Journal of Functional Analysis 282 (2022) 109333

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

The SHAI property for the operators on L^p

W.B. Johnson [a,*,1], N.C. Phillips [b,2], G. Schechtman [c,3]

[a] Department Mathematics, Texas A&M University, College Station, TX 77843–3368, USA
[b] Department of Mathematics, University of Oregon, Eugene, OR 97403-1222, USA
[c] Department of Mathematics, Weizmann Institute of Science, Rehovot, Israel

A R T I C L E   I N F O

Article history:
Received 24 January 2021
Accepted 6 November 2021
Available online 30 November 2021
Communicated by T. Schlumprecht

MSC:
47L20
46E30

Keywords:
Ideals of operators
L^p spaces
SHAI property

A B S T R A C T

A Banach space X has the SHAI (surjective homomorphisms are injective) property provided that for every Banach space Y, every continuous surjective algebra homomorphism from the bounded linear operators on X onto the bounded linear operators on Y is injective. The main result gives a sufficient condition for X to have the SHAI property. The condition is satisfied for $L^p(0,1)$ for $1 < p < \infty$, spaces with symmetric bases that have finite cotype, and the Schatten p-spaces for $1 < p < \infty$.

1. The main results

Following Horváth [9], we say that a Banach space X has the SHAI (surjective homomorphisms are injective) property provided that for every Banach space Y, every

* Corresponding author.
E-mail addresses: johnson@math.tamu.edu (W.B. Johnson), gideon@weizmann.ac.il (G. Schechtman).

1 Supported in part by NSF DMS-1900612.
2 Supported in part by NSF DMS-1501144 and by Simons Foundation Collaboration Grant for Mathematicians #587103.
3 Supported in part by the Israel Science Foundation.

https://doi.org/10.1016/j.jfa.2021.109333
0022-1236/

surjective continuous algebra homomorphism from the space $L(X)$ of bounded linear operators on X onto $L(Y)$ is injective, and hence by Eidelheit's [6] classical theorem, X is isomorphic as a Banach space to Y. The continuity assumption is redundant by an automatic continuity theorem of B. E. Johnson [5, Theorem 5.1.5], but we note in passing that for some Banach spaces X, there exist surjective discontinuous homomorphisms from $L(X)$ onto some Banach algebras [15]. The spaces ℓ^p for $1 \leq p \leq \infty$ are known to have the SHAI property [9, Proposition 1.2], as do some other classical spaces [9], [10], but there are many spaces that do not have the SHAI property [9]. Our research on the SHAI property was motivated by the problem mentioned by Horváth [9] whether $L^p = L^p(0,1)$ has the SHAI property. A consequence of our main results, Corollary 1.6, is that for $1 < p < \infty$, the space L^p has the SHAI property. We do not know whether L^1 has the SHAI property. The space L^∞ does have the SHAI property because L^∞ is isomorphic as a Banach space to ℓ^∞ [1, Theorem 4.3.10].

Before stating our theorems, we need to review the notion of an unconditional Schauder decomposition of a Banach space X. A family $(E_\alpha)_{\alpha\in A}$ of closed subspaces of X is called an unconditional Schauder decomposition for X provided every vector x in X has a unique representation $x = \sum_{\alpha\in A} x_\alpha$, where the convergence is unconditional and, for each $\alpha \in A$, the vector x_α is in E_α. Notice that by uniqueness of the representation, $E_\alpha \cap E_\beta = \{0\}$ when $\alpha \neq \beta$, and there are idempotents P_α on X such that $P_\alpha X = E_\alpha$ and $P_\alpha P_\beta = 0$ for $\alpha \neq \beta$. It is known that the P_α are in $L(X)$. Moreover, for any subset B of A, the net $\{\sum_{\alpha\in F} P_\alpha \colon F \subset B \text{ finite}\}$ is bounded in $L(X)$ and converges strongly to an idempotent P_B that has range $\overline{\operatorname{span}}_{\alpha\in B}\, E_\alpha$. The suppression constant of the decomposition is then defined to be $\sup\{\|\sum_{\alpha\in F} P_\alpha\| : F \subset A \text{ finite}\}$. Note that $\|P_B\|$ is bounded by this suppression constant for all subsets B of A. In practice, this theorem is rarely used, since typically one constructs the idempotents P_α and checks the uniform boundedness of the aforementioned nets and verifies the statement about the ranges of the strong limits of the nets. Finally, observe that a collection $(e_\alpha)_{\alpha\in A}$ forms an unconditional Schauder basis for X if and only if $(E_\alpha)_{\alpha\in A}$ is an unconditional Schauder decomposition of X, where $E_\alpha = \mathbb{K}e_\alpha$ ($\mathbb{K}$ is the scalar field). In the sequel, we will most often use an unconditional Schauder decomposition E_α where each E_α is finite dimensional. Such a decomposition is called an unconditional FDD. FDD stands for finite dimensional decomposition. Schauder decompositions and FDDs are discussed in the monograph [13, Section 1.g]. Schauder bases, type/cotype theory, and other concepts from Banach space theory that are used in this paper are treated in the textbook [1].

