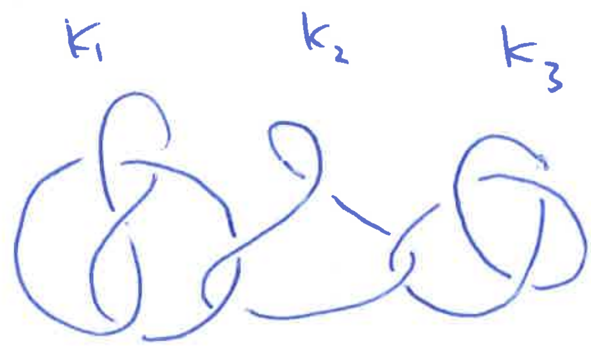
Lecture 7: Reshetikhin–Turaev Invariants.

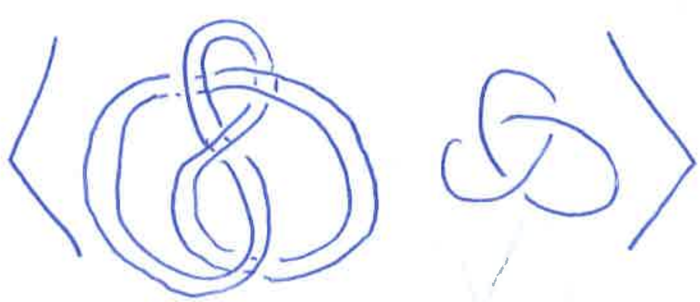
- $L \subset S^3$ link w/ n (ordered) components, $D = D(L)$. Let $b_1, \dots, b_n \in \mathcal{B}_r$. Denote by

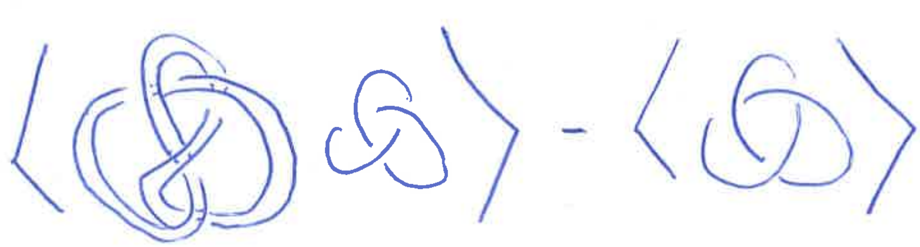
$$\langle b_1, \dots, b_n \rangle_D$$

the Kauffman bracket of the cabling of D by $b_1, \dots, b_n$ (put b_i on the i-th component).

Eg:

$D =$ 

$\langle z^2, 1, z \rangle_D =$ 

$\langle e_2, 1, z \rangle =$ 

- Let L be a framed link. $D = D(L)$ is called a standard diagram of L if the blackboard framing of D coincides w/ the framing of L.

Eg.

L	Standard D.
$U_+ =$  $+1$	
$U_- =$  -1	
 -3	
 -4	

Def / Thm (Reshetikhin–Turaev, Lickorish)

Suppose $M = M_L$, $D = D(L)$ is a standard diagram of L. Let b_+, b_- resp. be the number of positive, negative eigenvalues of the linking matrix $Lk(L)$. Then $\forall r \geq 3$, the following

$$\boxed{RT_r(M) \doteq \langle \omega_r, \cdots, \omega_r \rangle_D \langle \omega_r \rangle_{U_+}^{-b_+} \langle \omega_r \rangle_{U_-}^{-b_-}}$$

defines a complex valued invariant of M, ie, is invariant under KM I and KM II.

Moreover, $\forall r \geq 3$,

(1). $RT_r(M \# N) = RT_r(M) \cdot RT_r(N)$,

(2) $RT_r(-M) = \overline{RT_r(M)}$,

(3) $RT_r(S^3) = 1$.

• Recall $\omega_r = \sum_{n=0}^{r-2} \langle e_n \rangle e_n \in \mathcal{B}_r$.

