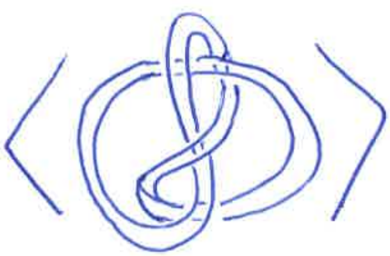
Lecture 3: Colored Jones Polynomials

Reference: BHMV: Three-Manifold Invariants derived from the Kauffman Bracket

Topology, 1992.

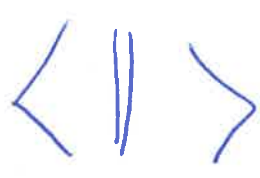
<u>Naive idea</u>: Consider $\langle\ \rangle$ of n parallel copies of K, and hope it is invariant under RM I, II, III.

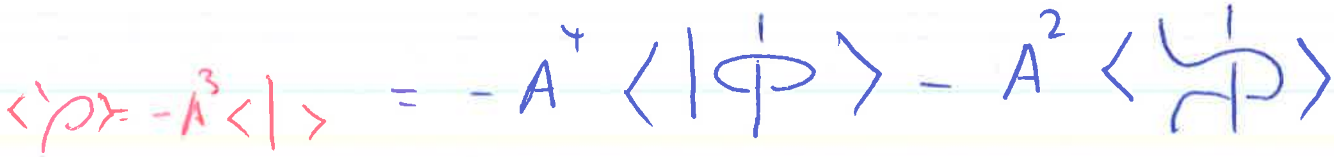


· True for RM II, III, since

are compositions of a sequence of RM II's & III's.

What about RM I?

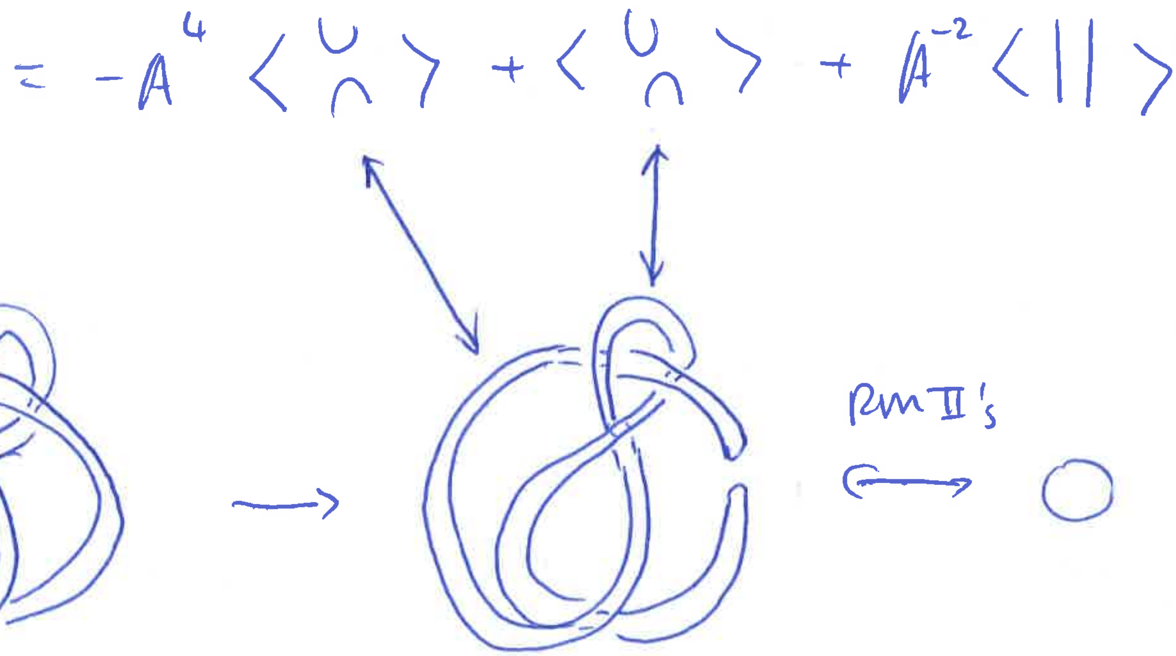
 $\neq c$ 



direct calculation $\underline{\underline{=}}$ $A^8 \langle || \rangle - A^8 + 1$

