CLOSED IDEALS OF OPERATORS ON AND COMPLEMENTED SUBSPACES OF BANACH SPACES OF FUNCTIONS WITH COUNTABLE SUPPORT

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ABSTRACT. Let λ be an infinite cardinal number and let $\ell_\infty^c(\lambda)$ denote the subspace of $\ell_\infty(\lambda)$ consisting of all functions which assume at most countably many non zero values. We classify all infinite dimensional complemented subspaces of $\ell_\infty^c(\lambda)$, proving that they are isomorphic to $\ell_\infty^c(\kappa)$ for some cardinal number κ. Then we show that the Banach algebra of all bounded linear operators on $\ell_\infty^c(\lambda)$ or $\ell_\infty(\lambda)$ has the unique maximal ideal consisting of operators through which the identity operator does not factor. Using similar techniques, we obtain an alternative to Daws' approach description of the lattice of all closed ideals of $\mathscr{B}(X)$, where $X = c_0(\lambda)$ or $X = \ell_p(\lambda)$ for some $p \in [1, \infty)$, and we classify the closed ideals of $\mathscr{B}(\ell_\infty^c(\lambda))$ that contain the ideal of weakly compact operators.

1. INTRODUCTION AND THE STATEMENTS OF THE MAIN RESULTS

The aim of this paper is to contribute to two the closely related programs of classifying complemented subspaces of Banach spaces and classifying the maximal ideals of (bounded, linear) operators acting thereon. Our results constitute a natural continuation of the research undertaken by the authors in [7, 14, 15, 16] concerning the maximal ideals of the Banach algebras $\mathscr{B}(X)$ for certain classical Banach spaces X. From this perspective, we complement results ([9, 22]) of Gramsch and Luft who classified all closed ideals of operators acting on Hilbert spaces and a result ([4]) of Daws, who classified all closed ideals of operators acting on $c_0(\lambda)$ and $\ell_p(\lambda)$ for λ uncountable and $p \in [1, \infty)$. Nevertheless, bearing in mind that the similar problem of classifying all closed ideals of operators acting on $\ell_\infty(\lambda)$ is most likely intractable (as this would require, in particular, understanding *all* injective Banach spaces), we offer instead a complete description of the maximal ideals for the space $\ell_\infty(\lambda)$ and its closed subspace

$$\ell_\infty^c(\lambda) = \{z \in \ell_\infty(\lambda) \colon z(\alpha) \neq 0 \text{ for at most countably many } \alpha < \lambda\}.$$

(For consistency of notation we sometimes denote ℓ_∞ by $\ell_\infty^c(\omega)$.) Our first result then reads as follows.

Theorem 1.1. *Let λ be an infinite cardinal number. Suppose that X is either $\ell_\infty(\lambda)$ or $\ell_\infty^c(\lambda)$. Then*

$$\begin{aligned} \mathscr{S}_X(X) &= \{T \in \mathscr{B}(X) \colon T \text{ is not bounded below on any copy of } X\} \\ &= \{T \in \mathscr{B}(X) \colon I_X \neq ATB \text{ for all } A, B \in \mathscr{B}(X)\} \end{aligned}$$

is the unique maximal ideal of $\mathscr{B}(X)$.

(We explain the required notation and terminology in the next section, while proofs are postponed to the final section.)

Actually, we are able to describe all the closed ideals of $\ell_\infty^c(\lambda)$ containing the ideal of weakly compact operatrs—they consist precisely of operators which do not preserve copies of $\ell_\infty^c(\kappa)$ for some κ not exceeding the cardinal succesor of λ.

Theorem 1.2. *Let λ be an infinite cardinal number. Then every closed ideal of $\mathscr{B}(\ell_\infty^c(\lambda))$ that contains the ideal of weakly compact operators is equal to $\mathscr{S}_{\ell_\infty^c(\kappa)}(\ell_\infty^c(\lambda))$ for some infinite cardinal number $\kappa \leqslant \lambda^+$.*

Remark. One cannot hope that a statement similar to Theorems 1.2 or 1.5 holds for the lattice of closed ideals of $\mathscr{B}(\ell_\infty(\lambda))$. Indeed, in general the lattice of closed ideals of $\mathscr{B}(\ell_\infty(\lambda))$ need not be linearly ordered. Let $E_1 = L_\infty(\{0,1\}^{\omega_2})$ and $E_2 = \ell_\infty(\omega_1)$. Then both E_1 and E_2 embed into $\ell_\infty(\omega_2)$, but E_1 does not embed into E_2 and E_2 does not embed into E_1 ([28, Theorem 1.7]). Consider the (not necessarily closed) ideals $\mathscr{J}_i$ consisting of operators which admit a factorisation through E_i ($i = 1, 2$). These sets are obviously closed under compositions from left and right, and, since $E_i \cong E_i \oplus E_i$ ($i = 1, 2$), they are also closed under addition, so they are indeed ideals of $\mathscr{B}(\ell_\infty(\omega_2))$. We *claim* that the closures of $\mathscr{J}_i$ ($i = 1, 2$) are incomparable. Indeed, let P_i be a projection on $\ell_\infty(\omega_2)$ with range isomorphic to E_i ($i = 1, 2$), and suppose that $P_1 \in \overline{\mathscr{J}_2}$. Since $P_1^2 = P_1$, we actually have $P_1 \in \mathscr{J}_2$ (see [19, Lemma 2.7]). This, however, is impossible because it would imply that the range of P_1 is isomorphic to a subspace of E_2. The other case is symmetric.

This observation shows even more: there are 1-injective Banach spaces E of the form $L_\infty(\mu)$ for which $\mathscr{B}(E)$ has more than one maximal ideal. It is readily seen that $E = E_1 \oplus E_2$ is such an example. Here $E = L_\infty(\mu)$, where μ is the direct product measure of the Haar measure on $\{0,1\}^{\omega_2}$ with the counting measure on ω_1.

A key result in the proof of Theorem 1.1 is the following theorem, which, we believe, is of interest in itself. In order to state it, we require a piece of terminology.

A Banach space X is *complementably homogeneous* if for each closed subspace Y of X such that Y is isomorphic to X, there exists a closed, complemented subspace Z of X such that Z is contained in Y and Z is isomorphic to X. That $\ell_1(\lambda)$ is complementably homogeneous follows directly from Lemma 2.1(ii). The Banach spaces $c_0(\lambda)$ have actually a stronger property: every isomorphic copy of $c_0(\lambda)$ in $c_0(\lambda)$ is complemented ([2, Proposition 2.8]). For $p \in (1, \infty)$ complete homogeneity of $\ell_p(\lambda)$ is also well-known and easy to prove; we shall include a proof of that fact (Proposition 3.8) for the sake of completeness.

Theorem 1.3. *For each infinite cardinal number λ, the Banach space $\ell_\infty^c(\lambda)$ is complementably homogeneous.*

We then employ Theorem 1.3 to characterise, up to isomorphism, all infinite-dimensional complemented subspaces of $\ell_\infty^c(\lambda)$, where λ is any uncountable cardinal number.

