# Lecture 4: Volume Conjecture

References: Thurston: The Geometry and Topology of 3-manifolds.

H. Murakami: An Introduction to the Volume Conjecture, Contemporary Math. 2011

<u>Volume Conjecture</u>: (Kashaev, Murakami-Murakami).

$K$ hyperbolic knot ($S^3 \setminus K$ admits a complete hyperbolic metric w/ finite volume). Then

$$\lim_{n\to+\infty} \frac{2\pi}{n} \ln \left| J_n'(K, e^{-\frac{\pi i}{2(n+1)}}) \right| = \mathrm{Vol}_{\mathbb{H}^3}(S^3 \setminus K)$$

Rm. In other (popular) literatures, people use $q = A^4$ and $V_n(K, q) = J_{n-1}'(K, A)$. Then

Volume Conj:

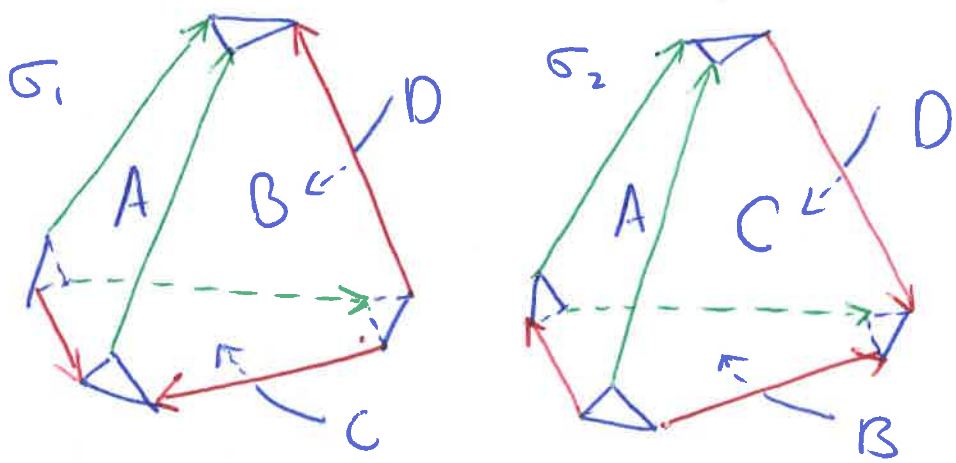
$$\boxed{\lim_{n\to+\infty} \frac{2\pi}{n} \ln \left| V_n(K, e^{\frac{2\pi i}{n}}) \right| = \mathrm{Vol}_{\mathbb{H}^3}(S^3 \setminus K).}$$

Goal Today:

① The Figure - 8 knot  is hyperbolic.

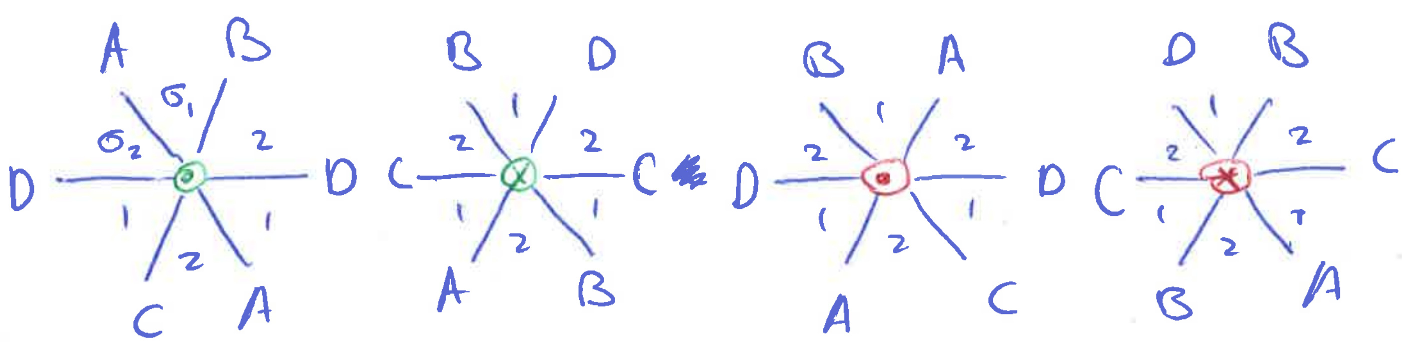
② Volume Conj is true for the Figure - 8 knot.

Thm (Thurston) The following gives an ideal triangulation of $S^3 \setminus$ .

