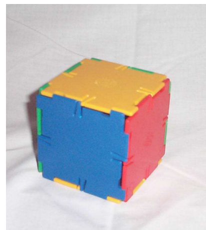
Summer Educational Enrichment in Math, 2022
Math Contest - Solutions

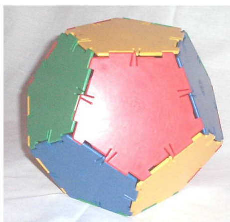
1. **Platonic Solids**: Name the 5 Platonic Solids and say how many faces they have. (Spelling does not matter.)

Name Tetrahedron
Faces 4

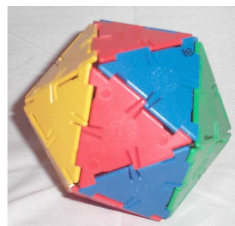


Name Hexahedron or Cube
Faces 6

Name Octahedron
Faces 8

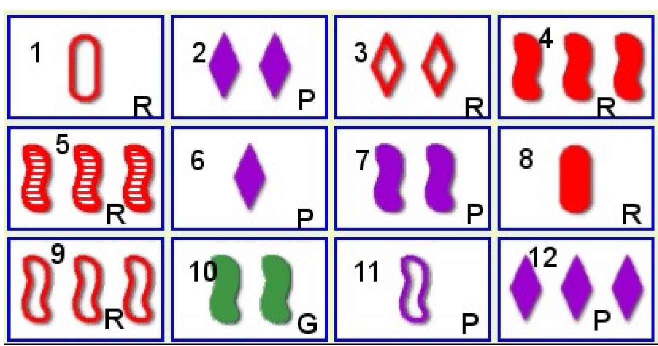


Name Dodecahedron
Faces 12



Name Icosahedron
Faces 20

2. **Sets**: Find as many sets as possible from the cards shown below. Recall: a set consists of 3 cards whose characteristics (color, shape, number, shading) are all the same or all different. Use the numbers in each square to form your sets (e.g. 1, 2, 3). The letters refer to the color of the card (G=green, P=purple, and R=red). There are **fewer** than 8 sets!



Set 1: 1, 3, 9	Set 2: 2, 6, 12	Set 3: 3, 5, 8	Set 4: 4, 5, 9
Set 5: 5, 10, 11	Set 6: 8, 10, 12	Set 7:	Set 8:

3. **Euler numbers**: Consider the octahexahedron made from 6 squares and 8 triangles:

The number of faces is: $\boxed{F = 14}$

The number of vertices is: $\boxed{V = 12}$ Explain below.

The number of edges is: $\boxed{E = 24}$ Explain below.

Explain V: **Solution**: The 6 squares have 4 vertices and the 8 triangles have 3 vertices for a total of $6 \times 4 + 8 \times 3 = 48$ vertices, counting each vertex for each face, but each vertex belongs to 4 faces. So we divide to get $48/4 = 12$ vertices.

Explain E: **Solution**: The 6 squares have 4 edges and the 8 triangles have 3 edges for a total of $6 \times 4 + 8 \times 3 = 48$ edges, counting each edge for each face, but each edge belongs to 2 faces. So we divide to get $48/2 = 24$ edges.

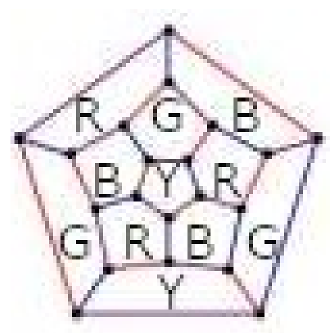
Calculate the Euler number: **Solution**: $F + V - E = 14 + 12 - 24 = 2$

Explain how you know the Euler number before counting F, V and E?

Solution: There are no holes.

4. **Map Coloring**: The map at the right has 11 countries. Color it with as few colors as possible. Countries with a common edge must have different colors. Use the abbreviations:

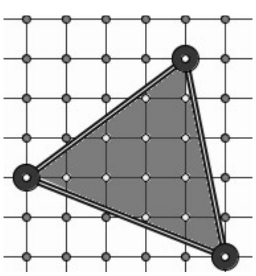
R=red B=blue G=green Y=yellow P=purple



Explain why you cannot do it with fewer colors.

Solution: It can be done with 4 colors. This is a posible solution.
It can't be done with 3 colors because the central yellow country is surrounded by 5 countries which can alternate between 2 colors, blue and red, but this cannot go all the way around because 5 is odd. So there must be a 4^{th} color, green.

5. **Pick's Theorem**: For the shaded region below, state the number of internal points and the number of boundary points, and find the area using Pick's Theorem:



$\boxed{I = 11}$ $\boxed{B = 3}$ $\boxed{A = 11 + \frac{3}{2} - 1 = 11.5}$

6. **Counting Cubes**:

a. The perfect squares are the sequence: $1, 4, 9, 16, 25, 36, \cdots$.

i. List the first 5 differencing numbers for the perfect squares.