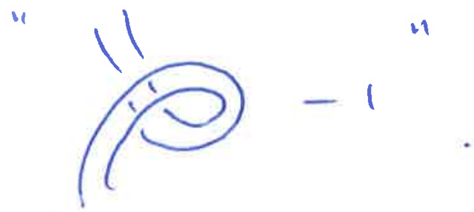
not $c \cdot \langle || \rangle$

Only non trivail pt in the calculation:

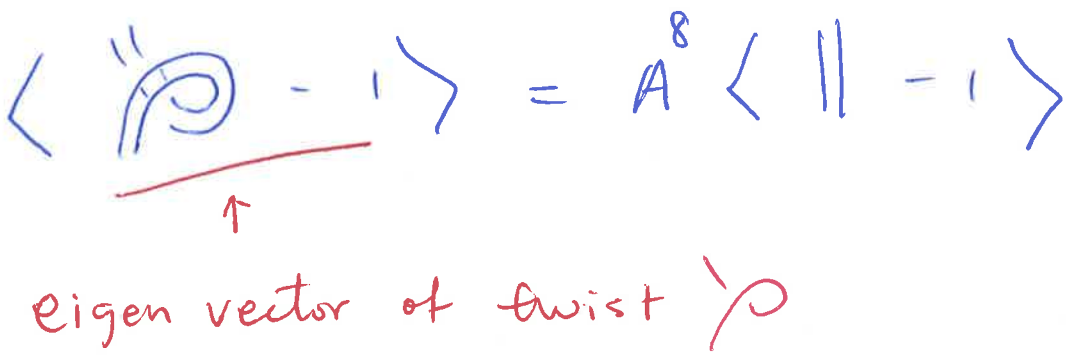


$= (-A^4 + 1)(-A^2 - A^{-2}) + A^2 \langle || \rangle$

Not so naive idea: consider



Then



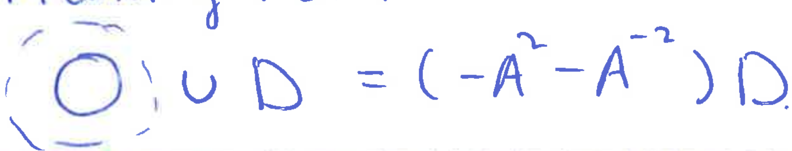
- Let $R = \mathbb{Z}[A^{\pm 1}]$, the ring of Laurent polynomials in A.

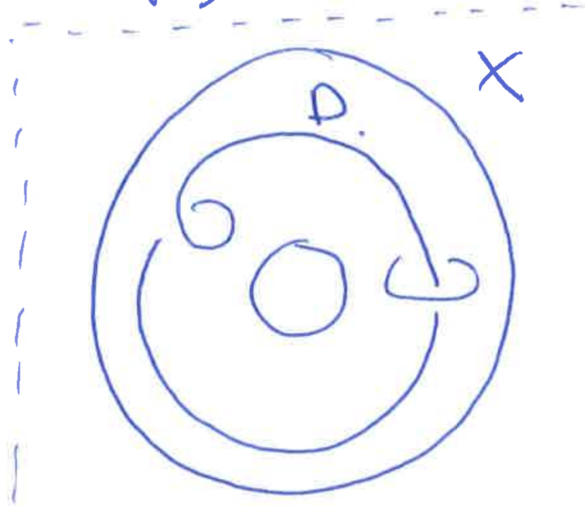
Def (Kauffman bracket skein module of annulus)

Let $\mathcal{B}$ be the R-module generated by all link diagrams in an annulus X, modulo

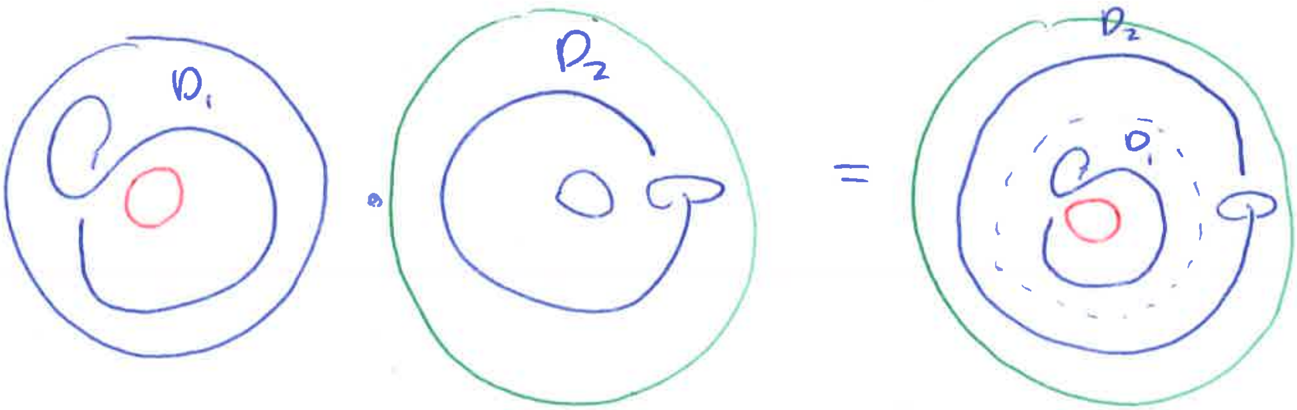
① Skein rel'n:

② Framing rel'n





<u>Algebra structure</u> on $\mathcal{B}$ is induced by gluing

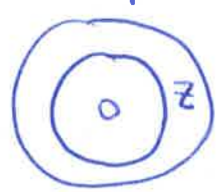


$D_1 \cdot D_2$.

Rm (i) "$\circ$" is commutative.

(ii) $\emptyset$ is considered as in $\mathcal{B}$ and is the <u>Unit</u> of $\circ$, hence is denoted by 1.

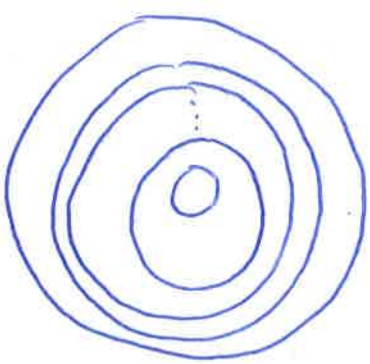
Thm. Let $z \in \mathcal{B}$ be the diagram represented by the core curve of X,



Then

$$\mathcal{B} \cong R[z] = \mathbb{Z}[A^{\pm 1}][z].$$

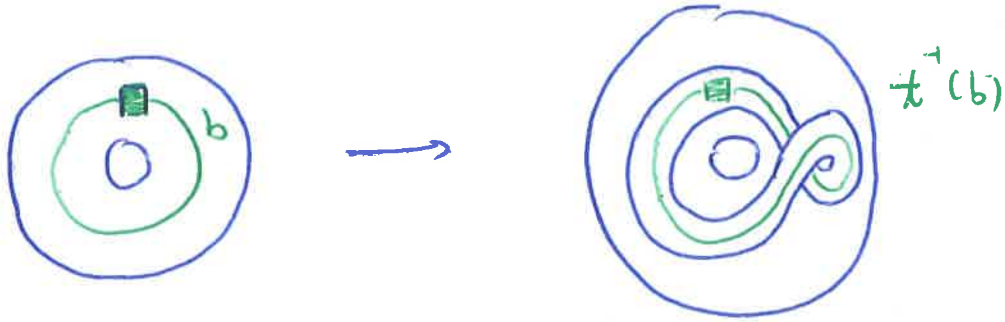
Eg:

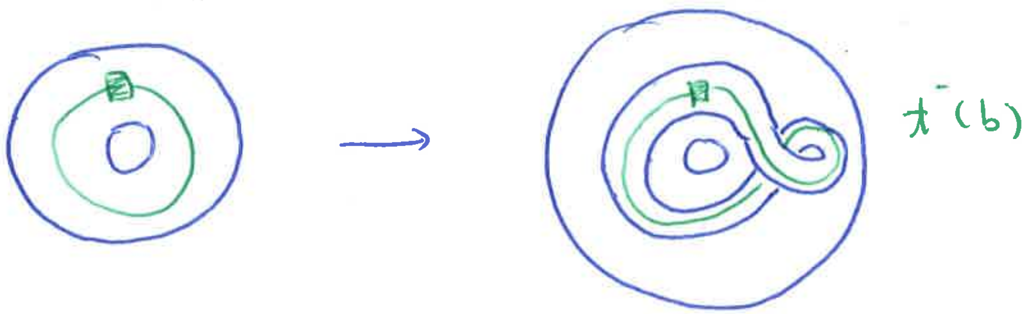


z^n = n parallel copies of z.

<u>Twist operators</u> $t^{\pm} : \mathcal{B} \to \mathcal{B}$ are resp. the ⑤

linear operators induced by





- $b \in \mathcal{B}$ is a R-linear combination of diagrams.
  Do the above operation to each diagram, then take the corresponding linear combination.