A concept that is particularly relevant for us is that of bounded completeness. An unconditional Schauder decomposition $(E_\alpha)_{\alpha\in A}$ for X is said to be boundedly complete provided that whenever $x_\alpha \in E_\alpha$ and $\{\|\sum_{\alpha\in F} x_\alpha\|_X : F \subset A \text{ finite}\}$ is bounded, then the formal sum $\sum_{\alpha\in A} x_\alpha$ converges in X, which is the same as saying that the net $\{\sum_{\alpha\in F} x_\alpha \colon F \subset A \text{ finite}\}$ converges. A convenient condition that obviously guarantees bounded completeness is that the decomposition has a disjoint lower p estimate for some $p < \infty$. The decomposition $(E_\alpha)_{\alpha\in A}$ is said to have a disjoint lower; respectively, upper; p estimate provided that there is $C < \infty$ so that whenever $x_1, \ldots, x_n$ are finitely many vectors in X such that for every $\alpha \in A$ there is at most one i with $1 \leq i \leq n$ for which $P_\alpha x_i \neq 0$, we have for $x = \sum_{i=1}^n x_i$ the inequality

$$\left\| \sum_{i=1}^n x_i \right\| \geq \frac{1}{C} \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}; \quad \text{respectively,} \quad \left\| \sum_{i=1}^n x_i \right\| \leq C \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}.$$

It is easy to see that the decomposition $(E_\alpha)_{\alpha \in A}$ has a disjoint lower p estimate with constant C if and only if whenever $F_1, \dots, F_n$ are disjoint finite subsets of A and x is in X, then

$$\|x\| \geq \frac{1}{C} \left(\sum_{j=1}^n \left\| \sum_{\alpha \in F_j} P_\alpha x \right\|^p \right)^{1/p},$$

where, as usual, P_α is the idempotent associated with the decomposition. Important for us is the following observation, which is very easy to prove. Suppose that $(E_\alpha)_{\alpha \in A}$ is an unconditional Schauder decomposition for a subspace X of a Banach space Y. Assume that the idempotents $\tilde{P}_\alpha$ associated with the decomposition extend to commuting idempotents P_α from Y onto E_α and that the net $\{\sum_{\alpha \in F} P_\alpha \colon F \subset A \text{ finite}\}$ is bounded in $L(Y)$. If $(E_\alpha)_{\alpha \in A}$ is a boundedly complete unconditional Schauder decomposition of X, then for each subset B of A, the net $\{\sum_{\alpha \in F} P_\alpha \colon F \subset B \text{ finite}\}$ converges strongly in $L(Y)$ to an idempotent P_B whose range is the closed linear span of the spaces E_α for $\alpha \in B$ (which, by abuse of notation, we abbreviate to $\overline{\operatorname{span}}\{E_\alpha \colon \alpha \in B\}$) and P_B extends the basis projection from X onto $\overline{\operatorname{span}}\{E_\alpha \colon \alpha \in B\}$. In particular, X is complemented in Y. Conversely, if X is known to be complemented in Y, then such extensions P_B of the basis projections $\tilde{P}_B$ from X onto $\overline{\operatorname{span}}\{E_\alpha \colon \alpha \in B\}$ obviously exist even when the decomposition is not boundedly complete. In general, to guarantee that X is complemented in Y, something is needed other than having commuting extensions P_α with $\{\sum_{\alpha \in F} P_\alpha \colon F \subset A \text{ finite}\}$ uniformly bounded: consider $X = c_0$, $Y = \ell^\infty$, and the unit vector basis of c_0.

From the definitions of type and cotype, it is clear that if X has type p and cotype q, then every unconditional Schauder decomposition for X has a disjoint upper p estimate and a disjoint lower q estimate, where the constants depend only on the suppression constant of the decomposition and the type p and cotype q constants of X. In particular, if $1 < p \leq 2$, then every unconditional Schauder decomposition for a subspace of a quotient of L^p has a disjoint upper p estimate and a disjoint lower 2 estimate, while if $2 \leq p < \infty$, then every unconditional Schauder decomposition for a subspace of a quotient of L^p has a disjoint upper 2 estimate and a disjoint lower p estimate [1, Theorem 6.2.14].

The observation in the following lemma will be used for transferring information from Y to X when there is a surjective homomorphism from $L(Y)$ onto $L(X)$.