Theorem 1.4. *Let λ be an infinite cardinal number. Then every infinite dimensional, complemented subspace of $\ell_\infty^c(\lambda)$ is isomorphic either to ℓ_∞ or to $\ell_\infty^c(\kappa)$ for some cardinal $\kappa \leqslant \lambda$. Consequently, $\ell_\infty^c(\lambda)$ is a primary Banach space.*

Let us remark that $\ell_\infty^c(\lambda)$ is a hyperplane in the space $E := \operatorname{span}\{\mathbf{1}_\lambda, \ell_\infty^c(\lambda)\}$. The space E in turn is easily seen to be isomorphic to its hyperplanes as it contains ℓ_∞. More importantly, E is a unital sub-C*-algebra of $\ell_\infty(\lambda)$ (the operations are pointwise), hence by the Gelfand–Naimark duality, $E \cong C(K)$ for some compact Hausdorff space K. The space E has one more incarnation—it is isometrically isomorphic to the space $L_\infty(\mu)$, where μ is the counting measure on λ restricted to the σ-field of sets that are countable or have countable complement. The space $L_\infty(\mu)$ is not injective and therefore is not isomorphic to a dual space. Indeed, it is an observation of Pełczyński and Sudakov ([24]) that $\ell_\infty^c(\lambda)$ (λ uncountable) is not complemented in $\ell_\infty(\lambda)$, so it is not injective. Since $C(K)$-spaces isomorphic to dual spaces are injective, we conclude that $L_\infty(\mu)$ is not a dual space.

It is noteworthy that there are very few kinds of $C(K)$-spaces with completely understood complemented subspaces. To the best of our knowledge, these are:

- $c_0(\lambda)$ for any set λ; these spaces are isomorphic to $C(\lambda^1)$ where λ^1 is the one-point compactification of the discrete set λ. Every infinite-dimensional, complemented subspace of $c_0(\lambda)$ is isomorphic to $c_0(\kappa)$ for some cardinal number $\kappa \leqslant \lambda$ (see [10] or [2, Proposition 2.8]).
- $C[0,\omega^\omega]$, where $[0,\omega^\omega]$ carries the order topology. Every infinite-dimensional, complemented subspace of $C[0,\omega^\omega]$ is isomorphic either to c_0 or to $C[0,\omega^\omega]$ ([3, Theorem 3]).

  No further separable examples—apart from c_0 and $C[0,\omega^\omega]$—of infinite dimensional $C(K)$-spaces with completely understood complemented subspaces are known.
- $C(\beta\mathbb{N}) \cong \ell_\infty$. Every infinite-dimensional, complemented subspace of ℓ_∞ is isomorphic to ℓ_∞; this is Lindenstrauss' theorem ([20]).
- $C(K)$, where K is the compact scattered space constructed by Koszmider under the Continuum Hypothesis ([17]); it is the Stone space of a Boolean algebra generated by a certain almost disjoint family of subsets of $\mathbb{N}$. Every infinite-dimensional, complemented subspace of $C(K)$ is isomorphic either to c_0 or $C(K)$.
- Spaces of the form $C(K \sqcup L) \cong C(K) \oplus C(L)$, where K is scattered, $C(L)$ is a Grothendieck space (equivalently, every operator from $C(L)$ to a separable space is weakly compact; for definition and basic properties of Grothendieck spaces see, e.g., [5, p. 150]), and all the complemented subspaces of $C(K)$ and $C(L)$ are classified. (So far, for K we can take the one-point compactification of a discrete set, $[0,\omega^\omega]$, or the above-mentioned space constructed by Koszmider; and for L we can take $\beta\mathbb{N}$, so that $C(L)$ is isometrically isomorphic to ℓ_∞, a prototypical example of a Grothendieck space; see [11].) Each complemented subspace of $C(K) \oplus C(L)$ is isomorphic to a space of the form $X \oplus Y$, where X is a complemented subspace of $C(K)$ and Y is complemented in $C(L)$.

Indeed, if K is scattered, then $C(K)^* \cong \ell_1(|K|)$, so it follows from [12, Corollary 2] that the unit ball of $C(K)^*$ is weak*-sequentially compact. Applying [5, Corollary 5 on p. 150], we infer that every operator $T \colon C(L) \to C(K)$ is weakly compact and hence strictly singular by [23]. It follows then from [30, Theorem 1.1] that every complemented subspace of $C(K) \oplus C(L)$ is of the desired form.

We therefore extend the above list by adding the following classes of examples:

- $\ell_\infty^c(\lambda)$, for any uncountable cardinal number λ,
- $\ell_\infty^c(\lambda) \oplus C(K)$, where $C(K)$ is isomorphic to one of the spaces $c_0(\kappa)$, $C[0, \omega^\omega]$ or the above-mentioned space constructed by Koszmider. Indeed, for any cardinal number λ, $\ell_\infty^c(\lambda)$ is a Grothendieck space (see Corollary 3.4 below), so we can use the antepenultimate clause to deduce the claim.

As a by-product of our investigations, we give an alternative approach to Daws' description of all closed ideals of $\mathscr{B}(c_0(\lambda))$ and $\mathscr{B}(\ell_p(\lambda))$ for λ uncountable and $p \in [1, \infty)$.

Theorem 1.5. *Let λ be an infinite cardinal number and let $p \in [1, \infty)$. Then every non-zero, proper closed ideal of $\mathscr{B}(c_0(\lambda))$ and $\mathscr{B}(\ell_p(\lambda))$ is of the form $\mathscr{S}_{c_0(\kappa)}(c_0(\lambda))$ and $\mathscr{S}_{\ell_p(\kappa)}(\ell_p(\lambda))$, respectively, for some infinite cardinal number $\kappa \leqslant \lambda$.*

2. Preliminaries

In this paper, we consider Banach spaces over the fixed scalar field either of real or complex numbers. Our terminology is standard and follows mainly [1] and [21]. By *operator*, we mean a bounded, linear operator acting between Banach spaces. Let X and Y be Banach spaces. We denote by $\mathscr{B}(X, Y)$ the Banach space of all operators from X to Y and write $\mathscr{B}(X)$ for $\mathscr{B}(X, X)$. An operator $P \in \mathscr{B}(X)$ is a *projection* if $P^2 = P$.

Let $T \colon X \to Y$ be an operator. We say that T is *bounded below* if there is a constant $c > 0$ such that $\|Tx\| \geqslant c\|x\|$ for all $x \in X$. It is clear an operator is bounded below if and only if it is injective and has closed range, in which case it is an isomorphism onto its range.

