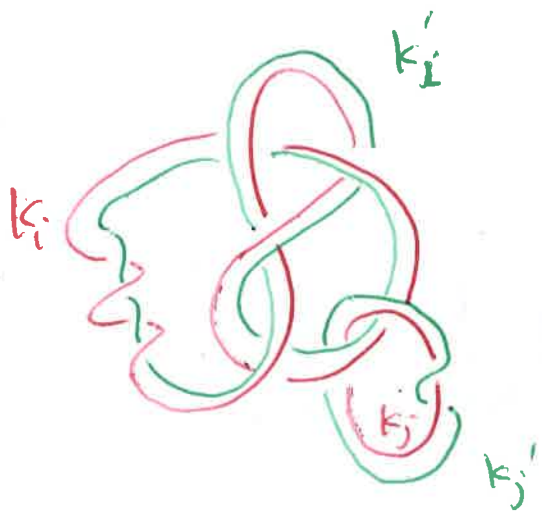
Lecture 5: Kirby Calculus (Surgery along framed links)

- <u>Framed link</u>: smooth embedding

$$L : \coprod_k (S^1 \times [0,1]) \longrightarrow S^3$$



- Let L_i be the i-th component of L.

  $K_i = L_i|_{S^1 \times \{0\}}$ is the <u>core</u> of L_i

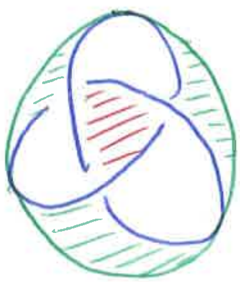
  $K_i' = L_i|_{S^1 \times \{1\}}$ is the <u>parallel copy</u> of K_i.

- Each L_i has a <u>framing</u> n_i ("number of twists").

Q: What is a 0-framing?

Def: A <u>Seifert surface</u> of a (unframed) link $L \subset S^3$ is a connected, oriented, compact surface $S \subset S^3$ w/ $\partial S = L$.

Eg:



S

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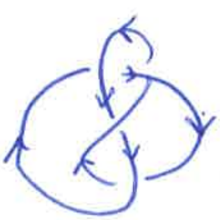
S

HW: $S \cong \Sigma_{1,1}$.

Thm (Seifert). Every L has a Seifert surface.

Seifert's algorithm:



orient L, ori pre resolution $\longrightarrow$

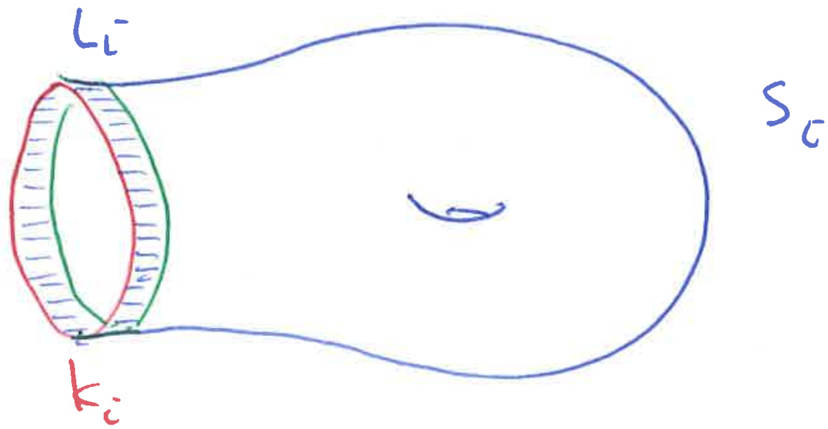
assign +/- to disks $\longrightarrow$ according to orientation

glue back $\longrightarrow$

$\longrightarrow$ connect different components by tubes, if any, to get a connected surface

③

- L_i is of <u>0-framing</u> if it is a collar nbhd of k_i in a Seifert surface S_i of k_i.



- L_i is of <u>n-framing</u> if the algebraic intersection number

$$i(k_i', S_i) = n.$$

- It is independent of choice of S_i, since all Seifert surfaces are homologous.

$$H_2(S^3 \setminus N(k_i), \partial N(k_i)) \cong \mathbb{Z} \text{ and}$$

$$[S_i] = 1 \in \mathbb{Z}.$$

Alternitive description of 0-framing:

Consider inclusion $i : \partial N(K_i) \to S^3 \setminus N(K_i)$

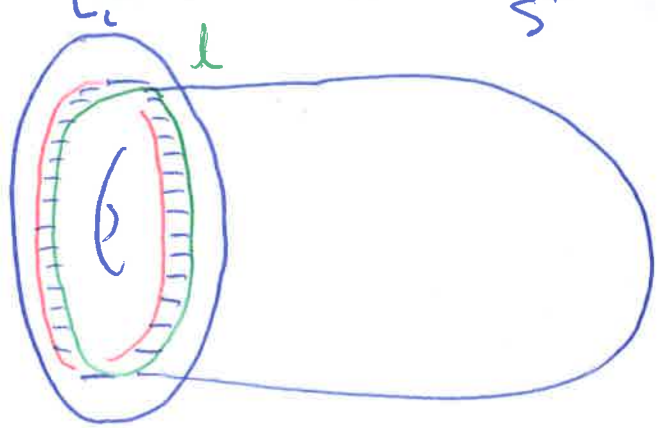
and the induced map

$$\begin{array}{ccc} i_* : H_1(\partial N(K_i)) & \to & H_1(S^3\setminus N(K_i)) \\ \wr\| & & \wr\| \\ \mathbb{Z}^2 & & \mathbb{Z}. \end{array}$$

The kernel $\ker i_* \cong \mathbb{Z}$ and is generated by

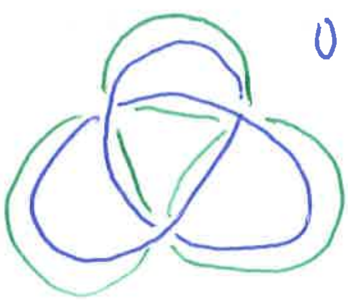
the <u>longitude</u> $l \subset N(K_i)$. l bounds a surface

$S' \subset S^3 \setminus N(K_i)$ (since $[l] \in \ker i_*$). The framed link L_i ~~determined~~ determined by K_i and l has framing 0, and

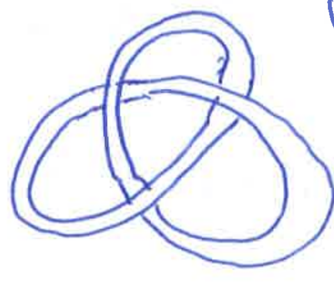


the Seifert surface $S_i = L \cup_l S'$.

Eg:



0-framing



(-3)-framing blackboard framing

· A framed link can also be represented by a link diagram w/ an ~~interger~~ integer n_i on each component K_i (<u>Kirby diagram</u>). (5)

· The <u>linking number</u> of K_i and K_j is

$$lk_{ij} \doteq i(K_i, S_j) = i(K_j, S_i).$$

· The <u>linking matrix</u> of L is the matrix

$LK(L) = (LK_{ij})$, where $LK_{ii} = n_i$ and $LK_{ij} = lk_{ij}$ if $i \neq j$.

· A framed link L (or a Kirby diagram D) determines

1) a 4-mfd $\underline{X_L}$ by attaching 2-handles to B^4 along L, and in turn

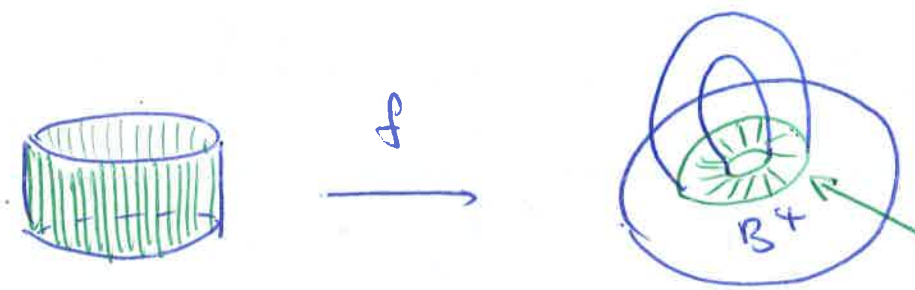
2) a 3-mfd $\underline{M_L} \doteq \partial X_L$.

Notation: $B^n = D^n$, $\forall n$.

- 4-dimensional 2-handle $= D^2 \times D^2$.

$\partial(D^2 \times D^2) = \underline{S^1 \times D^2} \cup_{S^1 \times S^1} D^2 \times S^1$

"attaching 2-handles"



$\partial B^4 = S^3$.