Lemma 1.1. *Suppose that $(E_\alpha)_{\alpha \in A}$ is an unconditional decomposition for X that has a disjoint lower p estimate with $1 \leq p < \infty$, and let $Y \supseteq X$. Then there is a constant*

$C < \infty$ such that if $A_1, \dots, A_n$ are disjoint subsets of A and P_{A_j} is the basis projection onto $E_{A_j} = \overline{\operatorname{span}}\,\{E_\alpha : \alpha \in A_j\}$ and $T_1, \dots, T_n$ are operators in $L(Y)$, then

$$\left\| \sum_{i=1}^n T_i P_i \right\| \le C \left(\sum_{i=1}^n \|T_i\|^q \right)^{1/q}, \text{ where } 1/p + 1/q = 1.$$

Proof. Suppose $x \in X$. Then

$$\begin{aligned} \left\| \sum_{i=1}^n T_i P_i x \right\| &\le \sum_{i=1}^n \|T_i\| \|P_i x\| \le \left(\sum_{i=1}^n \|T_i\|^q \right)^{1/q} \left(\sum_{i=1}^n \|P_i x\|^p \right)^{1/p} \\ &\le C \left(\sum_{i=1}^n \|T_i\|^q \right)^{1/q} \|x\|, \end{aligned}$$

where the constant C is the disjoint lower p constant of $(E_\alpha)_{\alpha \in A}$. $\square$

A family of sets is said to be *almost disjoint* provided the intersection of any two of them is finite.

Definition 1.2. Suppose that $(E_n)_{n=1}^\infty$ is an unconditional FDD for a Banach space X. We say that (E_n) has property (#) provided there is an almost disjoint continuum $\{N_\alpha : \alpha < c\}$ of infinite sets of natural numbers such that for each $\alpha < c$, X is isomorphic to the closed linear span of the subspaces E_n for $n \in N_\alpha$.

Subsymmetric bases are obvious examples of FDDs that have property (#). (A basis is subsymmetric if it is unconditional and every subsequence of the basis is equivalent to the basis. Symmetric bases are subsymmetric.) A second almost obvious example is the direct sum of two Banach spaces with subsymmetric bases. Such a space has an FDD with property (#) such that each space in the decomposition is two-dimensional. In Corollary 1.6 we point out that the Haar basis for L^p has property (#) when $1 < p < \infty$. In Proposition 1.3 and the sequel, when $(E_n)_{n=1}^\infty$ is an unconditional FDD for a Banach space X, we use the following notation. For $F \subset \mathbb{N}$, P_F denotes the basis projection from X onto the closed linear span E_F of the subspaces E_n for $n \in F$.

Proposition 1.3. *Let $(E_n)_{n=1}^\infty$ be an unconditional FDD for a Banach space X. Assume that (E_n) has property (#), witnessed by an almost disjoint family $\{N_\alpha : \alpha < c\}$ of infinite subsets of the natural numbers. Suppose that Φ is a non-zero, non-injective continuous homomorphism from $L(X)$ onto a Banach algebra $\mathcal{A}$. Then for each $\alpha < c$, $\Phi(P_{N_\alpha})$ is a non-zero idempotent in $\mathcal{A}$. Moreover, there is a constant $C < \infty$ such that if F is any finite subset of $[\alpha < c]$, then $\left\| \sum_{\alpha \in F} \Phi(P_{N_\alpha}) \right\|_{\mathcal{A}} \le C$. If $\mathcal{A}$ is a subalgebra of $L(Y)$ for some Banach space Y, then $(\Phi(P_{N_\alpha}))_{\alpha < c}$ is a family of commuting extensions to Y of the projections associated with an unconditional Schauder decomposition for a subspace Y_0 of Y.*

Proof. Since, for each α, the range of P_{N_α} is isomorphic to X, and Φ is not zero, $\Phi(P_{N_\alpha})$ is a non-zero idempotent in $\mathcal{A}$. Suppose that F is a finite subset of $\{\alpha\colon \alpha < c\}$. Take a finite set S of natural numbers so that $N_\alpha \cap N_\beta \subset S$ for all distinct α, β in F. For $\alpha \in F$, let $Q_\alpha = P_{N_\alpha \setminus S}$ be the basis projection from X onto $\overline{\operatorname{span}}\{E_n\colon n \in N_\alpha \setminus S\}$. The kernel of Φ is a non-trivial ideal in $L(X)$ and hence contains the finite rank operators. Since $P_{N_\alpha} - Q_\alpha$ is a finite rank operator, $\Phi(P_{N_\alpha}) = \Phi(Q_\alpha)$ for each $\alpha \in F$. But the projections Q_α, for $\alpha \in F$, are projections onto the closed spans of disjoint subsets of the FDD $(E_n)_{n=1}^\infty$, so

$$\Big\| \sum_{\alpha\in F} \Phi(Q_\alpha) \Big\|_{\mathcal{A}} \le \Big\| \sum_{\alpha\in F} Q_\alpha \Big\| \|\Phi\| \le C\|\Phi\|,$$

where C is the suppression constant of (E_n). The last statement is now obvious. $\square$

With the preliminaries out of the way, we state the main theorem in this article.