Def: Let $e_n \in \mathcal{B}$ defined recursively by

$e_0 = 1$, $e_1 = z$ and

$$e_n = z \cdot e_{n-1} - e_{n-2}.$$

Rm: e_n = n-th Chebyshev Polynomial in z

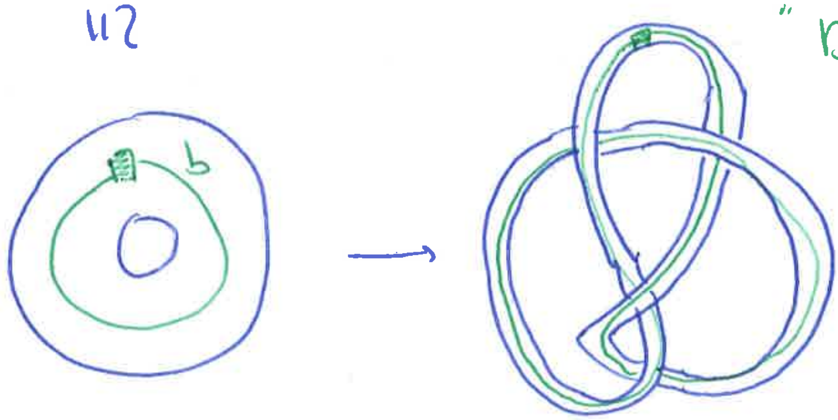
of the 2nd type.

Key Lemma:

$$\begin{cases} t^{+}(e_n) = (-1)^n A^{n^2+2n} \, e_n \\ t^{-}(e_n) = (-1)^n A^{-n^2-2n} \, e_n \end{cases},$$

- Let $D = D(K)$ and $b \in \mathcal{B}$. The <u>cabling of D by b</u> is the "skein" obtained by "putting" b on D.

$$D : \underset{\cong}{S^1 \times [0,1]} \longrightarrow \mathbb{R}^2$$



"$D(b)$" cabling of D by b.

- Denote by $\langle b \rangle_D$ the Kauffman bracket of the cabling of D by b.

$$\langle b \rangle_D = \langle \text{"}D(b)\text{"} \rangle \in \mathbb{Z}[A^{\pm 1}]$$

Def / Thm (Reshetikhin - Turaev, Wenzl)

$D = D(K)$. Then the following n-th colored Jones polynomial

$$J_n(K, A) = \left((-1)^n A^{n^2+2n} \right)^{-w(D)} \langle e_n \rangle_D$$

defines an invariant of K.

pf is similar to that of J.P + key lemma.

Rm: $J_1(K, A) = J(K, A)$, Since $e_1 = z$.

Eg: $J_n(O, A) = \langle e_n \rangle_O$

$$= e_n \Big|_{z = -A^2 - A^{-2}}$$

$$= (-1)^n \frac{A^{2n+2} - A^{-2n-2}}{A^2 - A^{-2}}.$$

Rm: Let $q = A^2$, then $J_n(O, A) = (-1)^n [n+1]$.

Quantum integer:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Normalized Colored Jones Polynomials

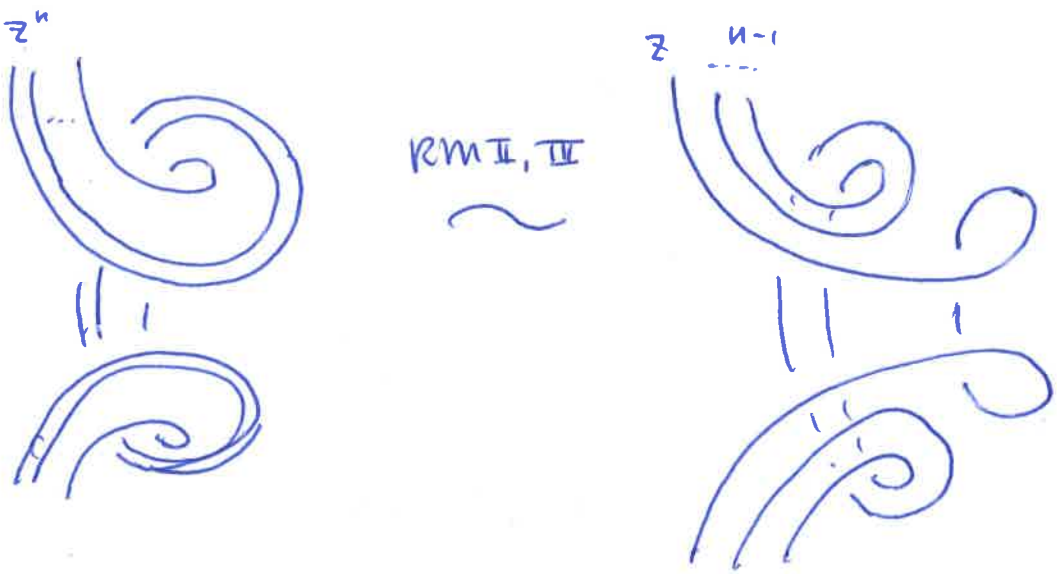
$$J_n'(K,A) = \frac{J_n(K,A)}{J_n(O,A)}$$