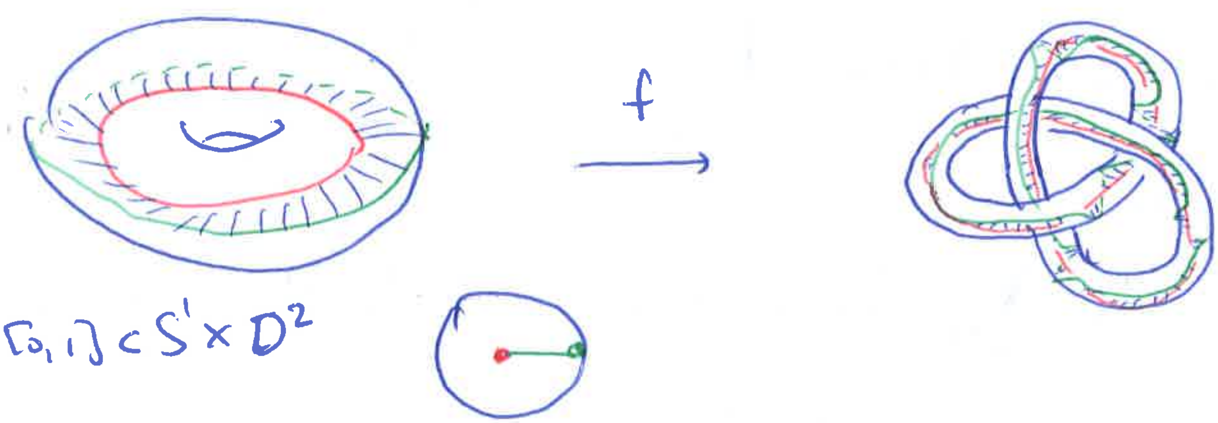
$f(S^1 \times D^2)$

attaching map $f: S^1 \times D^2 \longrightarrow S^3 = \partial B^4$

The resulting 4-mfd is $B^4 \cup_f (D^2 \times D^2)$.

- f is determined by $f|_{S^1 \times [0,1]}$ up to isotopy,

$S^1 \times D^2$



$S^1 \times [0,1] \subset S^1 \times D^2$

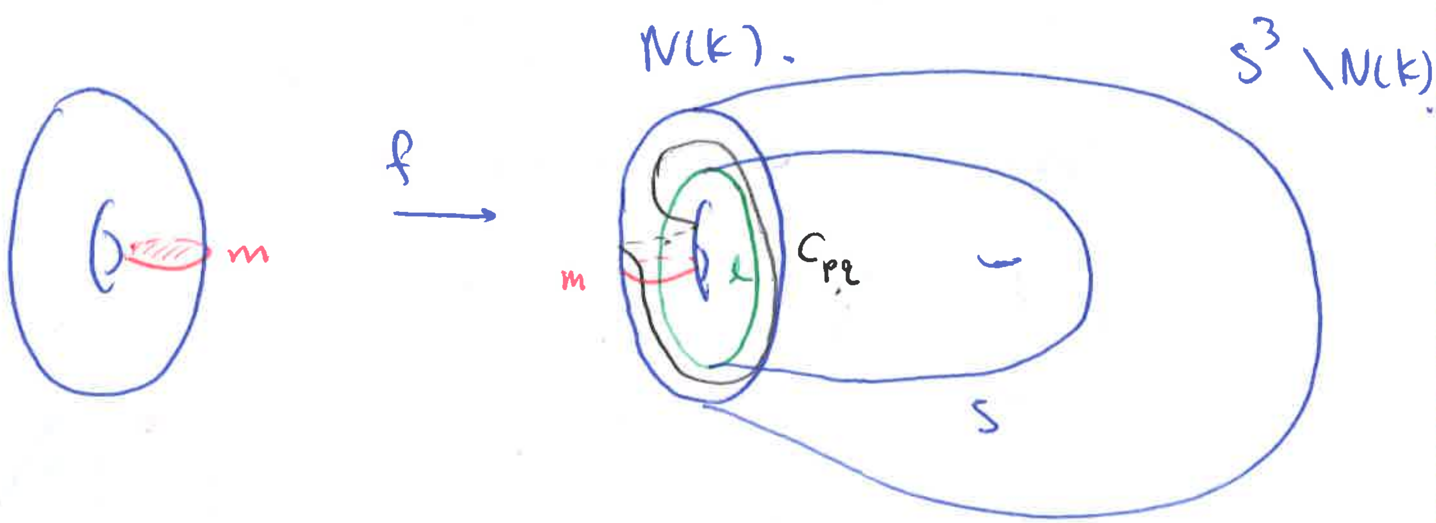
$[0,1] \subset D^2$.

and $f(S^1 \times [0,1]) \subset S^3$ is a framed link.

Rm: $M_L = \partial X_L$ is the 3-mfd obtained from S^3 by doing n_i - Dehn surgery along K_i. ⑦

Recall: p/q - Dehn surgery along K.

Let $H = D^2 \times S^1$ <u>meridian</u> $m = \partial D^2 \times \{1\} \subset \partial H$.



In $T^2 = \partial N(K)$, there are curves m, ℓ, where m bounds a disk in $N(K)$ and ℓ bounds a surface in $S^3 \setminus N(K)$.

Any $p, q \in \mathbb{Z}$ w/ $(p,q)=1$ determines $C_{pq} \subset T^2$, s.t $[C_{pq}] = p[m] + q[\ell] \in H_1(T^2)$.