Theorem 1.4. *Let $(E_n)_{n=1}^\infty$ be an unconditional FDD for a Banach space X. Assume that $(E_n)_{n=1}^\infty$ has property (#) (Definition 1.2) and $(E_n)_{n=1}^\infty$ has a disjoint lower p estimate for some $p < \infty$. Then X has the SHAI property.*

Proof. Suppose, for contradiction, that Φ is a non-injective continuous homomorphism from $L(X)$ onto $L(Y)$ for some non-zero Banach space Y. We continue with the set up in Proposition 1.3, where property (#) for (E_n) is witnessed by an almost disjoint family $\{N_\alpha\colon \alpha < c\}$ of infinite subsets of the natural numbers, and for $F \subset \mathbb{N}$, the basis projection from X onto the closed linear span E_F of $\{E_n\colon n \in F\}$ is denoted by P_F.
We claim that to get a contradiction it is enough to prove that the subspace Y_0 is complemented in Y. Indeed, if Y_0 is complemented in Y, then $L(Y_0)$ is isomorphic as a Banach algebra to a subalgebra of $L(Y)$. However, defining $Y_\alpha = \Phi(P_{N_\alpha})Y$ for $\alpha < c$, we know that $(Y_\alpha)_{\alpha<c}$ is an unconditional Schauder decomposition for Y_0. But then for every subset S of $\{\alpha\colon \alpha < c\}$ there is an idempotent Q_S from Y_0 onto $\overline{\operatorname{span}}\{Y_\alpha\colon \alpha \in S\}$ with Q_S zero on all Y_β for which $\beta \notin S$. Thus if S_1 and S_2 are different subsets of $\{\alpha\colon \alpha < c\}$, then $\|Q_{S_1} - Q_{S_2}\| \ge 1$, and hence the density character of $L(Y_0)$, whence also of $L(Y)$, is at least 2^c. However, since X is separable, the density character of $L(X)$ is at most c (actually, equal to c since X has an unconditional FDD), so $L(Y)$ cannot be a continuous image of $L(X)$. This completes the proof of the claim.
To show that Y_0 must be complemented in Y, we use the fact proved in Proposition 1.3 that there is a constant C such that for every finite subset F of $\{\alpha\colon \alpha < c\}$ we have $\big\|\sum_{\alpha\in F} \Phi(P_{N_\alpha})\big\|_{L(Y)} \le C$. It was remarked in the introduction to this section that this condition guarantees that Y_0 is complemented in Y when $(Y_\alpha)_{\alpha<c}$ is a boundedly complete decomposition. To see that $(Y_\alpha)_{\alpha<c}$ is boundedly complete, we use Lemma 1.1. We guarantee bounded completeness by proving that $(Y_\alpha)_{\alpha<c}$ has a disjoint lower p estimate. That is, we just need to find a constant C so that if $F_1, \dots, F_m$ are disjoint finite subsets of $\{\alpha\colon \alpha < c\}$ and y is in Y (or even just in Y_0), then

$$\|y\| \geq \frac{1}{C}\left(\sum_{j=1}^{m}\left\|\sum_{\alpha\in F_j}\Phi(P_{N_\alpha})y\right\|^p\right)^{1/p}. \tag{1}$$

Just as in the proof Proposition 1.3, we can write $\sum_{\alpha\in F_j}\Phi(P_{N_\alpha}) = \Phi(Q_j)$ with Q_j, for $1 \leq j \leq m$, being the basis projections onto the closed spans of disjoint sets of FDD basis spaces (E_n). So (1) can be rewritten as

$$\|y\| \geq \frac{1}{C}\left(\sum_{j=1}^{m}\|\Phi(Q_j)y\|^p\right)^{1/p}. \tag{2}$$

From Lemma 1.1 and the surjectivity of Φ, for any $T_1,\dots,T_m$ in $L(Y)$ we have

$$\left\|\sum_{j=1}^{m}T_j\Phi(Q_j)\right\| \leq C\left(\sum_{j=1}^{m}\|T_j\|^q\right)^{1/q}, \tag{3}$$

where C depends only on p and on $\|\Phi\|\cdot\|(\Phi^*)^{-1}\|$, and $1/p+1/q=1$. Take any $y\in Y$ and take $\beta_j \geq 0$ with

$$\sum_{j=1}^{m}\beta_j^q = 1 \qquad \text{and} \qquad \sum_{j=1}^{m}\beta_j\|\Phi(Q_j)y\| = \left(\sum_{j=1}^{m}\|\Phi(Q_j)y\|^p\right)^{1/p}.$$

Let y_0 be any unit vector in Y and let T_j be $\Phi(Q_j)$ followed by a norm (at most) one projection onto the (at most) one dimensional space $\mathbb{K}\Phi(Q_j)y$ followed by $\Phi(Q_j)y \mapsto \beta_j\|\Phi(Q_j)y\|y_0$. Then by (3),

$$\begin{aligned}\left(\sum_{j=1}^{m}\|\Phi(Q_j)y\|^p\right)^{1/p} &= \sum_{j=1}^{m}\beta_j\|\Phi(Q_j)y\| = \left\|\sum_{j=1}^{m}T_j\Phi(Q_j)y\right\| \\ &\leq C\left(\sum_{j=1}^{m}\|T_j\|^q\right)^{1/q}\|y\| = C\|y\|,\end{aligned}$$

which is (2). □

Our first corollary of Theorem 1.4 is immediate. Its hypothesis is satisfied by many spaces that are used in analysis, including most Orlicz and Lorentz sequence spaces.