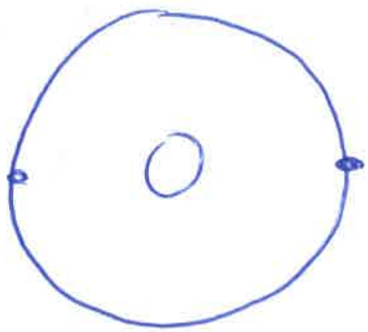
Pf of (1) (2) (3).

(1). If $\mathcal{M} = \mathcal{M}_L$, $N = \mathcal{M}_{L'}$, then $\mathcal{M} \# N = \mathcal{M}_{L \# L'}$.

(2) If $\mathcal{M} = \mathcal{M}_L$, then $-\mathcal{M} = \mathcal{M}_{\bar{L}}$, where $\bar{L}$ is the mirror image of L.

(3) ~~#~~ $S^3 = \mathcal{M}_\phi$.

To prove the well-definition, we need to consider the Kauffman bracket ~~of~~ skein module of.

$(X, 2) =$  ,

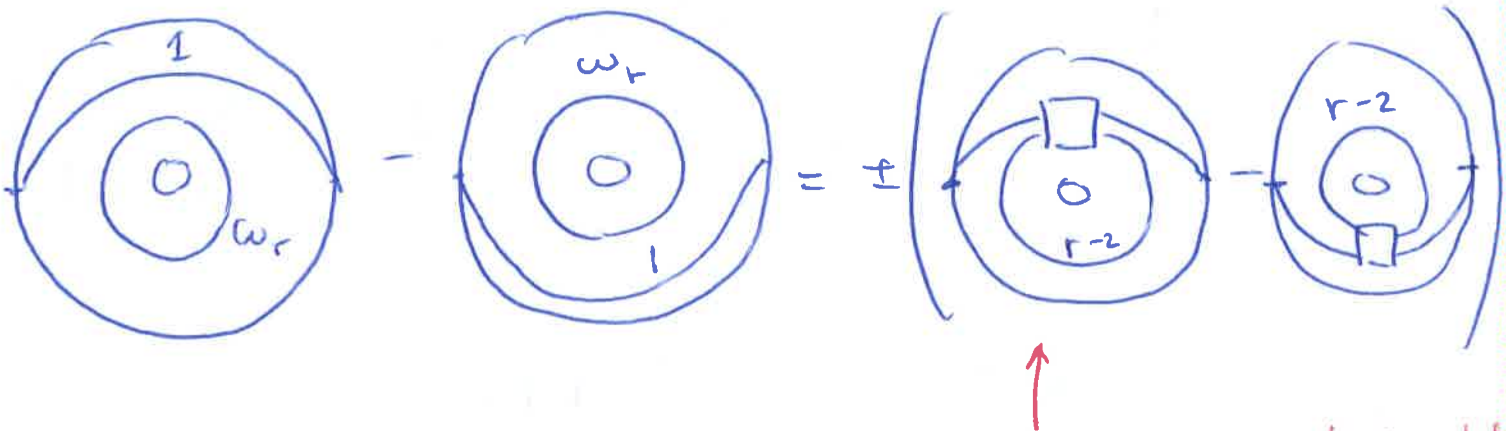
which is the $\mathbb{Z}[A_r^{\pm 1}]$-module generated by (isotopy classes) diagrams in $(X, 2)$ modulo

① Kauffman bracket skein rel'n

$$\times = A^{A})(+ A^{-1} \asymp$$

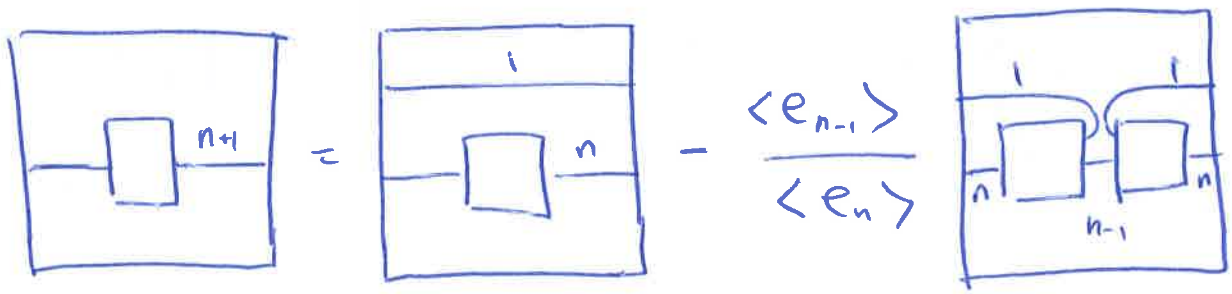
② Framing rel'n. $\bigcirc \sqcup D = (-A^2 - A^{-2}) D$.