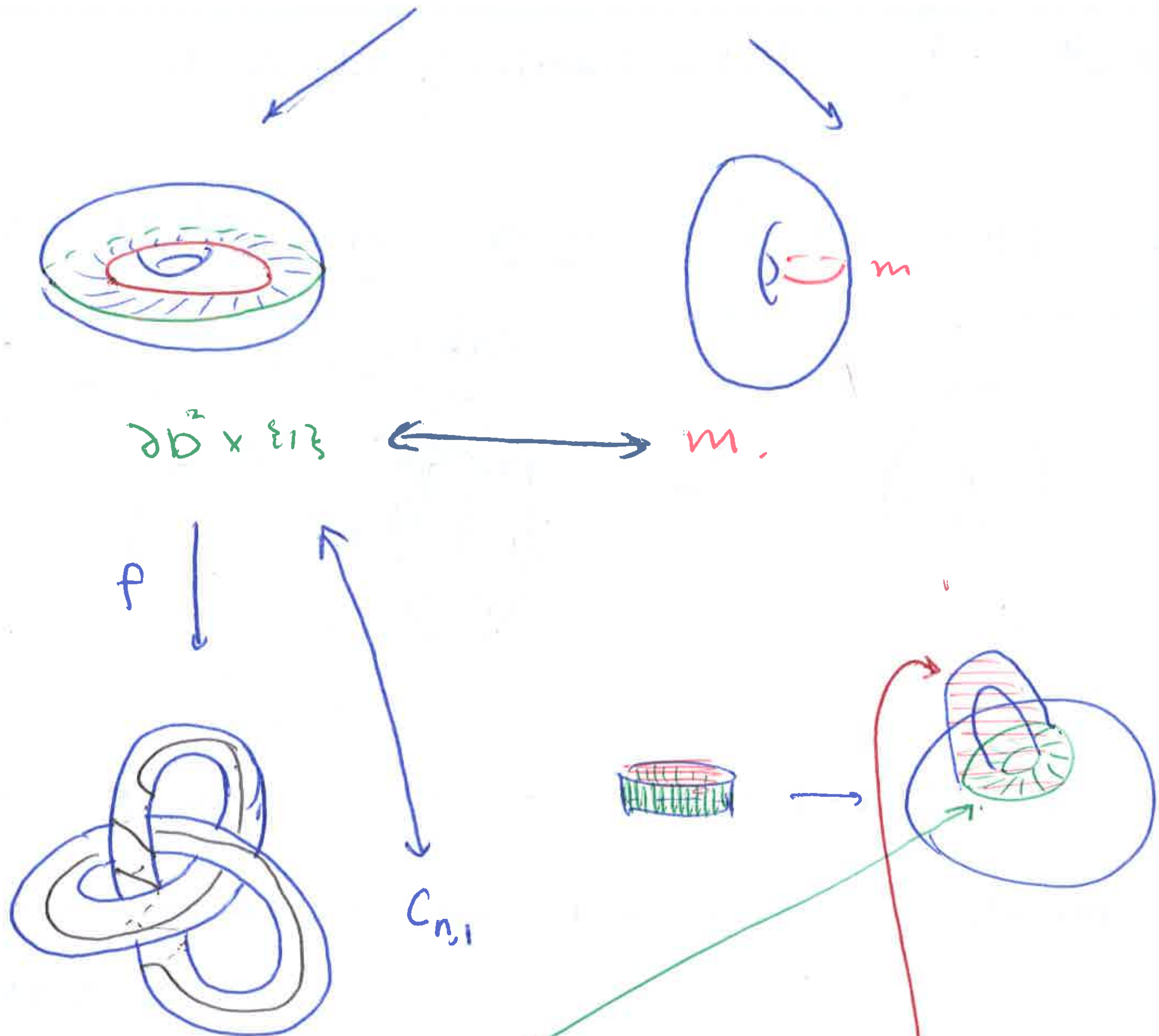
Let $f: T^2 = \partial H \longrightarrow T^2 = \partial N(K)$ determined by

$$m \longmapsto C_{pq}$$

Then $M_{K_{p/q}} \cong H \cup_f (S^3 \setminus N(K))$

Back to 2-hand $D^2 \times D^2$.

$$S^3 = \partial(D^2 \times D^2) = (S^1 \times D^2) \cup_{S^1 \times S^1} (D^2 \times S^1).$$



$$\partial X_L = \text{"replace } f(S^1 \times D^2) \text{ in } S^3 \text{ by } D^2 \times S^1\text{"}.$$

$$\| $$

$$\partial\left(B^4 \cup_f (D^2 \times D^2)\right) = \left(\partial B^4 \setminus N(K)\right) \cup_f (D^2 \times S^1).$$

$$f : m \longmapsto C_{n,1}.$$

$\Rightarrow$ M_L is obtained from S^3 by $n/1$ - Dehn surgery.