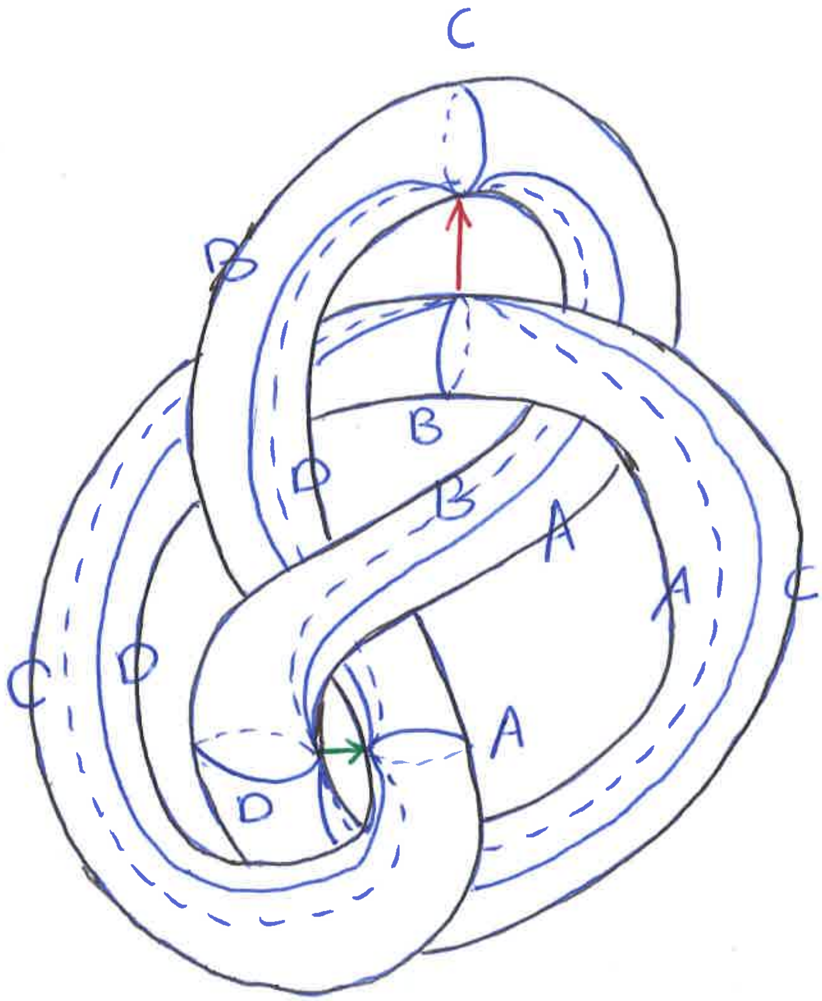



. That is, glue $\sigma_1, \sigma_2$ along pairs of faces according to color and direction of arrows.

. Combinatorics around edges.



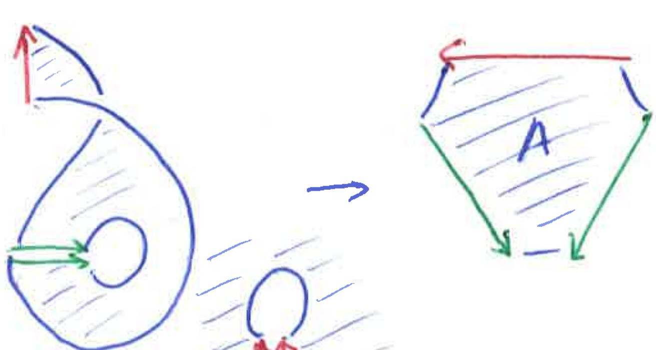
In the rest, let $K$ = Figure-8 knot. 



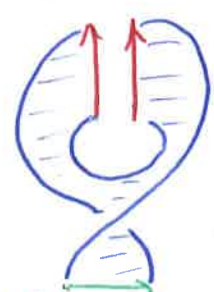

In the Figure, $\color{red}\uparrow$ $\color{green}\uparrow$ are the two edges, and the blue curves are intersections of faces w/ $\partial N(K)$

The Four faces are

A:

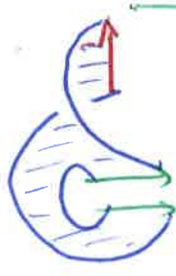



B:



C:

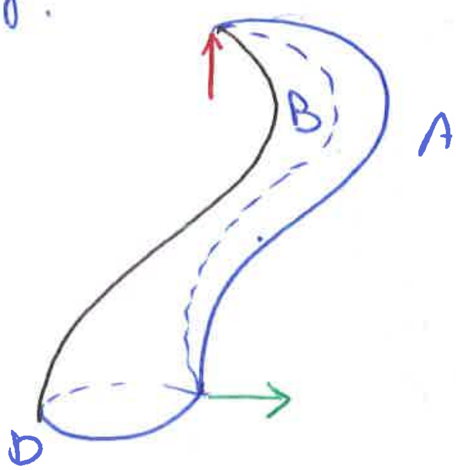
D:

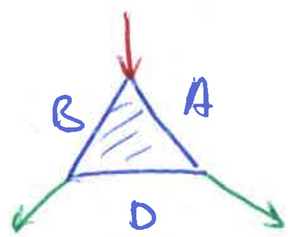


- A, B, C, D are disjoint in $S^3 \setminus N(K)$.