Let E be a Banach space. We say that an operator T acting between Banach spaces is *bounded below on a copy of E* if there exists a subspace E_0 of the domain of T such that E_0 is isomorphic to E and the operator $T|_{E_0}$ is bounded below. For each pair of Banach spaces X, Y we set

$$\mathscr{S}_E(X, Y) = \{T \in \mathscr{B}(X, Y) \colon T \text{ is not bounded below on any copy of } E\},$$

and write $\mathscr{S}_E(X) = \mathscr{S}_E(X, X)$. We call operators which belong to $\mathscr{S}_E(X, Y)$ *E-singular operators*. We denote by $\mathscr{S}_E$ the class of all E-singular operators acting between arbitrary Banach spaces. In general, the set $\mathscr{S}_E(X)$ need not be closed under addition. This is readily seen in the case where $E = X = \ell_p \oplus \ell_q$ for $1 \leqslant p < q < \infty$.

We use von Neumann's definition of an ordinal number. We identify cardinal numbers with initial ordinal numbers. For instance, ω_1 is the first uncountable ordinal, ω_2 is the second one and so on. If λ is a cardinal and $\alpha < \lambda$ we often write $[0, \alpha)$ and (α, λ) for the set of ordinals less than λ which are, respectively, less than α and greater than α. Even though α and $[0, \alpha)$ are equal, we distinguish these symbols to avoid confusion in the

situation when we think of α an element of λ and in the situation when it is considered a subset of it. The *cofinality* of a cardinal number λ is the minimal cardinality κ of a family of cardinals strictly less than λ, say $\{\lambda_\alpha : \alpha < \kappa\}$, such that $\lambda = \sum_{\alpha < \kappa} \lambda_\alpha$. The cofinality of ω_1 is ω_1 while the cofinality of ω_ω is countable. If Λ is a set, we denote by $|\Lambda|$ the cardinality of Λ. For a cardinal number λ, we denote by λ^+ its cardinal successor; for instance $\omega_1^+ = \omega_2$ etc.

For a subset Y of a topological space X we define the *density* of Y, dens Y, to be the minimal cardinality of a dense subset of Y. Let X be a Banach space and let $Y \subseteq X$ be a closed subspace. We then define the *codensity* of Y (relative to X) by codens $Y =$ dens X/Y.

Let λ be an infinite cardinal and let E_λ denote one of the spaces: $c_0(\lambda)$, $\ell_p(\lambda)$ $(p \in [1, \infty])$ or $\ell_\infty^c(\lambda)$. For each subset $\Lambda \subseteq \lambda$ the following formula defines a contractive projection P_Λ on E_λ:

$$(2.1) \qquad (P_\Lambda f)(\alpha) = \begin{cases} f(\alpha), & \text{if } \alpha \in \Lambda \\ 0, & \text{if } \alpha \notin \Lambda \end{cases} \qquad (f \in E_\lambda).$$

We then denote the range of P_Λ by E_Λ. Of course, the range of P_Λ is isometrically isomorphic to $E_{|\Lambda|}$. We denote by $R_\Lambda : E_\lambda \to E_\Lambda$ the *restriction operator* which identifies an element f in the range of P_Λ with the restriction of f, $R_\Lambda f$, to the set Λ. For $f \in E_\lambda$ we define the *support* of f by supp $f = \{\alpha < \lambda : f(\alpha) \neq 0\}$. A function f is *supported* in a set Λ if supp $f \subseteq \Lambda$.

We shall also require the following lemma due to Rosenthal ([26, Proposition 1.2, Corollary on p. 29 and Remark 1 on p. 30]).

Lemma 2.1. *Let λ be an infinite cardinal number and let X be a Banach space.*

   (i) *Suppose that $T : \ell_\infty(\lambda) \to X$ is an operator such that $T|_{c_0(\lambda)}$ is bounded below. Then there exists a set $\Lambda \subset \lambda$ of cardinality λ such that $T|_{\ell_\infty(\Lambda)}$ is bounded below. Consequently, if Z is an injective Banach space and $T : Z \to X$ is an operator that is bounded below on a copy of $c_0(\lambda)$, then T is bounded below on a copy of $\ell_\infty(\lambda)$.*

   (ii) *Suppose that $T : X \to \ell_1(\lambda)$ is an operator such that for some $\delta > 0$ and some bounded set $\{y_\alpha : \alpha < \lambda\} \subset X$ we have*

$$\delta \leqslant \|Ty_\alpha - Ty_\beta\| \qquad (\alpha, \beta < \lambda, \alpha \neq \beta).$$

     Then T is bounded below on some complemented subspace of X that is isomorphic to $\ell_1(\lambda)$.

   (iii) *Suppose that $T : c_0(\lambda) \to X$ is an operator such that $\inf\{\|Te_\alpha\| : \alpha < \lambda\} > 0$, where $\{e_\alpha : \alpha < \lambda\}$ is the canonical basis of $c_0(\lambda)$. Then there exists a set $\Lambda \subset \lambda$ of cardinality λ such that $T|_{c_0(\Lambda)}$ is bounded below.*

We say that a Banach space is *primary* if whenever X is isomorphic to $Y \oplus Z$, then at least one of the spaces Y or Z is isomorphic to X.

3. Auxiliary results

Proposition 3.1. *Let λ be a cardinal number and let $T\colon \ell_\infty^c(\lambda) \to \ell_\infty^c(\lambda)$ be an operator that is not bounded below on any sublattice isometric to $c_0(\lambda)$. Then for every $\varepsilon > 0$ there is subset Λ of λ so that $|\Lambda| < \lambda$ and*

$$\|TR_{\lambda\setminus\Lambda}\| \leqslant \varepsilon.$$

Consequently, if also T is a projection onto a subspace X, then X is isomorphic to a complemented subspace of $\ell_\infty^c(\kappa)$ for some cardinal number $\kappa < \lambda$.

Proof. Let $(f_\alpha)_{\alpha\in\Gamma}$ be a collection of disjointly supported unit vectors in $\ell_\infty^c(\lambda)$ that is maximal with respect to the property that $\|Tf_\alpha\| \geqslant \varepsilon$ for each α. By Lemma 2.1(iii) and the hypothesis on T we have $|\Gamma| < \lambda$. Let Λ be the union of the supports of the f_α ($\alpha \in \Gamma$), so that also $|\Lambda| < \lambda$. By the maximality of the collection $(f_\alpha)_{\alpha\in\Gamma}$, if f is a unit vector whose support is contained in $\lambda \setminus \Lambda$, then $\|Tf\| < \varepsilon$, which implies that $\|TR_{\lambda\setminus\Lambda}\| \leqslant \varepsilon$.