Rm.

components K_i of L $\longleftrightarrow$ 2-handles H_i $\longleftrightarrow$ basis $\{[H_i]\}$ of $H_2(X_L;\mathbb{Z})$.

linking matrix $Lk(L)$ $\longleftrightarrow$ intersection forms on $H_2(X_L;\mathbb{Z}_2)$

Thm (Lickorish, Wallace).

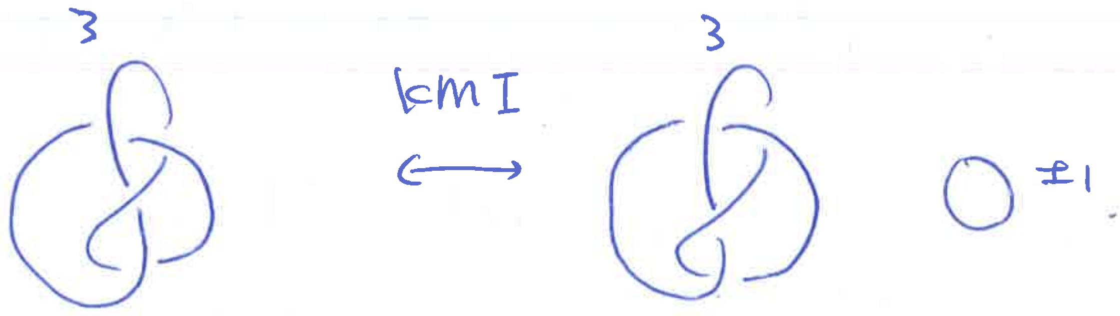
Any $M^3 \cong M_L$ for some $L \subset S^3$.

(Original statement: any M^3 can be obtained from S^3 by doing surgery along some L w/ integer coefficients).

Thm (Kirby, 1978 Invent, "A calculus for framed links in S^3")

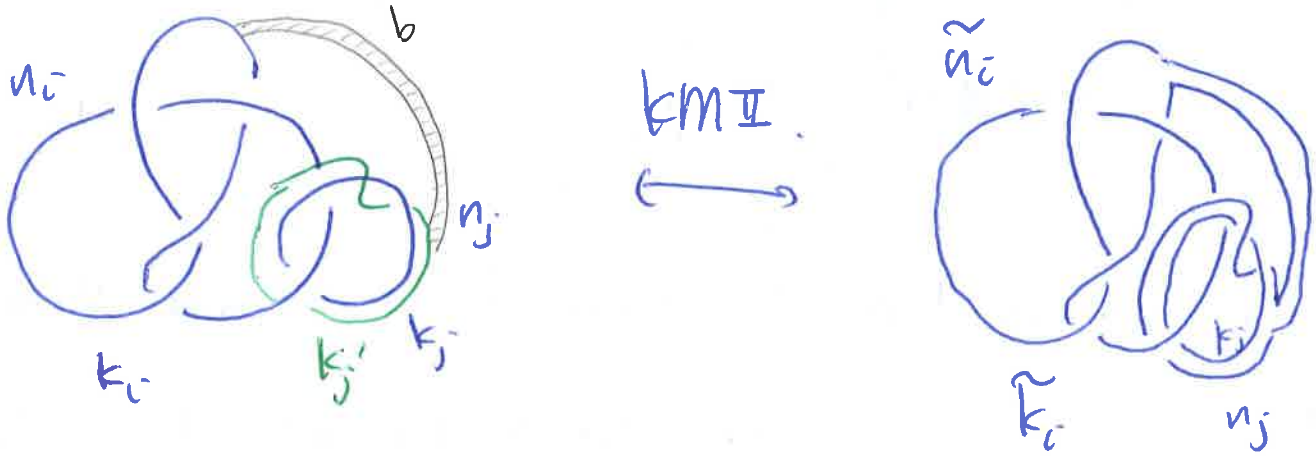
$M_{L_1} \cong M_{L_2}$ iff L_1 can be obtained from L_2 by a sequence of the following Kirby Moves KM I and KM II.

KM I (Blow up/down): Add or subtract an isolated copy of an unknot of framing ± 1.



Notation: $U_\pm \doteq \bigcirc^{\pm 1}$.

KM II (Handle slide): Replace K_i by a band sum $\tilde{K}_i = K_i \#_b K_j'$ of K_i and a parallel copy K_j' of K_j, w/ $\tilde{n}_i = n_i + n_j \pm 2 lk_{ij}$ ($\pm$ depends on b).