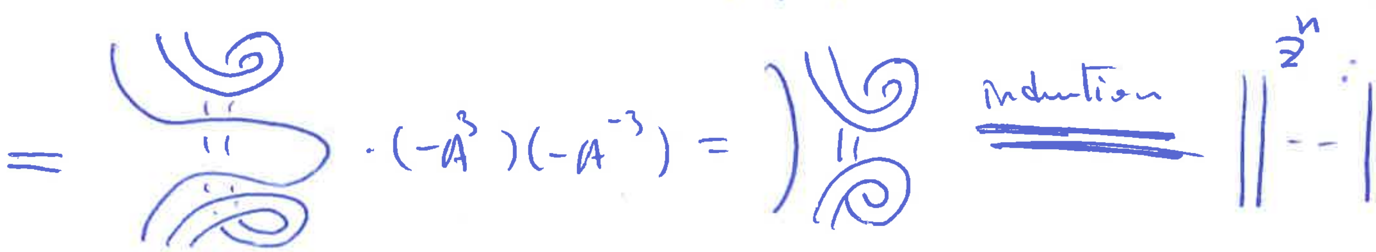
Pf of key Lemma:

Lemma 1: $\forall\, b \in \mathcal{B},\ t^+ t^-(b) = t^- t^+(b) = b$.

pf: Since $\mathcal{B} = R[z]$, it suffices to show it for $b = z^n$.