For the "consequently" statement, suppose that T is a projection onto a subspace X. Then

$$I_X = (TR_{\lambda\setminus\Lambda} + TR_\Lambda)|_X,$$

so

$$\|(I_X - TR_A)|_X\| \leqslant \varepsilon.$$

Hence if $\varepsilon < 1$, there is an operator U on X so that

$$I_X = (UTR_A)|_X.$$

Thus I_X factors through $\ell_\infty^c(|\Lambda|)$. $\qquad\square$

Corollary 3.2. *Every infinite-dimensional, complemented subspace of $\ell_\infty^c(\omega_1)$ that does not contain a subspace isomorphic to $c_0(\omega_1)$ is isomorphic to ℓ_∞.*

Proof. Let X be an infinite-dimensional, complemented subspace of $\ell_\infty^c(\omega_1)$ that does not contain a subspace isomorphic to $c_0(\omega_1)$. Let P be any projection onto X. By Proposition 3.1, X is isomorphic to $\ell_\infty^c(\omega) = \ell_\infty$. $\qquad\square$

In view of Corollary 3.2, in order to complete the classification of the complemented subspaces of $\ell_\infty^c(\lambda)$, that is to prove Theorem 1.4, it is enough to show that a complemented subspace X of $\ell_\infty^c(\lambda)$ that contains an isomorphic copy of $c_0(\lambda)$ must be isomorphic to $\ell_\infty^c(\lambda)$. We begin by showing that such an X must contain a subspace isomorphic to $\ell_\infty^c(\lambda)$.

Proposition 3.3. *Let κ and λ be infinite cardinal numbers such that $\kappa \leqslant \lambda$ and let $T\colon \ell_\infty^c(\lambda) \to \ell_\infty^c(\lambda)$ be an operator. Suppose that there is a transfinite sequence $(x_\beta)_{\beta<\kappa}$ of disjointly supported unit vectors in $\ell_\infty^c(\lambda)$ so that the restriction of T to $\overline{\operatorname{span}}\,\{x_\beta\colon \beta < \kappa\}$ is bounded below. Then there is a subspace Y of $\ell_\infty^c(\lambda)$ that is isometric to $\ell_\infty^c(\kappa)$ such that $T|_Y$ is bounded below.*

Proof. Define an isometry $S\colon \ell_\infty^c(\kappa) \to \ell_\infty^c(\lambda)$ by

$$(Sz)(\beta) = z(\beta)x_\beta \qquad (z \in \ell_\infty^c(\kappa), \beta < \kappa).$$

The assumption on T says that the operator $TS\colon c_0(\kappa) \to \ell_\infty^c(\lambda)$ is bounded below.

Regard $\ell_\infty^c(\kappa)$ as the subspace $\ell_\infty^c([0, \kappa))$ of $\ell_\infty(\lambda)$. By injectivity of $\ell_\infty(\lambda)$, the operator $(TS)|_{c_0(\kappa)}$ has a norm-preserving extension to an operator $U\colon \ell_\infty(\lambda) \to \ell_\infty(\lambda)$ (see [21, Proposition 2.f.2]). The operator U is an isomorphism on $c_0(\kappa)$, hence by Lemma 2.1(i), there is a subset Γ of κ with cardinality κ such that the restriction of U to $\ell_\infty(\Gamma)$ is an isomorphism. But U maps $\ell_\infty^c(\Gamma)$ into $\ell_\infty^c(\lambda)$, which is to say that TS is an isomorphism on $\ell_\infty^c(\Gamma)$. This implies that $Y := S[\ell_\infty^c(\Gamma)]$ is isomorphic to $\ell_\infty^c(\kappa)$ and $T|_Y$ is bounded below. $\qquad\square$

The following corollary to Proposition 3.3 is essentially known in the sense that it can be easily deduced from a string of standard results concerning Grothendieck spaces; however, we offer here a direct proof.

Corollary 3.4. *For every cardinal number λ, $\ell_\infty^c(\lambda)$ is a Grothendieck space.*

Proof. If $\ell_\infty^c(\lambda)$ is not a Grothendieck space there exists an operator

$$T\colon \ell_\infty^c(\lambda) \to c_0 \subset \ell_\infty^c(\lambda)$$

that is not weakly compact (see [5, p. 150]). Then by [23] there is a subspace Y of $\ell_\infty^c(\lambda)$ isomorphic to c_0 such that $T|_Y$ is bounded below. Without loss of generality we may assume that Y is spanned by disjointly supported functions. By Proposition 3.3, T is bounded below on some subspace isomorphic to $\ell_\infty^c(\lambda)$; a contradiction as T maps into c_0. $\qquad\square$

We need a consequence of [26, Lemma 1.1]

Lemma 3.5. *Let Γ be uncountable and let T be an operator on $\ell_\infty(\Gamma)$ such that T is the identity on $c_0(\Gamma)$. Then for every $\varepsilon > 0$ there is $\Gamma' \subset \Gamma$ with $|\Gamma'| = |\Gamma|$ so that*

$$\|(R_{\Gamma'}T)|_{\ell_\infty(\Gamma')} - I_{\ell_\infty(\Gamma')}\| < \varepsilon,$$

where $R_{\Gamma'}\colon \ell_\infty(\Gamma) \to \ell_\infty(\Gamma')$ is the restriction operator.

Proof. We identify $\ell_\infty(\Gamma)^*$ with the finitely additive measures on the σ-algebra of all subsets of Γ. For $\gamma \in \Gamma$, set $\mu_\gamma = T^*\delta_\gamma$, where δ_γ is the point mass measure at γ. By [26, Lemma 1.1] there is $\Gamma' \subset \Gamma$ with $|\Gamma'| = |\Gamma|$ so that for all $\gamma \in \Gamma'$ we have $|\mu_\gamma|(\Gamma' \setminus \{\gamma\}) < \varepsilon$. $\qquad\square$

Proof of Theorem 1.3. We need to show that if X is a subspace of $\ell_\infty^c(\lambda)$ isomorphic to $\ell_\infty^c(\lambda)$, then X contains a subspace isomorphic to $\ell_\infty^c(\lambda)$ that is complemented in $\ell_\infty^c(\lambda)$.

Let $\prec$ denote the order on X that is induced from its isomorphism with $\ell_\infty^c(\lambda)$. From (the proof of) Proposition 3.1 we have a transfinite sequence $\{f_\gamma\colon \gamma < \lambda\}$ of unit vectors in X that are disjoint both relative to the order structure on X given by $\prec$ and the order structure on $\ell_\infty^c(\lambda)$ given by its pointwise ordering $<$. Let X be the lattice closure in $(\ell_\infty^c(\lambda), <)$ of the linear span of $\{f_\gamma\colon \gamma < \lambda\}$ (so that Z is isometrically isomorphic to $\ell_\infty^c(\lambda)$ and there is a norm-one projection Q from $\ell_\infty^c(\lambda)$ onto Z). By replacing X itself with a subspace, we might as well assume that in $(X, \prec)$ the f_γ form the unit vector basis

for $c_0(\lambda)$ and hence that in $(X, \prec)$ the space X is the pointwise closure of the f_γ. Let $J \colon (Z, <) \to (X, \prec)$ denote the natural surjective order isomorphism that is the identity on the set $\{f_\gamma \colon \gamma < \lambda\}$. Now

$$(JQ)|_X \colon X \to X$$

is an operator on X that is the identity on $\{f_\gamma \colon \gamma < \lambda\}$, so by Lemma 3.5 (applied to any extension of the operator to an operator on the $\ell_\infty(\lambda)$ lattice generated by $(X, \prec)$) we have a subset Λ of λ of cardinality λ so that

$$(3.1) \qquad \qquad \qquad \|(R_\Lambda JQ)|_{X_\Lambda} - I_{X_\Lambda}\| < \varepsilon$$

where X_Λ is the pointwise closure of $\{f_\gamma \colon \gamma \in \Lambda\}$ in $(X, \prec)$, R_Λ is the restriction mapping on $(X, \prec)$, and $\varepsilon > 0$ can be as small as we want; in particular, $\varepsilon < 1$. So there is an automorphism U on X so that $UR_\Lambda JQ$ is the identity on X_Λ and hence $UR_\Lambda JQ$ is a projection from $\ell_\infty^c(\lambda)$ onto X_Λ. Since X_Λ is isomorphic to $\ell_\infty^c(\lambda)$, this completes the proof. $\qquad \square$