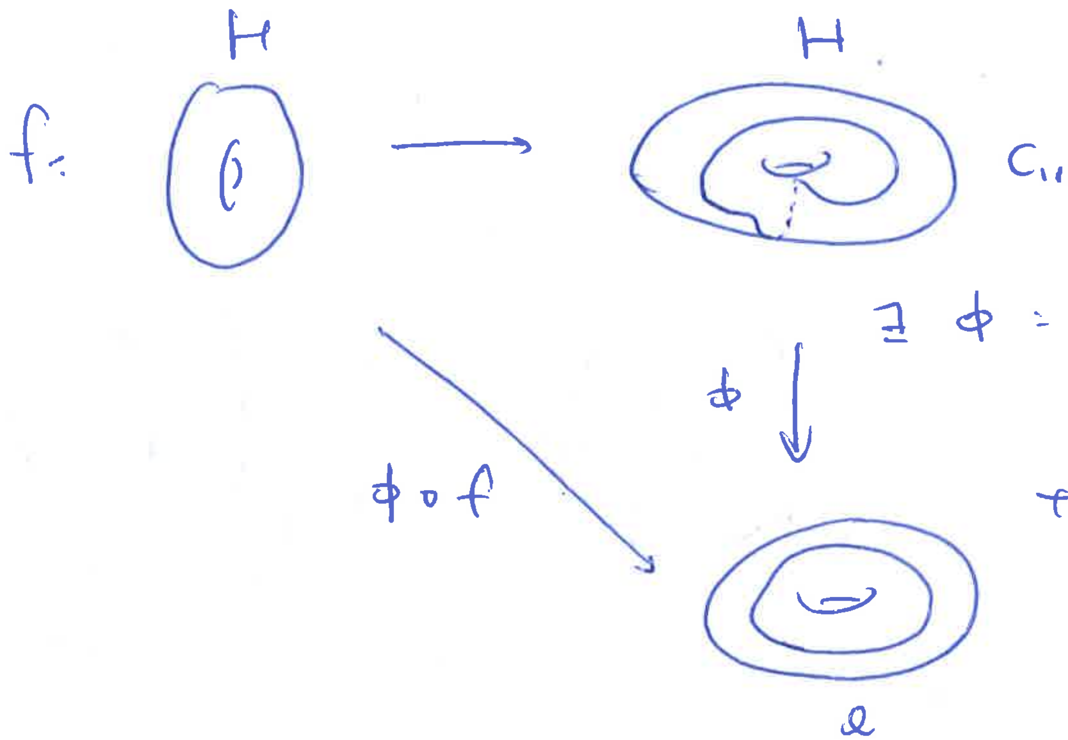
Band sum $K_0 \#_b K_1$ of K_0 and K_1.

$b: [0,1] \times [0,1] \to S^3$ embedding s.t. $b(I \times I) \cap K_i = b(\{i\} \times I)$

Then $K_0 \#_b K_1 \doteq (K_0 \cup K_1 - b(\partial I \times I)) \cup b(I \times \partial I)$.

Idea of pf: "$\Longrightarrow$"

RmI: $M_{U_\pm} \cong S^3 \Rightarrow M_{L \cup U_\pm} \cong M_L \# S^3 \cong M_L$



$\exists\ \phi : T^2 \to T^2$ homeo
$c_{11} \to \ell$
that extends to
$\phi' : H \to H$.

$$\Rightarrow M_{U_\pm} = H \cup_f H \underset{\mathrm{id} \cup \phi'}{\cong} H \cup_{\phi \circ f} H = S^3$$

Rm: Consider 4-mfd $X_{U_\pm}$.

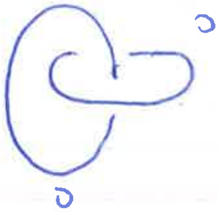
Since $\partial X_{U_\pm} \cong S^3$, can attach a 4-handle B^4!

Easy to see

0-handle $\cup_{U_\pm}$ 2-handle $\cup$ 4-handle $\cong \pm \mathbb{CP}^2$.

$\Rightarrow X_{L \cup U_\pm} \cong X_L \# (\pm \mathbb{CP}^2)$. Blow up/down.

