Orlicz space ℓ_M of type 2, non isomorphic to ℓ_2 so that $d_k(\ell_M) \leq \delta_k$ for every k.*

M will be defined by $M(x) = F(x^2)$ where $F : [0,1] \to [0,1]$ is a convex function such that $F(0) = 0, F(1) = 1$(i.e. ℓ_M is the 2-convexification of ℓ_F) .

Observe that for every x, y, $F(x) + F(y) \leq F(x+y)$ (in particular, $2F(x) \leq F(2x)$) and that $\frac{F(x)}{x}$ is an increasing function of x.

For $0 \leq a < b \leq 1$, let $F_{a,b}(t) = \frac{t-a}{b-a}F(b) + \frac{b-t}{b-a}F(a)$ and let us denote

$$\Phi_k(a) = \frac{1}{a}F^{-1}(2F_{\frac{a}{2},2ka}(a)).$$

Lemma 4.

$$d_k(\ell_M) \leq (\sup_{0<a\leq\frac{1}{2k}} \Phi_k(a) + 1)^{1/2}.$$

Proof. It is well known [24, Lemma 13.3(ii)] that $d_k(\ell_M) \leq \mu$, provided

$$(11) \qquad \|(\sum_{i=1}^{k} y_i^2)^{1/2}\| \geq \mu^{-1}(\sum_{i=1}^{k} \|y_i\|^2)^{1/2}$$

for every $y_1, \ldots, y_k \in X$ (notice that ℓ_M is 2-convex, so that

$$\|(\sum_{i=1}^{k} y_i^2)^{1/2}\| \leq (\sum_{i=1}^{k} \|y_i\|^2)^{1/2}).$$

Let us fix $y_1, \dots, y_k \in X$ with $\sum \|y_i\|^2 = 1$. Set $t_i = \|y_i\|^2$, thus $\sum t_i = 1$. Let us denote $a_i(j) = t_i^{-1} y_i(j)^2$. By the definition of the norm in X we have for $i = 1, \dots, k$

$$\sum_{j=1}^{\infty} F(a_i(j)) = 1. \tag{12}$$

Without loss of generality assume that $t_1 \geq t_2 \geq \cdots$. Let $1 \leq m \leq k$ be such that $t_1 + \cdots + t_{m-1} < \frac{1}{2} \leq t_1 + \cdots + t_m$. Then $t_m + t_{m+1} + \cdots + t_k > \frac{1}{2}$, hence $t_m \geq \frac{1}{2k}$, whence $t_1 \geq \cdots \geq t_m \geq \frac{1}{2k}$ and $t_1 + \cdots + t_m \geq \frac{1}{2}$.

Let $\alpha = 2\sup\{\Phi_k(a) : 0 < a < \frac{1}{2k}\}$. Then for every $0 \leq a \leq 1$, all $a_i \in (\frac{a}{2}, 2ka)$, and all t_i with $\Sigma t_i \leq 1$ and $\Sigma t_i a_i \leq a$,

$$\sum t_i F(a_i) \leq F(\alpha/2a).$$

We shall prove that

$$\sum_{j=1}^{\infty} F\left(\frac{\alpha+1}{2} \sum_{i=1}^{m} t_i x_i(j)\right) \geq \frac{1}{2}, \tag{13}$$

hence $\sum\limits_{j=1}^{\infty} F((\alpha+1) \sum\limits_{i=1}^{m} t_i a_i(j))) \geq 1$, thus $\| \sum\limits_{i=1}^{m} y_i^2 \| \geq (\alpha+1)^{-\frac{1}{2}}$.

Let us observe that, by (12),

$$\sum_{j=1}^{\infty} \sum_{i=1}^{m} t_i F(a_i(j)) = \sum_{i=1}^{m} t_i \geq \frac{1}{2}.$$

It is clear that (13) follows from the following inequality, valid for any $0 \leq a_1, \dots, a_m \leq 1$:

$$F\Big(\frac{\alpha+1}{2} \sum_{i=1}^{m} t_i a_i\Big) \geq \sum_{i=1}^{m} t_i F(a_i) \tag{14}$$