The above proof of Theorem 1.3 can be adjusted to get the following stronger result.

Proposition 3.6. *Let κ and λ be infinite cardinal numbers. If X is a subspace of $\ell_\infty^c(\lambda)$ that is isomorphic to $\ell_\infty^c(\kappa)$, then there is a subspace Y of X that is isomorphic to $\ell_\infty^c(\kappa)$ and complemented in $\ell_\infty^c(\lambda)$.*

Proposition 3.7. *Let λ be an uncountable cardinal number and let E be either $c_0(\lambda)$ or $\ell_p(\lambda)$ for some $p \in (1, \infty)$. Suppose that $X = \ell_\infty^c(\lambda)$ in the former case or $X = E$ in the latter case and let $T \colon E \to X$ be an operator that is not bounded below on any subspace Y with $\mathrm{codens}\, Y < \lambda$. Fix $\varepsilon > 0$. Then there exists a subspace $Z \subseteq X$ isomorphic to E such that $\|T|_Z\| \leqslant \varepsilon$.*

Proof. Take a unit vector $z_0 \in E$ such that $\|Tz_0\| < \varepsilon$. Fix $\gamma < \lambda$ and suppose that we have chosen already a family of unit vectors $\{z_\delta \colon \delta < \gamma\} \subset E$ with pairwise disjoint supports such that $\|Tz_\delta\| < \varepsilon$ for all $\delta < \gamma$ and

$$\mathrm{supp}\, Tz_{\delta_1} \cap \mathrm{supp}\, Tz_{\delta_2} = \varnothing$$

for all distinct $\delta_1, \delta_2 < \gamma$.

Let $\Gamma = \bigcup_{\delta < \gamma} \mathrm{supp}\, z_\delta$. Clearly, $|\Gamma| < \lambda$, so the range of $P_{\lambda \setminus \Gamma}$ (cf. (2.1)) has codensity less than λ. We claim that there exists a unit vector z_γ supported in $\lambda \setminus \Gamma$ such that $\|Tz_\gamma\| < \varepsilon$ and

$$\mathrm{supp}\, Tz_\gamma \cap \bigcup_{\delta < \gamma} \mathrm{supp}\, Tz_\delta = \varnothing.$$

Assume not. Then there exists a transfinite sequence of unit vectors $(w_\beta)_{\beta < \lambda}$ with with pairwise disjoint supports contained in $\lambda \setminus \Gamma$ such that the set $\mathrm{supp}\, Tw_\beta \cap \bigcup_{\delta < \gamma} \mathrm{supp}\, Tz_\delta$ is non-empty for each $\beta < \lambda$. This means that there is $\alpha < \lambda$ which belongs to uncountably many of the sets $\mathrm{supp}\, Tw_\beta$ and a constant $c > 0$ such that $|(Tw_\beta)(\alpha)| > c$. Let $w_{\beta_1}, w_{\beta_2}, \ldots$ be such that $|(Tw_{\beta_n})(\alpha)| > c$ for all n. Set

$$y_n = \frac{\varepsilon_{\beta_1} w_{\beta_1} + \ldots + \varepsilon_{\beta_n} w_{\beta_n}}{\|\varepsilon_{\beta_1} w_{\beta_1} + \ldots + \varepsilon_{\beta_n} w_{\beta_n}\|} \qquad (n \in \mathbb{N}),$$

where $\varepsilon_{\beta_k} = \overline{(Tw_{\beta_k})(\alpha)}/|(Tw_{\beta_k})(\alpha)|$ $(k = 1, \ldots, n)$.

Then
$$\|Ty_n\| \geqslant \begin{cases} cn, & \text{if } E = c_0(\lambda), \\ \frac{cn}{n^{1/p}}, & \text{if } E = \ell_p(\lambda). \end{cases}$$

In either case $\|Ty_n\| \to \infty$ as $n \to \infty$; a contradiction.

The subspace $Z = \overline{\text{span}}\{z_\beta \colon \beta < \lambda\}$ is isomorphic to E and $\|T|_Z\| \leqslant \varepsilon$. $\qquad \square$

Proposition 3.8. *For each uncountable cardinal number λ and $p \in [1, \infty)$ the Banach space $\ell_p(\lambda)$ is complementably homogeneous.*

Proof. The case $p = 1$ is covered by Lemma 2.1(ii). Let us consider then the case where $p \in (1, \infty)$.

Let Y be a subspace of $\ell_p(\lambda)$ isomorphic to $\ell_p(\lambda)$. Let $T \colon \ell_p(\lambda) \to Y$ be an isomorphism of norm one. We claim that there exists a transfinite sequence $\{z_\alpha \colon \alpha < \lambda\}$ of elements in $\ell_p(\lambda)$ with pairwise disjoint supports that spans a copy of $\ell_p(\lambda)$ and such that Tz_α $(\alpha < \lambda)$ have pairwise disjoint supports. If this were not true, we could proceed exactly as in the proof of Proposition 3.7 to get a contradiction with boundedness of T. Since T is an isomorphism, $Z = \overline{\text{span}}\{Tz_\alpha \colon \alpha < \lambda\}$ is a copy of $\ell_p(\lambda)$ in Y. Moreover, it is a standard fact that such a Z is complemented. Indeed, for each $\alpha < \lambda$ take $w_\alpha \in \ell_p(\lambda)^*$ such that $\langle w_\alpha, Tz_\alpha \rangle = 1 = \|w_\alpha\|$. Set
$$Px = \sum_{\alpha < \lambda} \langle w_\alpha, P_{\text{supp } Tz_\alpha} x \rangle Tz_\alpha \quad \big(x \in \ell_p(\lambda) \big).$$

Then P is a projection whose range is precisely equal to Z. $\qquad \square$

Proposition 3.9. *Let λ be an infinite cardinal number, $p \in [1, \infty)$ and let X be a Banach space. Then the sets*
$$\mathscr{S}_{c_0(\lambda)}\big(X, \ell_\infty^c(\lambda)\big) \ \text{ and } \ \mathscr{S}_{\ell_p(\lambda)}\big(X, \ell_p(\lambda)\big)$$
are closed under addition.