- Small triangles.

e.g.



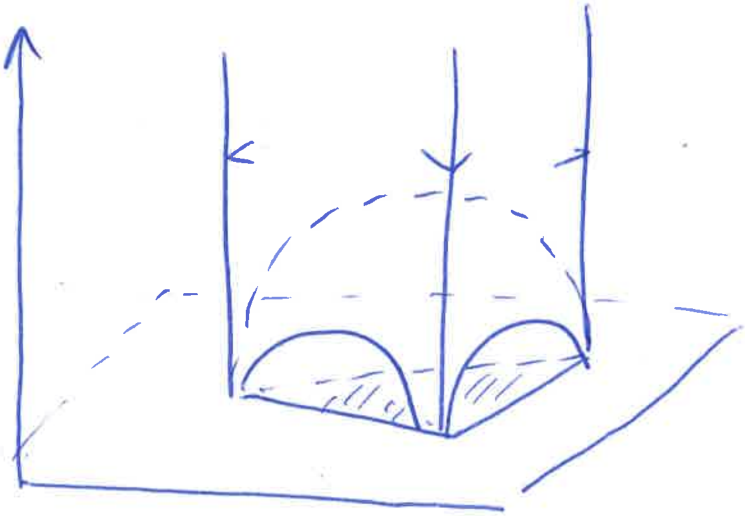

in $\sigma_2$.

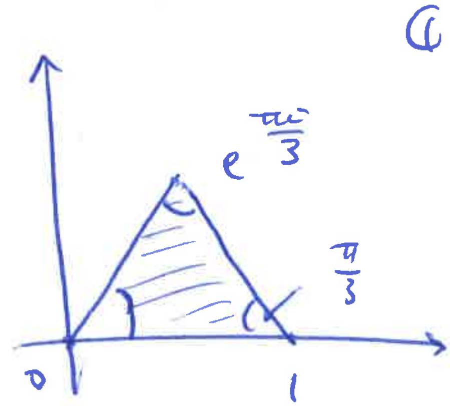
- Summary: Keep track of faces and small triangles, this gives "smooth" embedding of $\partial\sigma_1, \partial\sigma_2 (\cong S^2)$ into $S^3 \setminus N(K) \subset S^3$ preserving combinatorics.
By Schönflie Thm (every $C^\infty$ or PL embedding of $S^2$ in $S^3$ bounds a 3-ball), $S^3 \setminus N(K)$ is the union of $\sigma_1$ and $\sigma_2$.

## Hyperbolic structure:

Let $\sigma_1, \sigma_2$ be copies of regular hyperbolic ideal tetrahedra.

$\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_+$





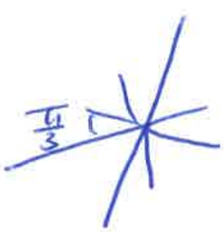

regular = all dihedral angles are $\frac{\pi}{3}$.

- glue faces of $\sigma_1\, \sigma_2$ by isometries

$\Rightarrow$ complete hyperbolic w/ finite volume.

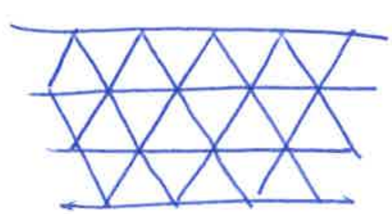
## Completeness:

around edges




$\Sigma \alpha = 2\pi$.

around the vertex: tiling of regular $\Delta$ of $\mathbb{R}^2$



Finite volume: volume of hyperbolic ideal tetrahedra w/ dihedral angles $(\alpha, \beta, \gamma)$ is

$$\mathrm{Vol}(\alpha, \beta, \gamma) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma),$$

where $\Lambda(x)$ is the <u>Lobachevsky function</u>

$$\Lambda(x) = -\int_0^x \ln|2\sin t|\, dt.$$

Functional Properties of $\Lambda(x)$

(i) $\Lambda(x)$ is odd w/ period $\pi$.

(ii) $\Lambda(2x) = 2\Lambda(x) + 2\Lambda(x + \frac{\pi}{2})$.