key lemma 1:

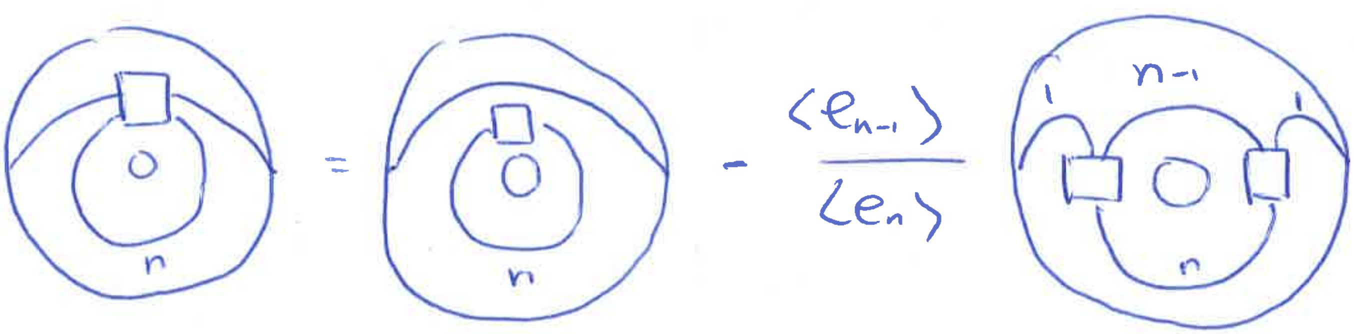


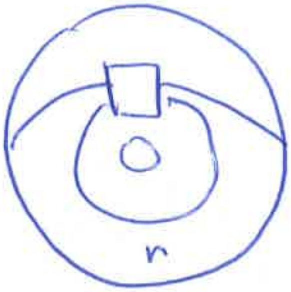
$(r-1)$-th Jones-Wenzl idempotent w/ first $(r-2)$-pts closed up

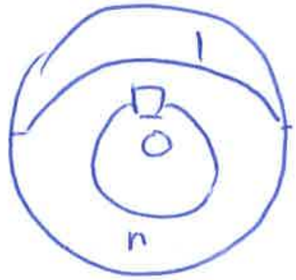
Pf: Recall the defining equation of f_{n+1}

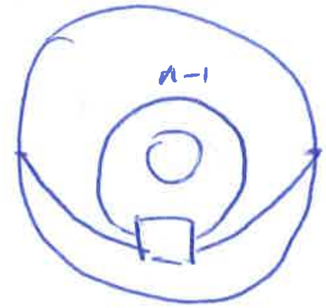


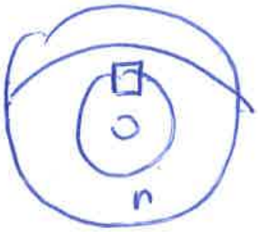
Closing up the first n-pts, we have

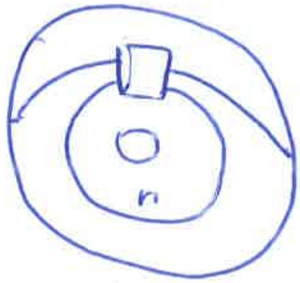


$f_n \cdot f_n = f_n$ $\Rightarrow$ 

$=$ 

$- \dfrac{\langle e_{n-1}\rangle}{\langle e_n\rangle}$ 

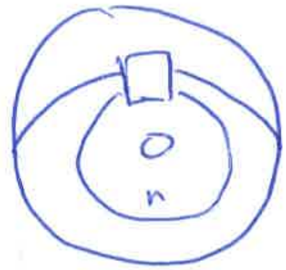
$\Rightarrow \langle e_n\rangle$ 

$= \langle e_n\rangle$ 

$+ \langle e_{n-1}\rangle$ 

Taking sum, $n = 0, \dots, r-2$, $\Rightarrow$

(I) 