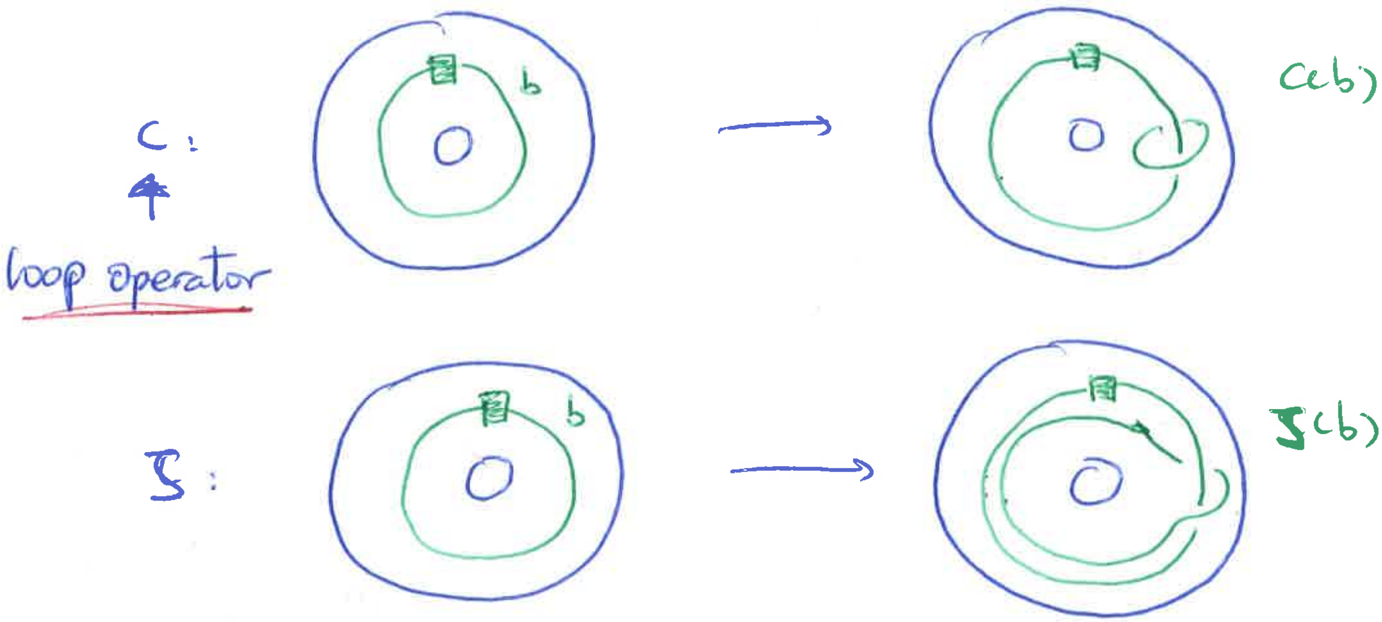
$t^+ t^-(z^n)$



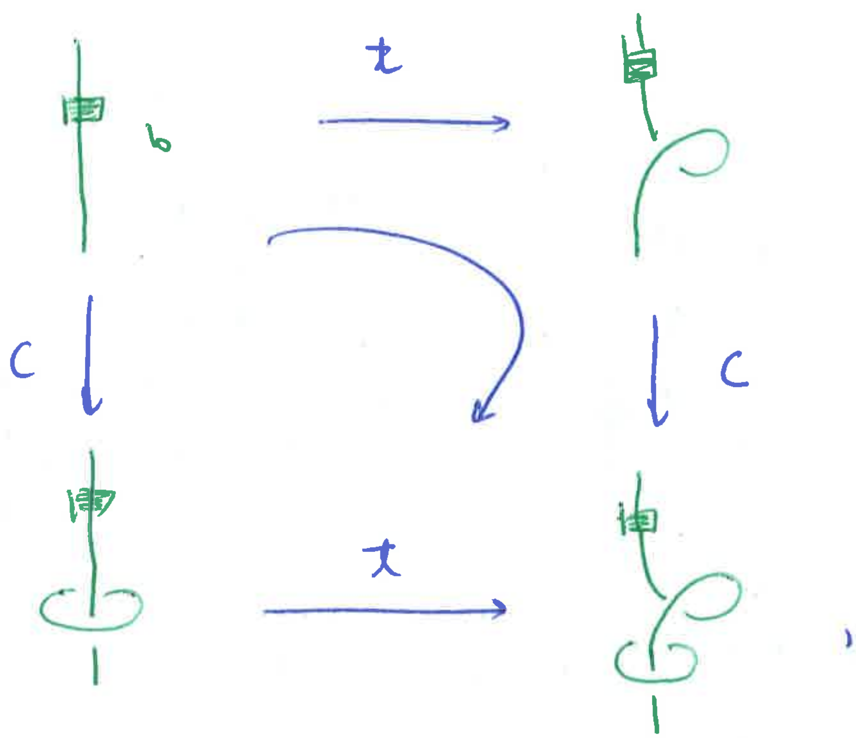


□

Consider two operators $c, \xi : \mathcal{B} \to \mathcal{B}$ by



Idea: c and $t^{\pm}$ commute



So eigenvectors of c are also eigenvectors of $t^{\pm}$, and vice versa.