To prove (14), let $a = \sum\limits_{i=1}^{m} t_i a_i$. Let us observe that for every i, $a_i \leq t_i^{-1} a \leq 2ka$. Since $\sum\limits_{i: a_i < \frac{a}{2}} t_i a_i < \frac{a}{2}$, we have $\sum\limits_{i: a_i \geq \frac{a}{2}} t_i a_i > \frac{a}{2}$. Therefore

$$\sum t_i F(a_i) = \sum_{i: a_i \geq \frac{a}{2}} + \sum_{i: a_i < \frac{a}{2}} \leq F(\frac{\alpha}{2} a) + F(\frac{a}{2}) \leq F(\frac{\alpha+1}{2} a).$$

■

Lemma 5. *Let $F(e^{-t}) = e^{-t-\varphi(t)}$ where $\varphi : [0,\infty) \to [0,\infty)$ is an increasing, convex, continuous function and $\lim_{t\to\infty} \varphi(t) = \infty$. Then*

$$d_k(\ell_M) \le 2kF^{-1}(\frac{1}{k}) \quad \text{for} \quad k = 1, 2, \dots$$

Proof. Let us first observe that for every $0 \le \gamma \le 1$, the function $\Psi(x) = \frac{F(\gamma x)}{F(x)}$ is decreasing on $[0,1]$. Indeed, put $\gamma = e^{-s}, x = e^{-t}$, then $\Psi(x) = \gamma e^{\varphi(t)-\varphi(s+t)}$ and the exponent is a decreasing function of t since $\varphi'(t) - \varphi'(s+t) < 0$.

We have

$$F_{\frac{a}{2},2ka}(a) = \frac{1}{4k-1}F(2ka) + (1 - \frac{1}{4k-1})F(\frac{a}{2}) \le \frac{2}{4k-1}F(2ka) \le \frac{1}{k}F(2ka),$$

because $F(\frac{a}{2}) \le \frac{1}{4k}F(2ka)$, since $\frac{F(x)}{x}$ is an increasing function of x. Consequently, $\Phi_k(a) \le \alpha$ if $F(\alpha a) \le \frac{1}{k}F(2ka)$, thus $d = \sup_{0\le\alpha\le\frac{1}{2k}} \Phi_k(a) \le \alpha$ provided $\frac{F(\alpha a)}{F(2ka)} \ge \frac{1}{k}$ for every $a \le \frac{1}{2k}$. Since $\frac{F(\alpha a)}{F(2ka)}$ is a decreasing function of a, its minimum in $[0, \frac{1}{2k}]$ is $F(\frac{\alpha}{2k})$, thus α is given by $F(\frac{\alpha}{2k}) = \frac{1}{k}$, i.e., $\alpha = 2kF^{-1}(\frac{1}{k})$. ■

Proof of Theorem 1.2. Let $t_k = -\ln n\frac{\delta_k}{2k}$. We can assume, without loss of generality, that $0 = t_0 < t_1 < t_2 < \dots$. Let φ be piecewise linear in the intervals $[t_{k-1}, t_k]$ and let it satisfy the conditions

$$e^{-\varphi(t_k)} \ge \tfrac{1}{2}\delta_k \tag{15}$$

$$\frac{\varphi(t_{k+1}) - \varphi(t_k)}{t_{k+1} - t_k} \ge \frac{\varphi(t_k) - \varphi(t_{k-1}}{t_k - t_{k-1}} \tag{16}$$

(condition (16) implies the convexity of φ). □

4. More Applications

Nielsen and Tomczak [21] proved that if X is a weak Hilbert space that has an unconditional basis, then $d_n(X)$ satisfies the estimate needed to apply Theorem 2.1. It is obvious that $d_n(\ell_2(X)) = d_n(X)$, so we get

Corollary 2. *If X is a weak Hilbert space that has an unconditional basis, then $\ell_2(X)$ has the HUAP.*

The interest in Corollary 2 is that $\ell_2(X)$ is a weak Hilbert space only when X is isomorphic to a Hilbert space [23, Theorem 12.3].