$= \displaystyle\sum_{n=0}^{r-2} \Big(\langle e_n\rangle$ 

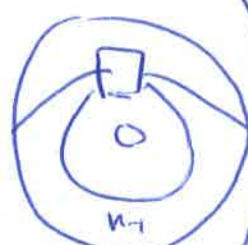
$+ \langle e_{n-1}\rangle$ 

$\Big)$

Rotating (I) by 180° $\Rightarrow$

(II) 

$= \displaystyle\sum_{n=0}^{r-2} \Big(\langle e_n\rangle$ 

$+ \langle e_{n-1}\rangle$ 

$\Big)$

(I) $-$ (II) $= \langle e_{r-2}\rangle \Big($ 

$-$ 

$\Big)$

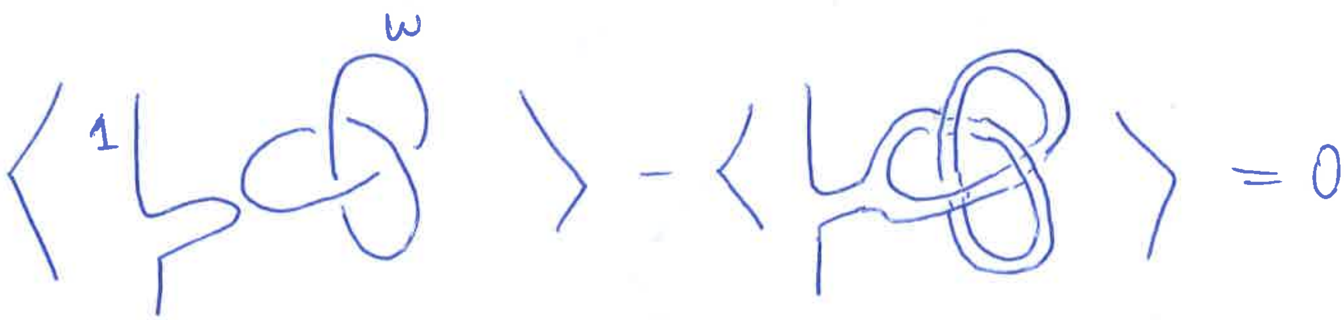
$$\langle e_{r-2}\rangle = (-1)^{r-2}\,\frac{A^{2(r-2)+2} - A^{-2(r-2)-2}}{A^2 - A^{-2}} = (-1)^{r-2} \qquad \square$$

Cor 1: If D and D' are two diagrams differed by a handle slid of some component over the component K_1 (w/ blackboard framing), then

$$\langle \omega_r, -, \dots, - \rangle_D = \langle \omega_r, -, \dots, - \rangle_{D'}$$