Corollary 1.5. *If X has a subsymmetric basis and has finite cotype, then X has SHAI.*

The next corollary solves the problem that motivated our research into the SHAI property.

Corollary 1.6. *For $1 < p < \infty$, the space L^p has the SHAI property.*

Proof. In view of Theorem 1.4, it is enough to prove that the Haar basis for L^p has property (#). Let $\{N_\alpha : \alpha < c\}$ be a continuum of almost disjoint infinite subsets of the natural numbers $\mathbb{N}$. Define for $\alpha < c$

$$X_\alpha = \overline{\operatorname{span}}\{h_{n,i} : n \in N_\alpha \text{ and } 1 \le i \le 2^n\},$$

where $\{h_{n,i} : n = 0, 1, \dots \text{ and } 1 \le i \le 2^n\}$ is the usual (unconditional) Haar basis for $L^p(0,1)$, indexed in its usual way, so that $\{|h_{n,i}| : 1 \le i \le 2^n\}$ is the set of indicator functions of the dyadic subintervals of $(0,1)$ that have length 2^{-n}. By the Gamlen–Gaudet theorem [7], X_α is isomorphic to L^p with the isomorphism constant depending only on p. □

Remark 1.7. *Although our proof that L^p has the SHAI property is simple enough, it is strange. The "natural" way of proving that a space X has the SHAI property is to verify that for any non-trivial closed ideal $\mathcal{I}$ in $L(X)$, the quotient algebra $L(X)/\mathcal{I}$ contains no minimal idempotents. (An idempotent P is called minimal provided $P \neq 0$ and the only idempotents Q for which $PQ = QP = Q$ are P and 0. Rank-one idempotents in $L(X)$ are minimal.) This suggests the following problem, which is related to the known problem whether every infinite dimensional complemented subspace of L^p is isomorphic to its square.*

Problem 1.8. Is there a non-trivial closed ideal $\mathcal{I}$ in $L(L^p)$ for which $L(L^p)/\mathcal{I}$ has a minimal idempotent?

If there is a positive answer to Problem 1.8, the witnessing ideal $\mathcal{I}$ cannot be contained in the ideal of strictly singular operators. This is because every infinite dimensional complemented subspace of L^p contains a complemented subspace that is isomorphic either to ℓ^p or to ℓ^2 [11], and the fact that idempotents in $L(X)/\mathcal{I}$ lift to idempotents in $L(X)$ when $\mathcal{I}$ is an ideal that is contained in $L(X)$ [3].

Problem 1.9. Does L^1 have the SHAI property?

2. Examples and permanence properties

Here we present some more examples of spaces with property (#) and with the SHAI property. We do not know whether every complemented subspace of L^p has the SHAI property, but we show that at least some of the known examples of such spaces do. Along the way we state and prove some permanence properties of (#).

The classical complemented subspaces of L^p have the SHAI property when $1 < p < \infty$. This was known for ℓ^2 and ℓ^p and proved above for L^p. The case of $\ell^p \oplus \ell^2$ follows easily from Theorem 1.4. That the remaining classical complemented subspace of L^p, $\ell^p(\ell^2)$, the ℓ^p sum of ℓ^2, has (#) and the SHAI property follows from Proposition 2.2 below. Before stating Proposition 2.2 we introduce a quantitative version of property (#).

Definition 2.1. Suppose that $(E_n)_{n=1}^{\infty}$ is an unconditional FDD for a Banach space X and K is a positive constant. We say that (E_n) has property (#) with constant K provided there is an almost disjoint continuum $\{N_\alpha \colon \alpha < c\}$ of infinite sets of natural numbers such that for each $\alpha < c$, X is K-isomorphic to the closed linear span of $\{E_n \colon n \in N_\alpha\}$.

Note that if $(E_n)_{n=1}^{\infty}$ has property (#) then it has property (#) for some positive constant K. Nevertheless, we need this quantitative notion for the full generality of Proposition 2.2.
Recall that if (e_i) is an unconditional basis for some Banach space Y and X_i, for $i = 1, 2, \dots$, is a Banach space, $(\bigoplus_{i=1}^{\infty} X_i)_Y$ is the space of sequences $\bar{x} = (x_1, x_2, \dots)$ whose norm, $\|\bar{x}\| = \left\| \sum_{i=1}^{\infty} \|x_i\| \cdot e_i \right\|_Y$, is finite. We denote the subspace of $(\bigoplus_{i=1}^{\infty} X_i)_Y$ of all sequences of the form $(0, \dots, 0, x_i, 0, \dots)$ by $X_i \otimes e_i$.