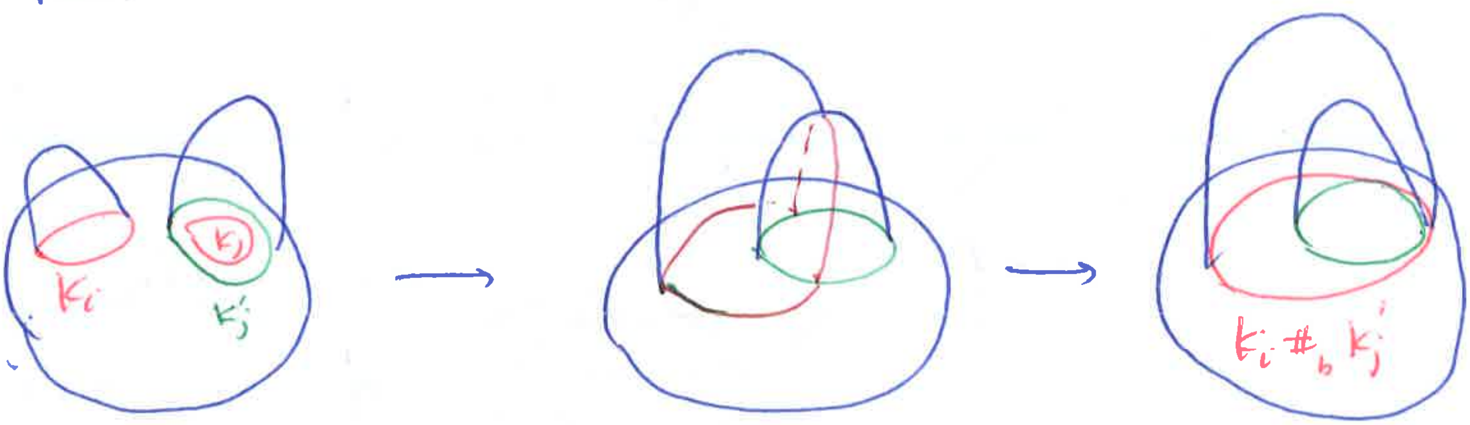
Similarly, one can see



$\longleftrightarrow S^2 \times S^2.$

$\longleftrightarrow S^2 \tilde{\times} S^2.$

KM II:



To see $\tilde{n}_j$, keep track of where K_i' goes. This can also be seen as follows.

Recall $\{K_i\} \longleftrightarrow$ basis $\{[H_i]\}$ of $H_2(X_L, \mathbb{Z})$.

$$K_i \longmapsto K_i \#_b K_j' \longleftrightarrow [H_i] \longmapsto [H_i] \pm [H_j].$$

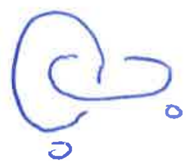
Then the intersection form, hence the linking matrix, changes as

$$\begin{pmatrix} n_i & lk_{ij} \\ lk_{ij} & n_j \end{pmatrix} \longmapsto \begin{pmatrix} \overbrace{n_i + n_j \pm 2\,lk_{ij}}^{\tilde{n}_i} & \overbrace{lk_{ij} \pm n_j}^{\tilde{lk}_{ij}} \\ lk_{ij} \pm n_j & n_j \end{pmatrix}.$$

"$\Leftarrow$". Suppose $M_{L_1} \cong M_{L_2} \cong M$.

Step ①: By possible $\#$ w/ $\pm \mathbb{CP}^2$, $S^2 \times S^2$, $S^2 \tilde{\times} S^2$, can assume $X_{L_1} \cong X_{L_2} \cong X$.

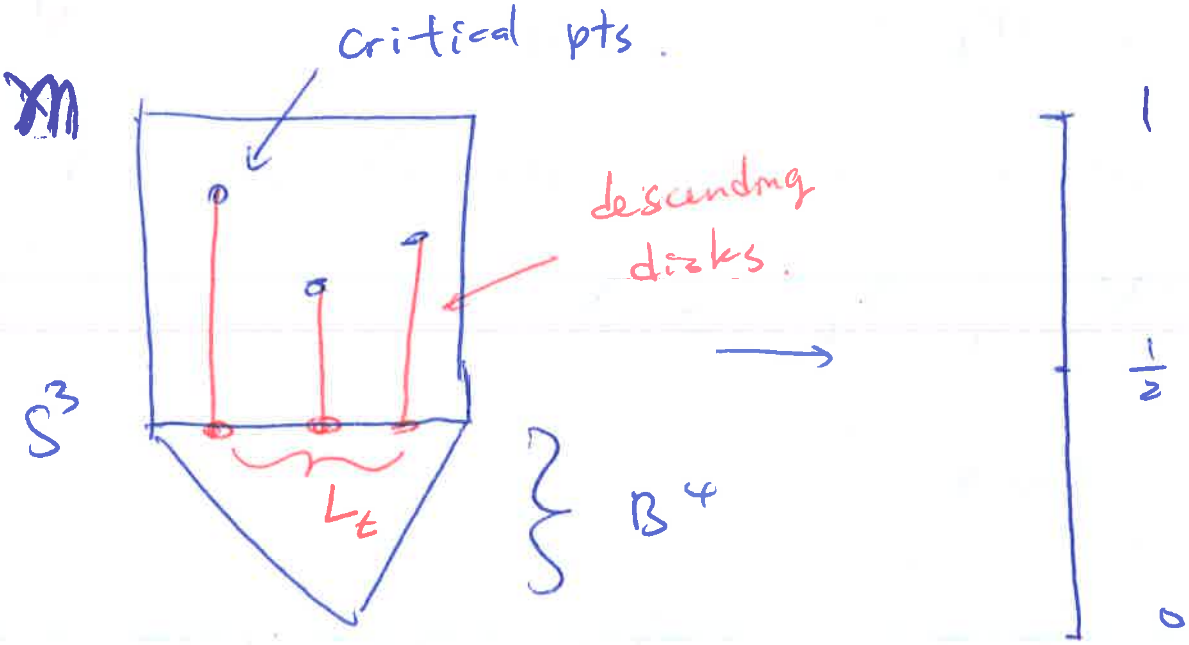
"Morse theory".