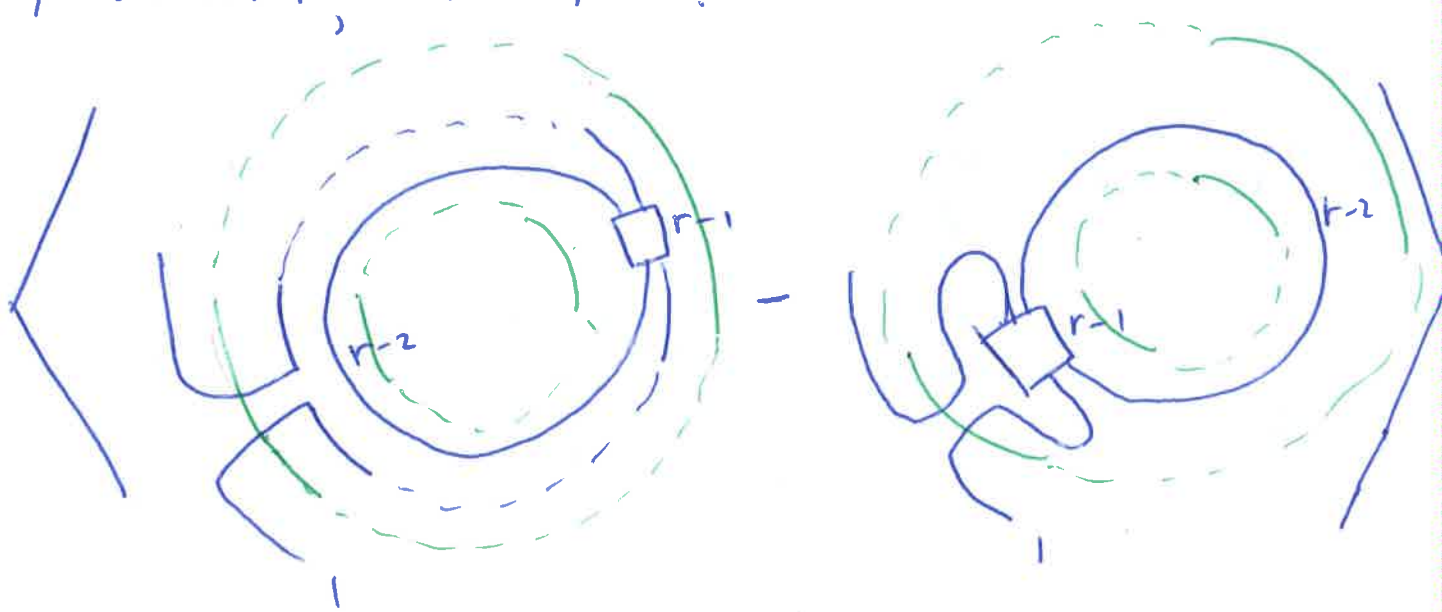
as multilinear forms on $\mathcal{B}_r$.