Proof. The case $\lambda = \omega$ follows from [16, Proposition 2.5]. Let us consider the case where λ is uncountable.

Suppose that T, S are in $\mathscr{S}_{c_0(\lambda)}\big(X, \ell_\infty^c(\lambda)\big)$ and assume, in search of contradiction, that that $T + S$ is not in $\mathscr{S}_{c_0(\lambda)}\big(X, \ell_\infty^c(\lambda)\big)$. Then there exists a subspace $X_1 \subseteq X$ isomorphic to $c_0(\lambda)$ and $\varepsilon > 0$ such that $\varepsilon\|z\| \leqslant \|(T + S)(z)\|$ for all $z \in X_1$. By Proposition 3.7, there exists a subspace $Z_1 \subset X_1$ isomorphic to $c_0(\lambda)$ such that $\|T|_{Z_1}\| \leqslant \frac{\varepsilon}{3}$. (Formally, we apply Proposition 3.7 to $T|_{X_1}$ acting on X_1 renormed to $c_0(\lambda)$.) Repeating the above argument, we conclude that there is a subspace $Z_2 \subseteq Z_1$ isomorphic to $c_0(\lambda)$ such that $\|S|_{Z_2}\| \leqslant \frac{\varepsilon}{3}$. Consequently, for all $z \in Z_2$ we have
$$\varepsilon\|z\| \leqslant \|Tz + Sz\| \leqslant \frac{\varepsilon}{3}\|z\| + \frac{\varepsilon}{3}\|z\| = \frac{2}{3}\varepsilon\|z\|;$$

a contradiction.

The proof in the case of $\ell_p(\lambda)$ for $p \in (1, \infty)$ is analogous. The proof of the case of $p = 1$ is actually contained in [26]. Indeed, suppose contrapositively that $T, S \colon X \to \ell_1(\lambda)$

are operators such that $T + S$ is not in $\mathscr{S}_{\ell_1(\lambda)}\big(X, \ell_1(\lambda)\big)$. Let Y be a copy of $\ell_1(\lambda)$ in X such that $(T + S)_{|Y}$ is an isomorphism. By [26, Theorem 3.3], there is a set of unit vectors $\{x_\alpha \colon \alpha < \lambda\}$ in Y and disjoint sets $E_\alpha \subset \lambda$ ($\alpha < \lambda$) such that for some $\delta > 0$ $\|((T + S)x_\alpha)|_{E_\alpha}\| \geqslant \delta$. It follows that for some subset $\Lambda \subseteq \lambda$ of the cardinality λ and for either T or S, say T, $\|(Tx_\alpha)|_{E_\alpha}\| \geqslant \delta/2$ for all $\alpha \in \Lambda$. It now follows from [26, Propositions 3.2 and 3.1] that Λ contains a subset Λ' of the same cardinality such that T is bounded below on the closed linear span of $\{x_\alpha \colon \alpha \in \Lambda'\}$. $\qquad\square$

Remark. One can improve a bit the ℓ_1-version of the previous proposition: If X contains an isomorphic copy of $\ell_1(\lambda)$ then for each $\varepsilon > 0$ it contains a $(1 + \varepsilon)$-isomorphic copy of $\ell_1(\lambda)$ that is complemented by means of a projection of norm at most $1 + \varepsilon$.

This follows from a simple combination of the non-separable counter-part of the James distortion theorem for ℓ_1 ([13], cf. [8]) which provides the $(1 + \varepsilon)$-isomorphic copy of $\ell_1(\lambda)$ together with Dor's result [6, Theorem A] which provides the needed projection.

This remark is provided here because it may be useful in questions related to the classification of commutators. Similar facts were useful when classifying the commutators in, say, the algebra of bounded operators on ℓ_1.

Proposition 3.9 can be strengthened in the case of $c_0(\lambda)$:

Proposition 3.10. *For each infinite cardinal number λ, $\mathscr{S}_{c_0(\lambda)}$ forms a closed operator ideal.*

Proof. Let X and Y be Banach spaces. The set $\mathscr{S}_{c_0(\lambda)}(X, Y)$ is closed in $\mathscr{B}(X, Y)$ and obviously is closed under taking compositions with other operators (whenever these make sense). The only non-trivial thing is to verify that it is indeed closed under addition.

Let $T, S \in \mathscr{B}(X, Y)$ be operators such that $T + S$ is bounded below by $\delta > 0$ on some subspace $X_0 \subseteq X$ isomorphic to $c_0(\lambda)$. Let $\{e_\alpha \colon \alpha < \lambda\}$ be a transfinite sequence in X_0 equivalent to the canonical basis of $c_0(\lambda)$. Since

$$\delta \leqslant \|Te_\alpha + Se_\alpha\|$$

for all $\alpha < \lambda$, there is a set Λ of cardinality λ such that $\|Te_\alpha\| \geqslant \frac{\delta}{2}$ or $\|Se_\alpha\| \geqslant \frac{\delta}{2}$ for all $\alpha \in \Lambda$. It follows then from Lemma 2.1(iii) that at least one of the operators T or S does not belong to $\mathscr{S}_{c_0(\lambda)}(X, Y)$. $\qquad\square$

Theorem 3.11. *Let κ and λ be infinite cardinals and let $p \in [1, \infty]$. Then then $\mathscr{A}$ is a closed ideal of $\mathscr{B}$ if*

$$\mathscr{A} = \mathscr{S}_{c_0(\kappa)}(c_0(\lambda)) \text{ and } \mathscr{B} = \mathscr{B}(c_0(\lambda));$$
$$\mathscr{A} = \mathscr{S}_{\ell_p(\kappa)}(\ell_p(\lambda)) \text{ and } \mathscr{B} = \mathscr{B}(\ell_p(\lambda));$$
$$\mathscr{A} = \mathscr{S}_{\ell_\infty^c(\kappa)}(\ell_\infty^c(\lambda)) \text{ and } \mathscr{B} = \mathscr{B}(\ell_\infty^c(\lambda)).$$

Moreover,

$$\mathscr{S}_{\ell_\infty^c(\kappa)}(\ell_\infty^c(\lambda)) = \mathscr{S}_{c_0(\kappa)}(\ell_\infty^c(\lambda)).$$

Proof. The first clause as well as the second one in the case of $p \in [1, \infty)$ follow directly from Proposition 3.9.