Recall that a Banach space X is *complementably universal* for a class $\mathcal{M}$ of Banach space provided that every space in $\mathcal{M}$ is isomorphic to a complemented subspace of X. Kadec [13] constructed a separable Banach space with the BAP that is complementably universal for all separable Banach spaces that have the BAP, while the authors [11] proved that there is no separable Banach space that is complementably universal for all separable Banach spaces that have the AP. Timur Oikhberg asked the authors whether there is a separable infinite dimensional Banach space not isomorphic to ℓ_2 that is complementably universal for all subspaces of itself. Notice that if such a space has the BAP, then it has the HAP, and hence must be "close" to a Hilbert space. Also notice that such a space cannot have all subspaces complemented, since that condition implies that the space is isomorphic to ℓ_2 [15]. Theorem 2.1 can be used to give an affirmative answer to Oikhberg's question.

Theorem 4.1. *There is a separable, infinite dimensional Banach space not isomorphic to ℓ_2 that is complementably universal for all subspaces of all of its quotients.*

Proof. Let X be any Banach space such that $d_{4^n}(X)$ satisfies the estimate assumed for $d_n(X)$ in Theorem 2.1. Let (E_k) be a sequence of finite dimensional spaces that is dense (in the sense of the Banach-Mazur distance) in the collection of all finite dimensional spaces that are contained in some quotient of $\ell_2(X)$ and let Y be the ℓ_2-sum of the E_k. Then $d_n(Y) \le 2d_{4^n}(X)$ because an n dimensional subspace of a quotient of a Banach space Z is 2-isomorphic to a quotient of a 4^n dimensional subspace of Z. By construction, $d_n(Y_1) \le d_n(Y) \le 2d_{4^n}(X)$ for any quotient Y_1 of Y, hence every subspace of every quotient of Y has the AP and hence the BAP since Y is forced to be superreflexive. The technique at the end of [7] (which also uses a result from [10]) then yields that if Z is a subspace of a quotient of Y, then $Z \oplus Y$ has a finite dimensional decomposition. The main result in [12] implies that $Z \oplus Y$ is isomorphic to $(\sum H_n)_2$ for some sequence H_n of finite dimensional spaces. By construction, the H_n are uniformly isomorphic to a subsequence of E_n, which gives that Z is isomorphic to a complemented subspace of Y.

∎

5. Open questions

Question 1. Does the HAP imply the HUAP?

Question 2. If X has the HAP, do all quotients of X have the AP? (if yes, then the two conditions would be, of course, equivalent.)

Question 3. Is every HAP space reflexive?

Let us recall that there exist non-reflexive spaces that are of type $2-\varepsilon$ and of cotype $2+\varepsilon$ for every $\varepsilon > 0$ (cf.[25]).

Question 4. Is the HAP preserved under ultrapowers?

Question 5. If X has the HAP, does $X \oplus \ell_2$ necessarily have the HAP?

Question 6. Does every HAPpy space belong to $\mathcal{L}$?

Question 7. If $X \notin \mathcal{L}$, does there exist a nilpotent operator on X with non zero trace?

Question 8. Can the space $X^{(2)}$ (see A. above) be modified so that no subspace of it is isomorphic to ℓ_2?

Question 9. If $X \in HAP$, is $\ell_2(X) \in HAP$?

We do not know the answer to the following special case of Question 9:

Question 9.1. If X is a weak Hilbert space, is $\ell_2(X)$ HAPpy?

Question 10. Is every quotient of a HAPpy space again HAPpy?

In connection with Questions 9.1 and 10, we recall the result of Mankiewicz and Tomczak-Jaegerman [20] that if X is not isomorphic to ℓ_2, then $\ell_2(X)$ has a quotient whose subspace does not have a basis. On the other hand, some of the spaces constructed in [8] have the property that every subspace of every quotient has a basis. This suggests

Question 11. If $d_n(X)$ goes to infinity sufficiently slowly and X is separable, must X have a finite dimensional decomposition?

The result of Maurey and Pisier included in [19] shows that every weak Hilbert space has a finite dimensional decomposition.

The rate of growth of $d_n(X)$ needed in Theorem 2.1 is of (inverse) Ackermann type. It is interesting to know whether this rate can be improved significantly. We even do not know the answer to the following

Question 12. Must X be HAPpy if $d_n(X) = o(\log(n))$?

In connection with Theorem 4.1 we have the following

Question 13. Does there exist a space with symmetric basis X such that every subspace of X is isomorphic to a complemented subspace of X, but X is not isomorphic to a Hilbert space?