Proposition 2.2. *For $i = 1, 2, \dots$ let $(E_n^i)_{n=1}^{\infty}$ be an unconditional FDD for a Banach space X_i, all satisfying property (#) with a common K. Then for each subsymmetric basis (e_i) of some Banach space Y, the unconditional FDD $(E_n^i \otimes e_i)_{i,n=1}^{\infty}$ of $(\bigoplus_{i=1}^{\infty} X_i)_Y$ satisfies (#). If, in addition, the decompositions $(E_n^i)_{n=1}^{\infty}$ have disjoint lower p estimates with uniform constant and (e_i) also has such an estimate, then $(\bigoplus_{i=1}^{\infty} X_i)_Y$ has the SHAI property.*

Proof. For each i, let $\{N_\alpha^i \colon \alpha < c\}$ be an almost disjoint continuum of infinite sets of natural numbers such that for every $\alpha < c$, X_α is K-isomorphic to the closed linear span of the subspaces E_n^i for $n \in N_\alpha$. Also, let $\{N_\alpha \colon \alpha < c\}$ be an almost disjoint continuum of infinite sets of natural numbers. Then

$$\left\{(i, n) \colon i \in N_\alpha \text{ and } n \in N_\alpha^i\right\}$$

is a continuum of almost disjoint subsets of $\mathbb{N} \times \mathbb{N}$. It is easy to see that this continuum satisfies what is required of the unconditional FDD $(E_n^i \otimes e_i)_{i,n=1}^{\infty}$ to satisfy (#). If the decompositions $(E_n^i)_{n=1}^{\infty}$ have disjoint lower p estimates with uniform constant and (e_i) also has such an estimate, then the FDD $(E_n^i \otimes e_i)_{i,n=1}^{\infty}$ clearly has a disjoint lower p estimate as well, so the SHAI property follows from Theorem 1.4. $\square$

Remark 2.3. Note that the proof above works with only notational differences if we deal with only finitely many X_i (and here we do not need to assume the uniformity of the (#) property). In particular, if each of X and Y has an unconditional FDD with (#), then so does $X \oplus Y$.

As we said above, this takes care of the space $\ell^p(\ell^2)$. The first non-classical complemented subspace of L^p is the space X_p of Rosenthal [16]. We recall its definition.

Let $p > 2$ and let $\bar{w} = (w_i)_{i=1}^{\infty}$ be a bounded sequence of positive real numbers. Let $(e_i)_{i=1}^{\infty}$ and $(f_i)_{i=1}^{\infty}$ be the unit vector bases of ℓ^p and ℓ^2. Let $X_{p,\bar{w}}$ be the closed span of $(e_i \oplus w_i f_i)_{i=1}^{\infty}$ in $\ell^p \oplus \ell^2$. If the w_i are bounded away from zero, then $X_{p,\bar{w}}$ is isomorphic to ℓ^2. If $\sum_{i=1}^{\infty} w_i^{\frac{2p}{p-2}} < \infty$, then $X_{p,\bar{w}}$ is isomorphic to ℓ^p. If one can split the sequence $\bar{w}$ into two subsequences, one bounded away from zero and the other such that the sum of the $\frac{2p}{p-2}$ powers of its elements converges, then $X_{p,\bar{w}}$ is isomorphic to $\ell^p \oplus \ell^2$. Rosenthal proved that in all other situations one gets a new space, isomorphically unique (i.e., any, two spaces corresponding to two choices of $\bar{w}$ with this condition are isomorphic). Moreover, $X_{p,w}$ is isomorphic to a complemented subspace of L^p. The constants involved (isomorphisms and complementations) are bounded by a constant depending only on p. This common (class of) space(s) is denoted by X_p. For $1 < p < 2$, X_p is defined to be $X^*_{p/(p-1)}$.

Proposition 2.4. *Let $p \in (1, \infty) \setminus \{2\}$. Then X_p has (#) and has the SHAI property.*

Proof. Let $p > 2$. Write $\mathbb{N}$ as a disjoint union of finite subsets σ_j for $j = 1, 2, \ldots,$ with $|\sigma_j| \to \infty$. For $i \in \sigma_j$ put $w_i = |\sigma_j|^{\frac{2-p}{2p}}$, so $w_i \to 0$ and for each j, $\sum_{i\in\sigma_j} w_i^{\frac{2p}{p-2}} = 1$. Set $E_j = \operatorname{span}(e_i \oplus w_i f_i)_{i\in\sigma_j}$. It follows that for any infinite subsequence of the unconditional FDD (E_j), the closed span of this subsequence is isomorphic to X_p. The FDD is unconditional and, as it lives in L^p, has a lower p estimate. So the result in this case follows from Theorem 1.4. The case $1 < p < 2$ follows by looking at the dual FDD. $\square$

Building on X_p and the classical complemented subspaces of L^p, Rosenthal [16] lists a few more isomorphically distinct spaces that are isomorphic to complemented subspaces of L^p when $p \in (1, \infty) \setminus \{2\}$. Using the discussion above one can easily show that they all have (#) and the SHAI property. Here we just comment on one of them for which the full power of Proposition 2.2 is needed. This is the space denoted in [16] by B_p. It is the ℓ^p sum of spaces X_i each having a 1-symmetric basis, and thus having (#) with uniform constant. Each X_i is isomorphic to ℓ^2, but the isomorphism constant tends to infinity as $i \to \infty$. By Proposition 2.2, B_p has (#) and the SHAI property.