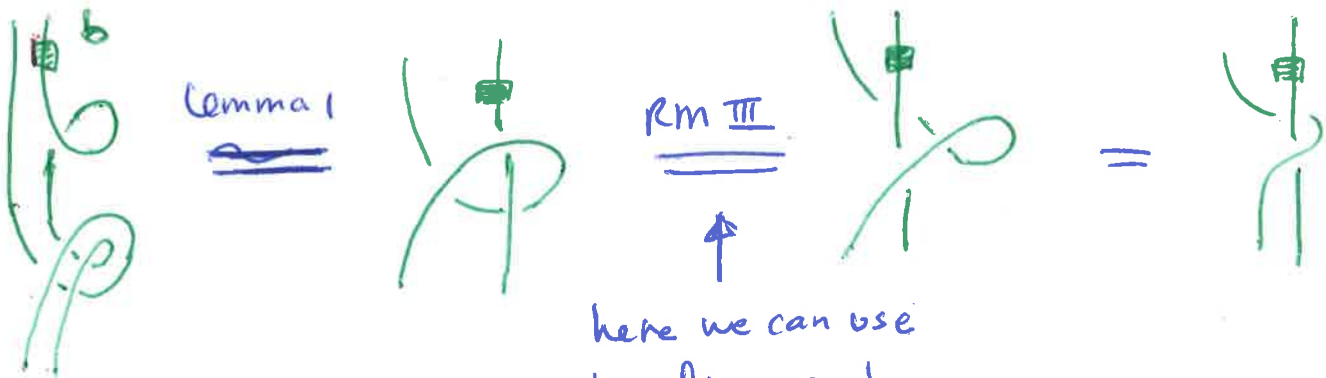
Lemma 2: $\forall b \in \mathcal{B}$

(i) $t^{+} z t^{-}(b) = -A^{3} S(b)$

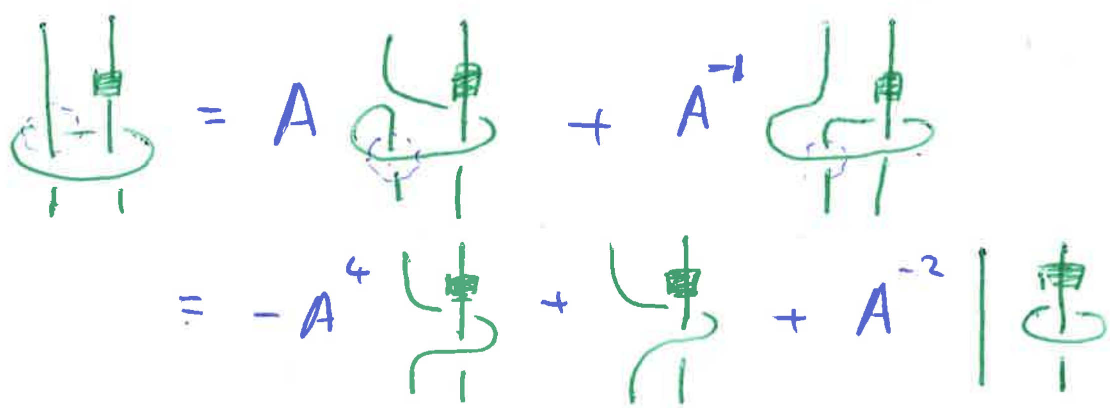
(ii) $c(zb) = A^{-2} z\, c(b) + (1 - A^{4}) S(b)$

(iii) $S(zb) = A^{2} z\, S(b) + (1 - A^{-4})\, c(b)$

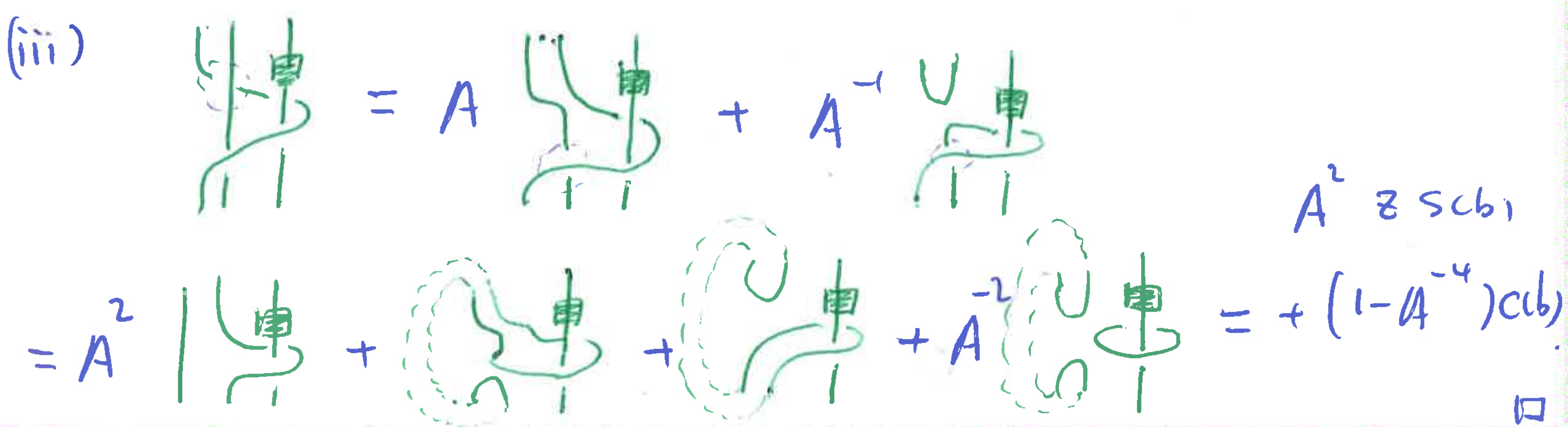
pf: (i)



(ii)



(iii)