② $\bigcirc_{\pm 1}$,  ,  can be obtained from $\emptyset$ by KM I and KM II

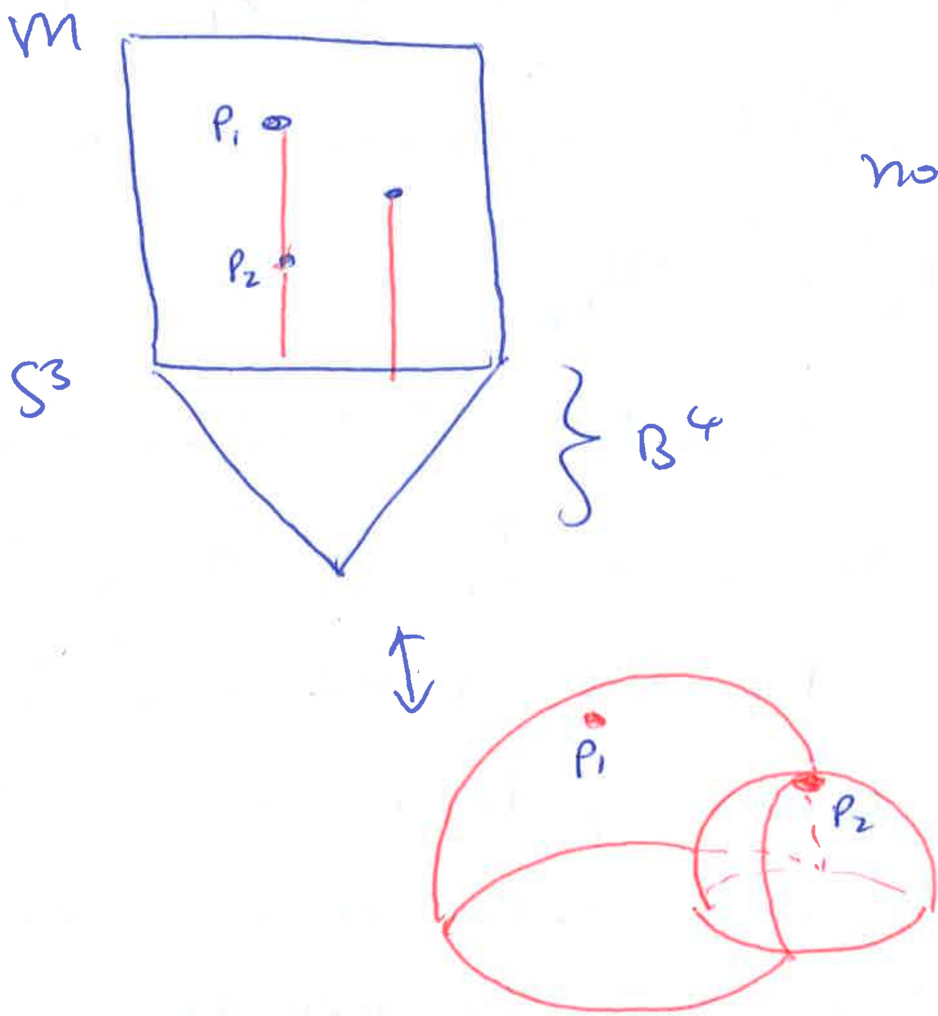
③ Consider Morse functions $f_i : X_{L_i} \to [0,1]$ w/ $f_i^{-1}(0) = \{pt\}$, $f_i^{-1}(\frac{1}{2}) \cong S^3$, no critical pts in $f_i^{-1}(0, \frac{1}{2})$, $f_i^{-1}(1) = M_{L_i}$.

"Cerf theory" $\Rightarrow$ $\exists$ homotopy $f : X \times [0,1] \to [0,1]$ from f_1 to f_2 s.t. each f_t is Morse.

For generic t, descending disks intersect S^3, gives isotopy between L_1 and L_2; for non-generic t, descending disk intersects critical pt of smaller value, that's where handle slid happens.



generic t.



non-generic.