The first infinite collection of mutually non-isomorphic complemented subspaces of L^p for $p \in (1, \infty) \setminus \{2\}$ was constructed in [17]. We recall the simple construction. Given two subspaces X and Y of $L^p(\Omega)$ with $1 \le p \le \infty$, $X \otimes_p Y$ denotes the subspace of $L^p(\Omega^2)$ that is the closed span of all functions of the form $h(s,t) = f(s)g(t)$ with $f \in X$ and $g \in Y$. It is easy to see (and was done in [17]) that the isomorphism class of $X \otimes_p Y$ depends only on the isomorphism classes of X and Y and that, if X and Y are complemented in $L^p(\Omega)$, then $X \otimes_p Y$ is complemented in $L^p(\Omega^2)$. More generally, if X_1, X_2, Y_1, Y_2 are subspaces of $L^p(\Omega)$ and $T_i \in L(X_i, Y_i)$, then $T_1 \otimes_p T_2 \in L(X_1 \otimes_p X_2, Y_1 \otimes_p Y_2)$. Note also that if $(E_n^i)_{n=1}^{\infty}$ is an unconditional FDD for X_i for $i = 1, 2$, then $(E_n^1 \otimes_p E_m^2)_{n,m=1}^{\infty}$ is an unconditional FDD for $X_1 \otimes_p X_2$. This follows from iterating Khinchine's inequality.

With a little abuse of notation we denote by X_p some isomorph of X_p that is complemented in $L^p[0,1]$. Set $Y_1 = X_p$, and for $n = 2, 3, \dots$, let $Y_n = Y_{n-1} \otimes_p X_p$. From the above it is clear that the spaces Y_n are complemented (alas, with norm of projection depending on n) in some L^p space isometric to $L^p[0,1]$. The main point in [17] was to prove that the spaces Y_n are isomorphically different. That all the spaces Y_n have (#) follows now from the following general proposition, because it is clear that $\otimes_p$ satisfies Conditions (1) and (2) in Proposition 2.5 for the class of all m tuples of subspaces of $L^p(\mu)$ spaces.

Proposition 2.5. *Assume that $X_1, \dots, X_m$ are Banach spaces, each of which has an unconditional FDD satisfying (#). Let $Y_1 \otimes \cdots \otimes Y_m$ denote an m fold tensor product endowed with norm defined on some class of m tuples of Banach spaces with the following two properties:*

1. *If $T_i \in L(Y_i, Z_i)$ for $i = 1, \dots, m$, then*
$$T_1 \otimes \cdots \otimes T_m \colon Y_1 \otimes \cdots \otimes Y_m \to Z_1 \otimes \cdots \otimes Z_m$$
is bounded.
2. *If Y_i has an unconditional FDD $(F_n^i)_{n=1}^{\infty}$ for each i, then $(F_{n_1}^1 \otimes \cdots \otimes F_{n_m}^m)_{n_1,\dots,n_m=1}^{\infty}$ is an unconditional FDD for the completion of $Y_1 \otimes \cdots \otimes_m Y_m$.*

Then, if we assume in addition that $(X_1, \dots, X_m)$ is in this class, the completion of $X_1 \otimes \cdots \otimes X_m$ has an unconditional FDD with (#).

Proof. For each $i = 1, \dots, m$, let $(E_n^i)_{n=1}^{\infty}$ be an unconditional FDD for a Banach space X_i such that there is an almost disjoint continuum $\{N_\alpha^i \colon \alpha < c\}$ of infinite sets of $\mathbb{N}$ such that for each $\alpha < c$, X_i is isomorphic to the closed linear span of the spaces E_n^i for $n \in N_\alpha^i$.

Consider the continuum
$$\{N_\alpha^1 \times \cdots \times N_\alpha^m \colon \alpha < c\}$$
of subsets of $\mathbb{N}^m$. This is an almost disjoint family whose cardinality is the continuum. Property (2) of the tensor norms we consider guarantees that $(E_{n_1}^1 \otimes \cdots \otimes E_{n_m}^m)_{n_1,\dots,n_m=1}^{\infty}$ is an unconditional FDD for the completion of $X_1 \otimes \cdots \otimes X_m$. Property (1) implies that for each $\alpha < c$, the closed linear span of
$$(E_{n_1}^1 \otimes \cdots \otimes E_{n_m}^m)_{(n_1,\dots,n_m) \in N_\alpha^1 \times \cdots \times N_\alpha^m}$$
is isomorphic to the completion of $X_1 \otimes \cdots \otimes X_m$. $\square$

Remark 2.6. Note that in general Property (1) does not imply Property (2). The Schatten classes C_p for $p \neq 2$ are examples of tensor norms that satisfy (1) but not (2).