We claim that $\mathscr{S}_{\ell^c_\infty(\kappa)}(\ell^c_\infty(\lambda)) = \mathscr{S}_{c_0(\kappa)}(\ell^c_\infty(\lambda))$, which is plainly enough as Proposition 3.9 asserts that $\mathscr{S}_{c_0(\kappa)}(\ell^c_\infty(\lambda))$ is a closed ideal of $\mathscr{B}(\ell^c_\infty(\lambda))$. Let us suppose that $T \notin \mathscr{S}_{c_0(\kappa)}(\ell^c_\infty(\lambda))$. By Proposition 3.3, T is bounded below on some copy of $\ell^c_\infty(\kappa)$, hence $T \notin \mathscr{S}_{\ell^c_\infty(\kappa)}(\ell^c_\infty(\lambda))$. On the other hand, $\mathscr{S}_{\ell^c_\infty(\kappa)}(\ell^c_\infty(\lambda)) \supseteq \mathscr{S}_{c_0(\kappa)}(\ell^c_\infty(\lambda))$ trivially, so the proof of the claim is complete.

Let us consider now the remaining case of $\mathscr{A} = \mathscr{S}_{\ell_\infty(\kappa)}(\ell_\infty(\lambda))$ and $\mathscr{B} = \mathscr{B}(\ell_\infty(\lambda))$. Suppose contrapositively that T, S are operators on $\ell_\infty(\lambda)$ such that $T+S \notin \mathscr{S}_{c_0(\kappa)}(\ell_\infty(\lambda))$. Let $X \subseteq \ell_\infty(\lambda)$ be a copy of $c_0(\kappa)$ on which $T + S$ is bounded below. Since $\mathscr{S}_{c_0(\lambda)}$ is an operator ideal (Proposition 3.10), either T or S is bounded below on some copy of $c_0(\lambda)$. By Lemma 2.1(i), at least one of those operators is bounded below on a copy of $\ell_\infty(\lambda)$, which means that either T or S is not in $\mathscr{S}_{\ell_\infty(\kappa)}(\ell_\infty(\lambda))$. $\qquad\square$

The next lemma is a counter-part of [4, Proposition 5.1].

Lemma 3.12. *Let κ and λ be cardinal numbers and suppose that κ is not a successor of any cardinal number. If E_λ is one of the spaces $c_0(\lambda)$, $\ell^c_\infty(\lambda)$ or $\ell_p(\lambda)$ for some $p \in [1, \infty)$, then*

$$\overline{\bigcup_{\rho < \kappa} \mathscr{S}_{E_\rho}(E_\lambda)} = \mathscr{S}_{E_\kappa}(E_\lambda).$$

Consequently, if the cofinality of κ is uncountable

$$\bigcup_{\rho < \kappa} \mathscr{S}_{E_\rho}(E_\lambda) = \mathscr{S}_{E_\kappa}(E_\lambda).$$

Proof. Only the inclusion from left to right requires justification.

Take $T \in \mathscr{S}_{E_\kappa}(E_\lambda)$. We claim that for each $\varepsilon > 0$ there is a set Λ with $|\Lambda| < \kappa$ such that $\|P_\Lambda T - T\| < \varepsilon$. We split the proof of this claim into two subcases.

- Let us consider first the case where $E_\lambda = \ell_1(\lambda)$ and assume that the assertion does not hold for some $\varepsilon > 0$. By [26, Theorem 3.3], there is a set of unit vectors $\{x_\alpha : \alpha < \kappa\}$ in Y and disjoint sets $E_\alpha \subset \lambda$ $(\alpha < \kappa)$ such that for some $\delta \in (0, \varepsilon)$ we have $\|Tx_\alpha|_{E_\alpha}\| \geqslant \delta$ $(\alpha < \kappa)$. It now follows from [26, Propositions 3.2 and 3.1] that κ contains a subset of the same cardinality, K say, such that T is bounded below on $\overline{\operatorname{span}}\{x_\alpha : \alpha \in \mathrm{K}\} \cong \ell_1(\kappa)$.
- Assume that our assertion is not true in the remaining cases, which means that for some $\varepsilon > 0$ and all sets Λ with cardinality less than κ we have

$$\|P_{\lambda \setminus \Lambda} T\| = \|P_\Lambda T - T\| \geqslant \varepsilon.$$

If $E_\lambda = c_0(\lambda)$ or $E_\lambda = \ell_p(\lambda)$ for some $p \in (1, \infty)$, this is a contradiction. Indeed, otherwise arguing exactly as in the proof of Proposition 3.7, we would have constructed a family of disjointly supported unit vectors $\{z_\gamma : \gamma < \kappa\}$ such that $\{Tz_\gamma : \gamma < \kappa\}$ are also disjointly supported and have norm at least $\varepsilon/2$. In particular, T would be bounded below on a copy of E_κ against the assumption. For $E_\lambda = \ell^c_\infty(\lambda)$ this is a contradiction with the last assertion of Proposition 3.11 combined with the already proved case of $c_0(\lambda)$.

Since $P_\Lambda T \in \mathscr{S}_{E_\rho}(E_\lambda)$ for some $\rho < \kappa$, we conclude that T belongs to the closure of the ideal $\bigcup_{\rho < \kappa} \mathscr{S}_{E_\rho}(E_\lambda)$ as desired.

The final assertion follows from the well-known fact that in first-countable spaces increasing unions of uncountably many closed sets that are well-ordered by inclusion are closed. $\qquad\square$

Lemma 3.13. *For each uncountable cardinal number we have*

$$\mathscr{W}(\ell_\infty^c(\lambda)) = \mathscr{S}_{\ell_\infty}(\ell_\infty(\lambda)),$$

where $\mathscr{W}$ denotes the ideal of weakly compact operators.

Proof. Let T be an operator on $\ell_\infty^c(\lambda)$. Then by [23], T is not weakly compact if and only if T is bounded below on some copy of c_0. By Proposition 3.3, being bounded below on a copy of c_0 is equivalent to being bounded below on a copy of $\ell_\infty = \ell_\infty^c(\omega)$. We then have $\mathscr{W}(\ell_\infty^c(\lambda)) = \mathscr{S}_{\ell_\infty}(\ell_\infty(\lambda))$. $\qquad\square$

Lemma 3.14. *Let κ and λ be infinite cardinals and let $T \in \mathscr{B}(\ell_\infty^c)$. If $T \notin \mathscr{S}_{\ell_\infty^c(\kappa)}(\ell_\infty^c(\lambda))$, then $\mathscr{S}_{\ell_\infty^c(\kappa^+)}(\ell_\infty^c(\lambda))$ is contained in the ideal generated by T.*

Proof. Note that $\mathscr{B}(\ell_\infty^c(\lambda))$ is nothing but $\mathscr{S}_{\ell_\infty^c(\lambda^+)}(\ell_\infty^c(\lambda))$. (Recall that λ^+ denotes the immediate cardinal successor of λ.) It is then enough to consider only the case where $\kappa \leqslant \lambda$. For the simplicity of notation, for any cardinal number κ set $E_\kappa = \ell_\infty^c(\kappa)$.

Fix $T \notin \mathscr{S}_{E_\kappa}(E_\lambda)$ and let E be a subspace of E_λ isomorphic to E_κ on which T is bounded below and such that $F = T[E]$ is complemented (such E exists by Proposition 3.6).