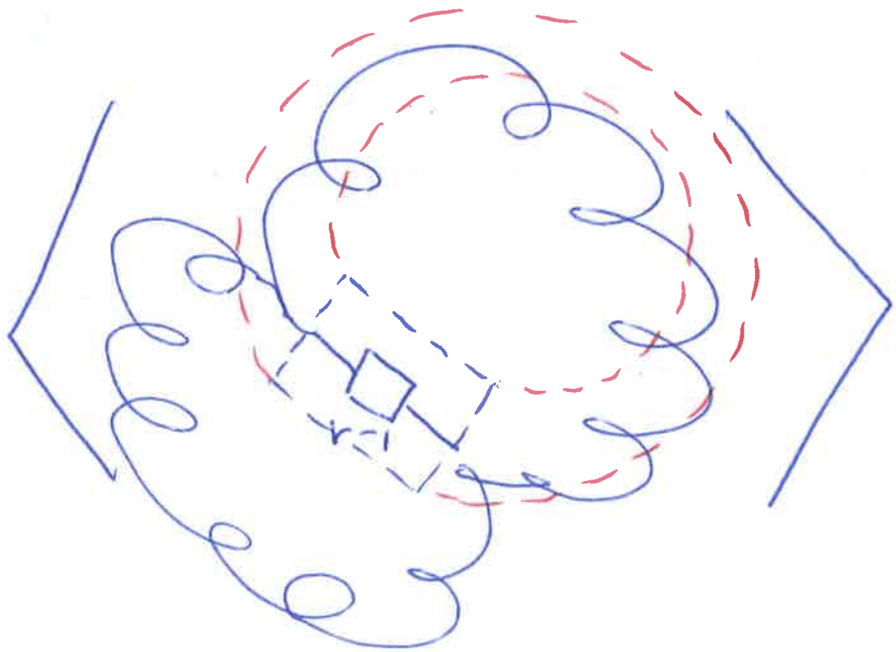
Pf: It suffices to prove



By Key Lemma 1, it equals

$\pm$

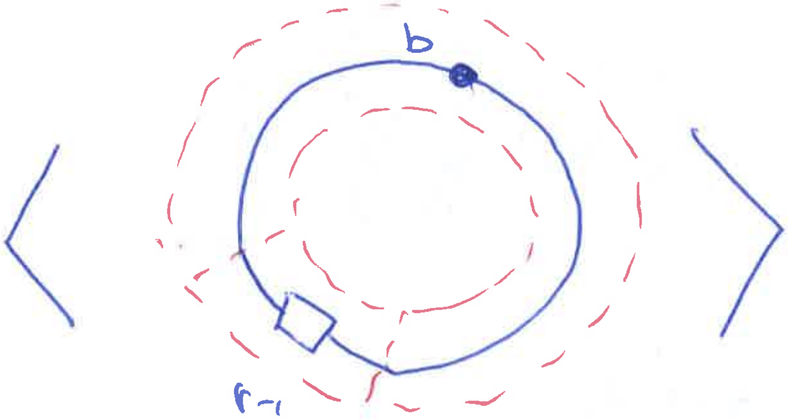


$=$ 

push everthing into the annulus by RM II and RM III

RM II 

RM III 

$=$ 

$$= c\,\langle e_{r-1} \rangle$$

$$= c\cdot(-1)^{r-1}\,\frac{A^{2(r-1)+2} - A^{-2(r-1)-2}}{A^2 - A^{-2}} = 0$$

□

Cor 2: $RT_r(M)$ is invariant under KM I.

Pf: If D, D' differs by a handle slide, then by Cor 1, $\langle w_r, \cdots, w_r \rangle_D = \langle w_r, \cdots, w_r \rangle_{D'}$.

The linking matrix $Lk(D')$ is obtained from $Lk(D)$ by adding the j-th row to the i-th, then adding the new j-th column to the i-th, which has the same b_+ and b_- as $Lk(D)$. □

To prove the invariance under KM I, we need

Key Lemma 2:

(1) $\langle w_r \rangle = \dfrac{-2r}{(A^2 - A^{-2})^2} \neq 0$,

(2) $\langle w_r \rangle_{u_+} \langle w_r \rangle_{u_-} = \langle w_r \rangle$.

Pf :

(1) $$\langle \omega_r \rangle = \left\langle \sum_{n=0}^{r-2} \langle e_n \rangle e_n \right\rangle = \sum_{n=0}^{r-2} \langle e_n \rangle^2$$

$$= \sum_{n=0}^{r-2} (-1)^{2n} \left(\frac{A^{2n+2} - A^{-2n-2}}{A^2 - A^{-2}} \right)^2$$

$$= \frac{1}{(A^2 - A^{-2})^2} \left(-2r + \left(\sum_{n=0}^{r-2} A^{4n+4} + 1 \right) + \left(\sum_{n=0}^{r-2} A^{-4n-4} + 1 \right) \right)$$

$$= \frac{1}{(A^2 - A^{-2})^2} \left(-2r + \underbrace{\frac{A^{4r} - 1}{A^4 - 1}}_{= 0} + \underbrace{\frac{A^{-4r} - 1}{A^4 - 1}}_{= 0} \right)$$

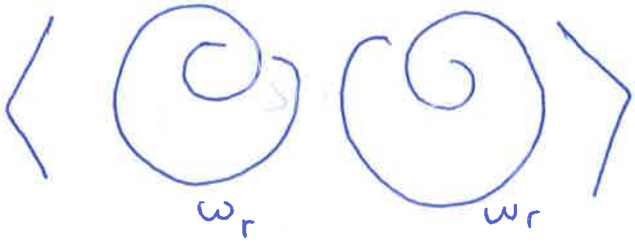
$$= \frac{-2r}{(A^2 - A^{-2})^2}$$ since $A^{4r} = 1$.