We note that it is clear from Proposition 2.5 that if $X_1, \dots, X_m$ are subspaces of L^p for $1 \le p < 2$ that have (sub)symmetric bases, then $X_1 \otimes_p \cdots \otimes_p X_m$ has (#) and the SHAI property. The class of subspaces of L^p for $1 \le p < 2$ that have a symmetric basis (i.e., the norm of a vector is invariant, up to a constant, under all permutations and changes of signs of its coefficients) is a rich family. (For $p > 2$, up to isomorphism it includes only ℓ^p and ℓ^2.) Thus the class of tensor products above includes, for example, $\ell_{p_1}(\ell_{p_2}(\dots(\ell_{p_m})\dots))$ whenever $p \le p_1 < p_2 < \cdots < p_m \le 2$.

Problem 2.7. Suppose $p \in (1, \infty) \setminus \{2\}$ and let X be a complemented subspace of L^p. Does X have the SHAI property? What if, in addition, X has an unconditional basis? What if, in addition, X is one of the $\aleph_1$ spaces constructed in [4]?

We complete this section with a discussion of another class of classical Banach spaces that have property (#) and thus also the SHAI property; namely, the Schatten ideals C_p of compact operators T on ℓ^2 for which the eigenvalues of $(T^*T)^{1/2}$ are p-summable. We treat the case $1 < p < \infty$ but remark afterwards how one can prove that C_1 (trace class operators on ℓ^2) has the SHAI property. Neither C_1 nor its predual C_∞ (compact operators on ℓ^2) has an unconditional FDD [12] and hence these spaces do not have property (#). In the sequel we also assume $p \neq 2$ because C_2, being isometrically isomorphic to ℓ^2, has already been discussed.
First, consider the subspace T_p of C_p consisting of the lower triangular matrices in C_p. Here we exclude $p = 1$, $p = \infty$, and $p = 2$. Neither T_p nor C_p has an unconditional basis [12], but T_p has an obvious unconditional FDD (E_n); namely, $E_n = \mathrm{span}_{1 \le j \le n} e_n \otimes e_j$; that is, a matrix is in E_n if and only if the only non-zero terms are in the first n entries of the n-th row. Since multiplying all entries in a row by the same scalar of magnitude one is an isometry on C_p, (E_n) is even 1-unconditional. If M is an infinite subset of $\mathbb{N}$, let $T_p(M)$ be the closed span in T_p of $(E_n)_{n \in M}$. Since (E_n) is 1-unconditional, $T_p(M)$ is norm one complemented in T_p. We claim (and justify at the end of this discussion) that T_p is isometric to a K_p-complemented subspace of $T_p(M)$ with K_p independent of M. The space T_p is isomorphic to $\ell^p(T_p)$ [2, p. 85], so the decomposition method [1, Theorem 2.2.3] shows that T_p is isomorphic to $T_p(M)$. Thus every almost disjoint family of infinite subsets of $\mathbb{N}$ witnesses that T_p has property (#). Theorem 1.4 applies because C_p has finite cotype when $p < \infty$, so T_p has the SHAI property when $1 < p < \infty$. Now for $1 < p < \infty$, T_p is complemented in C_p via the projection that zeroes out the entries that lie above the diagonal [14], [8], from which it follows easily [2] that T_p is isomorphic to C_p. Moreover, for M an infinite set of natural numbers, there is an obvious subspace Y of C_p that is isometric to $T_p(M)$ such that $T_p \subset Y$, which gives the claim. We record these observations in Proposition 2.8.

Proposition 2.8. *For $1 < p < \infty$, the space T_p has property (#). Consequently, for $1 < p < \infty$, the space C_p has the SHAI property.*

As we mentioned above, it can be proved that C_1 has the SHAI property even though it does not have an unconditional FDD. However, the C_p norms for $1 \leq p \leq \infty$ are what Kwapień and Pełczyński [12] call unconditional matrix norms; i.e., the norm $\left\|\sum_{i,j} a_{i,j} e_{i,j}\right\|$ of a linear combination $\sum_{i,j} a_{i,j} e_{i,j}$ of the natural basis elements $(e_{i,j})_{i,j=1}^{\infty}$, is equivalent (in our case even equal) to the norm of $\sum_{i,j} \varepsilon_i \delta_j a_{i,j} e_{i,j}$ for all sequences of signs $(\varepsilon_i)_{i=1}^{\infty}$ and $(\delta_j)_{j=1}^{\infty}$. One can define a variation of property (#) for bases with this unconditionality property, check that the natural bases for C_p, for $1 \leq p \leq \infty$, satisfy this property, and prove a version of Theorem 1.4. This shows that C_1 has the SHAI property (and gives an alternative proof also for C_p for $1 < p < \infty$). This variation of Theorem 1.4 does not apply to C_∞, which does not have finite cotype, and we do not know whether C_∞ has the SHAI property. Since our focus in this paper is on spaces that are more closely related to L^p than are the C_p spaces, we do not go into more detail. Our main reason for bringing up C_p is to point out why the definition of property (#) is made for unconditional FDDs rather than just for unconditional bases.

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