Let $S \in \mathscr{S}_{E_{\kappa^+}}(E_\lambda)$. By Lemma 2.1(iii) and Proposition 3.3, we conclude that

$$|\{\alpha < \lambda \colon Se_\alpha \neq 0\}| \leqslant \kappa,$$

so in particular,

$$\Big| \bigcup_{z \in E_\lambda} \operatorname{supp} Sz \Big| \leqslant \kappa.$$

Consider the set

$$\Gamma = [0, \kappa) \cup \bigcup_{z \in c_0(\lambda)} \operatorname{supp} Sz.$$

It is clear that the range of P_Γ is isomorphic to E_κ and $P_\Gamma S = S$. Let W be an operator such that $TWz = z$ for all $z \in F$. Furthermore, let $U \colon E_\Gamma \to F$ be an isomorphism. Since both E_Γ and F are complemented, U and U^{-1} can be extended to operators on E_λ. We shall keep the symbols U, U^{-1} for fixed extensions of these maps. Consequently,

$$S = P_\Gamma S = U^{-1}TWUP_\Gamma S$$

belongs to the ideal generated by T. $\qquad\square$

4. Proofs of Theorems 1.1, 1.2, 1.4 and 1.5

We are now in a position to prove Theorems 1.1, 1.2, 1.4 and 1.5 (we have already proved Theorem 1.3 in the previous section).

Proof of Theorem 1.1. The case of $\lambda = \omega$ follows from [21, Proposition 2.f.4] as explained in [18, p. 253]. Suppose then that λ is uncountable.

The set $\mathscr{S}_X(X)$ is indeed a closed ideal of $\mathscr{B}(X)$ by Theorem 3.11.

- If $X = \ell_\infty(\lambda)$ and $T \notin \mathscr{S}_X(X)$, then the range of T contains a (complemented) copy of $\ell_\infty(\lambda)$, say F.
- If $X = \ell_\infty^c(\lambda)$ and $T \notin \mathscr{S}_X(X)$, then it follows from Theorem 1.3 that there is a complemented copy of $\ell_\infty^c(\lambda)$, call it F, such that T is bounded below on F and $T[F]$ is complemented in X.

In either case, let W be an operator such that $TWz = z$ for all $z \in F$. (The operator W can be taken to be any extension of $(T|_F)^{-1}$ to the whole space.) Furthermore, let $U \colon X \to F$ be an isomorphism. Since F is complemented, U^{-1} can be extended to an operator on X, say V. Consequently,

$$I_X = VT(WU)$$

belongs to the ideal generated by T. For this reason, $I_X = ATB$ for some $A, B \in \mathscr{B}(X)$ if and only if $T \notin \mathscr{S}_X(X)$. $\qquad\square$

Proof of Theorem 1.2. We keep the notation of the proof of Lemma 3.14.

Let κ be an infinite cardinal. By Lemma 3.14, if $T \notin \mathscr{S}_{E_\kappa}(E_\lambda)$, then $\mathscr{S}_{E_{\kappa^+}}(E_\lambda)$ is contained in the ideal generated by T. We will show that if $\mathscr{J}$ is a closed ideal of $\mathscr{B}(E_\lambda)$, then $\mathscr{J} = \mathscr{S}_{E_\kappa}(E_\lambda)$ for some cardinal κ.

Set

$$\tau = \sup\{\rho \colon \rho \text{ is a cardinal number and } \mathscr{S}_{E_\rho}(E_\lambda) \subseteq \mathscr{J}\}.$$

This number is well defined as, by Lemma 3.13, $\mathscr{J}$ contains $\mathscr{W}(E_\lambda) = \mathscr{S}_{E_\omega}(E_\lambda)$. If the supremum τ is attained and $S \in \mathscr{J} \setminus \mathscr{S}_{E_\tau}(E_\lambda)$, then the ideal generated by S, hence $\mathscr{J}$ as well, would contain $\mathscr{S}_{E_\tau^+}(E_\lambda)$, which is impossible by the definition of τ. Consequently, $\mathscr{J} = \mathscr{S}_{E_\tau}(E_\lambda)$. Consider now the case where $\tau > \rho$ for each ρ such that $\mathscr{S}_{E_\rho}(E_\lambda)$ is contained in $\mathscr{J}$. Since $\mathscr{J}$ is closed it contains the closure of $\bigcup_{\rho < \kappa} \mathscr{S}_{E_\rho}(E_\lambda)$, which is to say that $\mathscr{S}_{E_\kappa}(E_\lambda) \subseteq \mathscr{J}$ by Lemma 3.12. However, if this inclusion were strict $\mathscr{S}_{E_{\kappa^+}}(E_\lambda)$ would be contained in $\mathscr{J}$, which would again contradict the definition of τ. $\qquad\square$

Proof of Theorem 1.4. Suppose that X is an infinite-dimensional complemented subspace of $\ell_\infty^c(\lambda)$ that is not isomorphic to ℓ_∞. By iterating Proposition 3.1, we infer that there is a cardinal number $\kappa \leqslant \lambda$ such that X is isomorphic to a complemented subspace of $\ell_\infty^c(\kappa)$ and $c_0(\kappa)$ embeds into X. Consequently, Proposition 3.3 applies and so X contains a subspace isomorphic to $\ell_\infty^c(\kappa)$. By Theorem 1.3, X contains a complemented subspace isomorphic to $\ell_\infty^c(\kappa)$. Since $\ell_\infty^c(\kappa)$ is isomorphic to $\ell_\infty(\ell_\infty^c(\kappa))$, the ℓ_∞-sum of countably many copies of itself, the Pełczyński decomposition method (cf. [1, Theorem 2.2.3]) yields that X is isomorphic to $\ell_\infty^c(\kappa)$. $\qquad\square$

Proof of Theorem 1.5. Let E_λ denote one of the spaces $c_0(\lambda)$ or $\ell_p(\lambda)$ for some $p \in [1, \infty)$. For $\lambda = \omega$ the result is well-known (see, *e.g.*, [25, Theorem on p. 82]). Suppose then that λ is uncountable.

Since E_λ has the approximation property, the ideal of compact operators $\mathscr{K}(E_\lambda)$ is the smallest closed non-zero ideal of $\mathscr{B}(E_\lambda)$ and $\mathscr{K}(E_\lambda) = \mathscr{S}_{E_\omega}(E_\lambda)$ ([25, Theorem on p. 82]). On the other hand $\mathscr{B}(E_\lambda) = \mathscr{S}_{E_{\lambda+}}(E_\lambda)$.

Let κ be an infinite cardinal. All we need to do is to show that if $T \notin \mathscr{S}_{E_\kappa}(E_\lambda)$, then $\mathscr{S}_{E_{\kappa+}}(E_\lambda)$ is contained in the ideal generated by T. However, the remainder of the proof is completely analogous to the proof of Theorem 1.2, so we leave the details to the reader. $\square$

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