- Vol (regular ideal tetra) $= 3\Lambda(\frac{\pi}{3})$.

$$\boxed{\mathrm{Vol}_{\mathbb{H}^3}(S^3 \setminus K) = 6\Lambda(\tfrac{\pi}{3}).}$$

Habiro's Formula (using $q = A^{-4}$, $V_n(K,q) = J'_{n-1}(K,A)$). 7

$$V_n(K,q) = 1 + \sum_{j=1}^{n-1} \prod_{k=1}^{j} \left(q^{\frac{n-k}{2}} - q^{-\frac{n-k}{2}}\right)\left(q^{\frac{n+k}{2}} - q^{-\frac{n+k}{2}}\right)$$

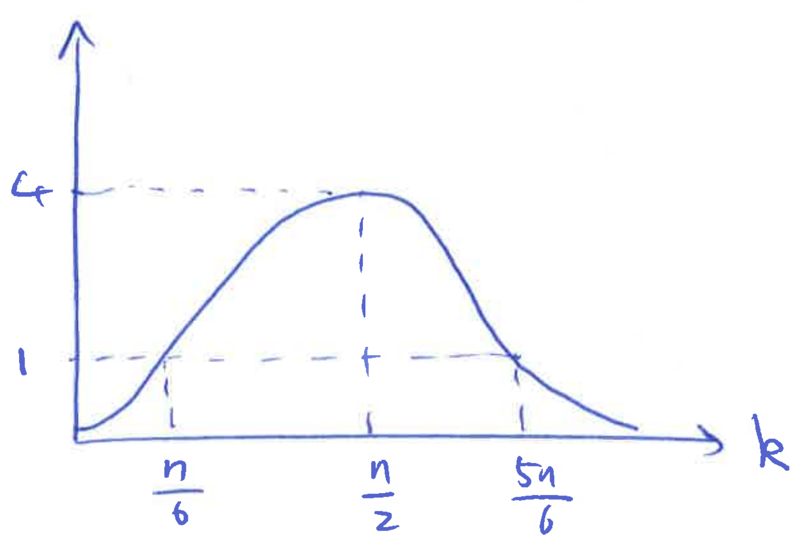
Let $q = e^{\frac{2\pi i}{n}}$. Then

$$V_n(K, e^{\frac{2\pi i}{n}}) = 1 + \sum_{j=1}^{n-1} \prod_{k=1}^{j} 4 \sin^2 \frac{k\pi}{n}.$$

Let $g_n(j) = \prod_{k=1}^{j} 4 \sin^2 \frac{k\pi}{n}$. Then

$$V_n(K, e^{\frac{2\pi i}{n}}) = 1 + \sum_{j=1}^{n-1} g_n(j).$$

Graph of $4 \sin^2 \frac{k\pi}{n}$



This implies $g_n(j)$ is decreasing if $0 < j < \frac{n}{6}$

or $\frac{5n}{6} < j$,

and is increasing if $\frac{n}{6} < j < \frac{5n}{6}$.

Thus, $g_n(j)$ achieves the maximum at $j = \lfloor \frac{5n}{6} \rfloor$.

$$\Rightarrow \quad g_n(\lfloor \tfrac{5n}{6} \rfloor) \leq V_n(K, e^{\frac{2\pi i}{n}}) \leq n \cdot g_n(\lfloor \tfrac{5n}{6} \rfloor).$$

$$\frac{1}{n} \ln g_n(\lfloor \tfrac{5n}{6} \rfloor) \leq \frac{1}{n} \ln |V_n(K, e^{\frac{2\pi i}{n}})| \leq \frac{\ln n}{n} + \frac{1}{n} \ln g_n(\lfloor \tfrac{5n}{6} \rfloor)$$

$$\Rightarrow \quad \lim_{n \to +\infty} \frac{1}{n} \ln |V_n(K, e^{\frac{2\pi i}{n}})|$$

$$= \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{\lfloor \frac{5n}{6} \rfloor} \ln \left(4 \sin^2 \frac{k\pi}{n}\right)$$

$$= 2 \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{\lfloor \frac{5n}{6} \rfloor} \ln 2 \sin \frac{k\pi}{n}$$

$$= \frac{2}{\pi} \int_0^{\frac{5\pi}{6}} \ln(2 \sin t)\, dt = -\frac{2}{\pi} \Lambda\left(\frac{5\pi}{6}\right).$$

Functional Prop of $\Lambda$ $\quad \frac{3}{\pi} \Lambda\left(\frac{\pi}{3}\right) = \frac{\mathrm{Vol}_{H^3}(S^3 \setminus K)}{2\pi}$

$\square$