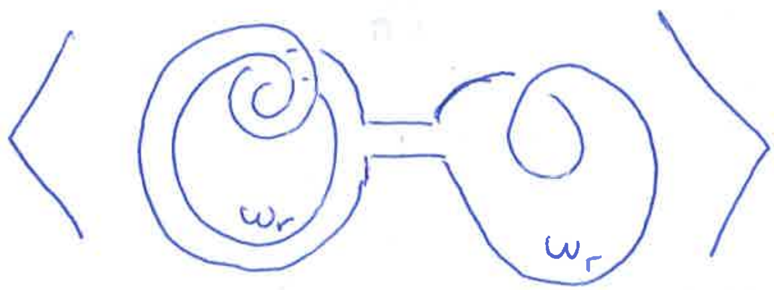
For (2), we recall that

(*)  $= (-1)^n A^{n^2 + 2n}$ 

(**)  $= \left(-A^{2n+2} - A^{-2n-2}\right)$ 

Then $\langle \omega_r \rangle_{u_+} \langle \omega_r \rangle_{u_-}$

$=$ 

Cor 1 

$\tilde{n}_i = n_i + n_j \pm 2 lk_{ij}$

$\tilde{lk}_{ij} = lk_{ij} \pm n_j$

$=$ 

$\underset{\star}{=} \sum_{n=0}^{r-2} \langle e_n \rangle$ 

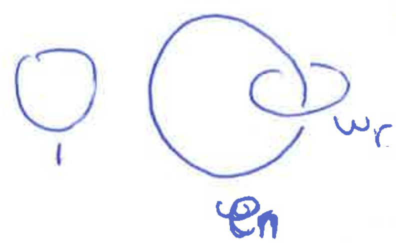
$\overset{(*)}{=} \sum_{n=1}^{r-2} (-1)^n A^{n^2+2n} \langle e_n \rangle$ 

by the following lemma

$= \langle \omega_r \rangle$ $\square$

Lemma:

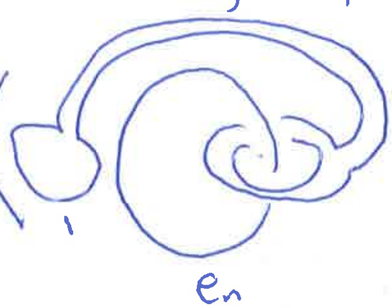
$\langle$  $\rangle = \begin{cases} \langle w_r \rangle & n = 0 \\ 0, & 0 < n \leq r-2. \end{cases}$

Pf. Consider $Q = \langle$  $\rangle$.

On the one hand, by Framing rel'n,

$Q = (-A^2 - A^{-2})\,\langle$  $\rangle$.

On the other hand, by Cor 1,

$Q = \langle$  $\rangle = \langle$  $\rangle$

$\overset{(**)}{=} (-A^{2n+2} - A^{-2n-2})\,\langle$  $\rangle$.

For $0 < n \leq r-2$, since $-A^2 - A^{-2} \neq -A^{2n+2} - A^{-2n-2}$,

$\langle$  $\rangle = 0$. For $n = 0$, $e_0 = 1$. $\square$

Cor 3: $RT_r(M)$ is well-defined and is invariant under KMI.

pf: By Key Lemma (1), (2), $\langle \omega_r \rangle_{u_+}$, $\langle \omega_r \rangle_{u_-} \neq 0$, hence $RT_r(M)$ is well-defined.

For KMI, if D and D' are differed by a KMI, say, $D' = D \amalg u_+$, then $b'_+ = b_+ + 1$, $b'_- = b_-$, and $\langle \omega_r, \ldots, \omega_r \rangle_{D'} = \langle \omega_r, \ldots, \omega_r \rangle_D \langle \omega_r \rangle_{u_+}$. $\square$

Rm: There is a theory for

$$A = e^{\frac{\pi i}{r}}, \quad r \text{ odd},$$

and A any primitive $2r$-th root of 1, due to BHMV and also Lickorish.

Normalization of RT-invariants.

Let $\mu = \dfrac{A^2 - A^{-2}}{\sqrt{-2r}}$, i.e., $\mu^{-2} = \langle \omega_r \rangle = \langle \omega_r \rangle_{u_+} \langle \omega_r \rangle_{u_-}$.

$\Rightarrow \quad \langle \mu \omega_r \rangle_{u_+} = \langle \mu \omega_r \rangle_{u_-}^{-1}$ and

$\langle \mu \omega_r \rangle = \mu^{-1}$.

Let $\boxed{\Omega_r = \mu \omega_r}$

Def \ Thm: $\forall r \geq 3$

$$\boxed{I_r(M) = \mu \langle \Omega_r, \cdots, \Omega_r \rangle_D \langle \Omega_r \rangle_{u_+}^{-\sigma}}$$

defines an invariant of M, where $M = M_L$, $D = D(L)$ standard, $\sigma = b_+ - b_-$.

Eg. $I_r(S^3) = \mu$.

$I_r(S^2 \times S^1) = 1$.

$I_r(\Sigma_g \times S^1) = (-2r)^{g-1} \sum_{n=0}^{r-2} \left(A^{2n+2} - A^{-2n-2}\right)^{2-2g} \in \mathbb{N}$.

Fact: it's the dimension of the RT-TQFT vector space