References

[1] P. G. Casazza, C. L. Garcia, and W. B. Johnson, An example of an asymptotically Hilbertian space which fails the approximation property, Proc. AMS 129 (2001), 3017-3023.

[2] P. G. Casazza, N. J. Nielsen, A Banach space with symmetric basis which is of weak cotype 2 but not of cotype 2, Studia Math. 157 (2003), 1-16.

[3] J. Diestel, H. Jarchow, and A. Tonge, Absolutely summing operators, Cambridge University Press 1995.

[4] T. Figiel and W. B. Johnson, A uniformly convex Banach space which contains no l_p, Compositio Math. 29 (1974), 179–190.

[5] A. Grothendieck, Produits Tensoriels Topologiques et Espaces Nucleaires, Memoirs AMS 16 (1955).

[6] R. C. James, Uniformly Non-Square Banach Spaces, Annals of Math. 80 (1964), 542-550.

[7] W. B. Johnson, Factoring compact operators, Israel J. Math. 9 (1971), 337–345.

[8] W. B. Johnson, Banach spaces all of whose subspaces have the approximation property, (Proc. Sem., Ges. Math. Datenverarb., Bonn, 1979), North-Holland, Amsterdam-New York, (1980), 15–26.

[9] W. B. Johnson, H. König, B. Maurey, J. R. Retherford, Eigenvalues of p-summing and l_p-type operators in Banach spaces, J. Funct. Anal. 32 (1979), 353–380.

[10] W. B. Johnson, H. P. Rosenthal, and M. Zippin, On bases, finite dimensional decompositions and weaker structures in Banach spaces, Israel J. Math. 9 (1971), 488–506.

[11] W. B. Johnson and A. Szankowski, Complementably universal Banach spaces, Studia Math. 58 (1976), no. 1, 91–97.

[12] W. B. Johnson and M. Zippin, On subspaces of quotients of $(\sum G_n)_{\ell_p}$ and $(\sum G_n)_{c_0}$, Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972). Israel J. Math. 13 (1972), 311–316 .

[13] M. I. Kadec, On complementably universal Banach spaces, Studia Math. 40 (1971), 85–89.

[14] V. Lidskii, Non-self adjoint operators with a trace, Dokl.Akad.Nauk SSSR 125 (1959), 485–487.

[15] J. Lindenstrauss and L. Tzafriri, On the complemented subspaces problem, Israel J. Math. 9 (1971), 263–269.

[16] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces I, Sequence spaces, Springer, Berlin 1977.

[17] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces II, Function spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], 97. Springer-Verlag, Berlin-New York, 1979.

[18] J. Lindenstrauss and L. Tzafriri, The uniform approximation property in Orlicz spaces, Israel J. Math.23 (1976),142–155.

[19] V. Mascioni, On Banach spaces isomorphic to their duals, Houston J. Math. 19 (1993), 27-38

[20] P. Mankiewicz and N. Tomczak-Jaegermann, Schauder bases in quotients of subspaces of $\ell_2(X)$, American J. Math. 116 (1994), 1341–1363. 15–26,

[21] N. J. Nielsen and N. Tomczak-Jaegermann, Banach lattices with property (H) and weak Hilbert spaces, Illinois J. Math. 36 (1992), no. 3, 345–371.

[22] G. Pisier, Martingales with values in uniformly convex spaces, Israel J. Math. 20 (1975), 326–350.

[23] G. Pisier, Weak Hilbert spaces, Proc. London Math. Soc. (3) 56 (1988), no. 3, 547–579.

[24] G. Pisier, The volume of convex bodies and Banach space geometry, Cambridge Tracts in Math. 94 (1989).

[25] G. Pisier and Q.Xu, Random series in the real interpolation spaces between the spaces v_p, Geometrical Aspects of Functional Analysis (1985/86), Lecture Notes in Math. 1267, Springer, Berlin (1987), 185–209.

[26] T. Schlumprecht, An arbitrarily distortable Banach space, Israel J. Math. 76 (1991), 81–95.

[27] N. Tomczak-Jaegermann, Banach-Mazur distances and finite-dimensional operator ideals , Longman Scientific and Technical, 1989.

[28] M. Zippin, Interpolation of operators of weak type between rearrangement invariant function spaces, J. Functional Analysis 7 (1971), 267–284.

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