□

Let $\lambda_n = -A^{2n+2} - A^{-2n-2}$, $\mu_n = (-1)^n A^{n^2+2n}$

Lemma 3:

(i) $s(z^n) = A^{2n} z^{n+1} + \cdots$

(ii) $c(z^n) = \lambda_n z^n + \cdots$

(iii) $t^{\pm}(z^n) = \mu_n^{\pm 1} z^n + \cdots$

pf (i):

$$s(z^n) = S(z \cdot z^{n-1}) \overset{\text{Lemma 2(iii)}}{=\!=} A^2 z\, s(z^{n-1}) + (1-A^{-4})\, c(z^{n-1})$$

$$\overset{\text{induction}}{=\!=} A^{2n} z^{n+1} + \cdots$$

(ii):

$$c(z^n) = c(z \cdot z^{n-1}) \overset{\text{Lemma 2(ii)}}{=\!=} A^{-2} z\, c(z^{n-1}) + (1-A^4)\, s(z^{n-1})$$

$$\overset{\text{induction}}{=\!=} A^{-2} \lambda_{n-1} z^n + (1-A^4) A^{2n-2} z^n + \cdots$$

$$= \lambda_n z^n + \cdots$$

(iii): Let $b = t^+(z^{n-1})$.

By induction, $b = \mu_{n-1} z^{n-1} + \cdots$ $\quad (*)$

$$t^+(z^n) = t^+ z\, t^-(b) \overset{\text{Lemma 2(i)}}{=\!=} -A^3 s(b)$$

$$\overset{(*)}{=\!=} -A^3 s(\mu_{n-1} z^{n-1} + \cdots)$$

$$\overset{\text{induction}}{=\!=} -A^3 \mu_{n-1} A^{2n-2} z^n + \cdots$$

$$= \mu_n z^n + \cdots$$

∎

Lemma 4:

Important Formulas !

(i) $c(e_n) = \lambda_n e_n$

(ii) $t^{\pm}(e_n) = \mu_n^{\pm 1} e_n$ (Key Lemma).

Pf (i): • easy to see $\lambda_{n+k} + \lambda_{n-k} = -\lambda_{k-1}\lambda_n, \ \forall k \quad (*)$.

• Lemma 2 (ii) $\Rightarrow c(z^2 b) = -\lambda_0 z\, c(zb) + (z + \lambda_1 - z^2)\, c(b) \quad (**)$

Now by induction and recurrence rel'n

$$c(e_{n+1}) = c(z e_n - e_{n-1}) = c(z(z e_{n-1} - e_{n-2}) - e_{n-1})$$

$$= c(z^2 e_{n-1}) - c(z e_{n-2}) - c(e_{n-1})$$

$$\overset{(**)}{=} -\lambda_0 z\, c(z e_{n-1}) + (z + \lambda_1 - z^2)\, c(e_{n-1}) - c(z e_{n-2}) - c(e_{n-1})$$

↓ recurrence rel'n

$$= -\lambda_0 z\, c(z e_{n-1}) + (z + \lambda_1 - z^2)\, c(e_{n-1}) - c(e_{n-1}) - c(e_{n-3}) - c(e_{n-1})$$

↓ recurrence rel'n — cancellation

$$= -\lambda_0 z\, c(e_n - e_{n-2}) + (\lambda_1 - z^2)\, c(e_{n-1}) - c(e_{n-3})$$

$$\overset{\text{induction}}{=} -\lambda_0 \lambda_n z e_n + \lambda_0 \lambda_{n-2} z e_{n-2} + \lambda_1 \lambda_{n-1} e_{n-1} - \lambda_{n-1} z^2 e_{n-1} - \lambda_{n-3} e_{n-3}$$

$$\overset{(*)}{=} \lambda_{n+1} z e_n + \lambda_{n-1} z e_n - \lambda_{n-1} z e_{n-2} - \lambda_{n-3} z e_{n-2} - \lambda_{n+2} e_{n-1}$$

$$- \lambda_{n-3} e_{n-1} - \lambda_{n-1} z^2 e_{n-1} - \lambda_{n-3} e_{n-3}$$

cancellation cancellation

$$= \lambda_{n+1}(z e_n - e_{n-1}) = \lambda_{n+1} e_{n+1}.$$

(ii). Since $c, t^{\pm 1}$ commute, e_n is an eigenvector of $t^{\pm}$.

By Lemma 3 (iii), $t^{\pm}(e_n) = \mu_n^{\pm 1} e_n$ □