Solution: $3, 5, 7, 9, 11$

ii. What is another name for the differencing numbers for the perfect squares?

Solution: odd numbers

b. The perfect cubes are the sequence: $1, 8, 27, 64, 125, 216, \cdots$.

i. List the first 5 differencing numbers for the perfect cubes.

Solution: $7, 19, 37, 61, 91$

ii. What is another name for the differencing numbers for the perfect cubes?

Solution: hexagon (hex) numbers

7. **Rational Tangles**: You have two ropes which were tangled using Twists (T) and Rotations (R). The tangle is assigned the rational number $\frac{-4}{3}$. Write down the list of Twists and Rotations which will untangle the ropes and the rational number assigned to each intermediate step. (There may be more blanks than you need.)

R/T T T R T T R T T

\# $\frac{-4}{3} \Rightarrow \frac{-1}{3} \Rightarrow \frac{2}{3} \Rightarrow \frac{-3}{2} \Rightarrow \frac{-1}{2} \Rightarrow \frac{1}{2} \Rightarrow -2 \Rightarrow -1 \Rightarrow 0$

8. **Euler Circuits**: Here is a map of the 5 bridges crossing the San Francisco Bay connecting 3 land masses.

a. Is it possible to travel so that you cross all 5 bridges without crossing any bridge twice?

$\boxed{\text{Yes}}$ No

b. Is it possible to travel so that you cross all 5 bridges without crossing any bridge twice and you end up where you started?

$\boxed{\text{Yes}}$ No

c. Draw your route. Put an A where you start and a B where you finish.

Solution: Start in Richmond. Cross the bridges in the order: Richmond-San Rafael, Golden Gate, SF-Oakland, San Mateo-Hayward, Dumbarton, and return to Richmond.

9. **Logic Problem**: A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie.

You meet two inhabitants: Betty and Peggy. Betty tells you "Peggy is a knave." Peggy tells you, "Betty and I are both knights." Can you determine who is a knight and who is a knave? Explain your reasoning.

Solution: If Peggy is a knight, then both she and Betty are knights. But then Betty is telling the truth that Peggy is a knave. This is a contradiction. So Peggy cannot be a knight; **Peggy is a knave**. Since Betty told you Peggy is a knave, which is true, **Betty is a knight**.

10. **Probability**:

a. **Birthday Problem**: What is the probability that Polly and Jason have different birthdays (assuming neither was born in a leap year)?

Solution: $1 \cdot \frac{364}{365} = \frac{364}{365}$

b. **Birthday Problem**: If 5 people are in a room, what is the probability that at least 2 of them have birthdays in the same month?

Solution: The probability they all have birthdays in different months.is

$$p = \frac{11}{12} \times \frac{10}{12} \times \frac{9}{12} \times \frac{8}{12} = \frac{11}{12} \times \frac{5}{6} \times \frac{3}{4} \times \frac{2}{3} = \frac{55}{144}$$

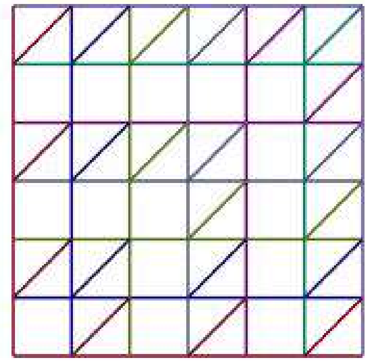
So the probability they are not all different is $1 - p = 1 - \frac{55}{144} = \frac{89}{144}$.

c. **Strings**: You hold 4 strings in your hand. You tie off 2 pairs at the top and 2 pairs at the bottom. When you let go, what is the probability that the strings are all in one loop?

Solution: The top pairings don't matter. When you connect 1 bottom string to 1 of the 3 others, 2 will allow for a single loop. So the probability is $\frac{2}{3}$. The remaining 2 strings are then tied. So the overall probability is $\frac{2}{3}$.

11. **Pigeonhole Principle**: There are 49 students at SEE-Math this year. Is it true that at least 5 of you are born in the same month? Explain why or why not.

Solution: Yes! There are 12 months. To avoid having 5 students in any month, you first assign 4 students to each month. That accounts for 48 students. The remaining 1 student must be assigned some month. But then at least one month has 5 students.



12. **Counting Diagonals**: What is the largest number of diagonals that can be drawn in a 6×6 grid with no 2 touching? $D_6 =$ ______

Draw them.

Solution: $D_6 =$ ___21___

Other diagonal drawings are possible.

13. **Magic Cards**:

a. Convert 46 base 10 to base 3.

Solution: $46 = 27 + 18 + 1 = 27 + 2 \cdot 9 + 1$. So $46_{10} = 1201_3$

b. Convert 34 base 5 to base 2.

Solution: $34_5 = 3 \cdot 5 + 4 = 19_{10}$. $19 = 16 + 2 + 1$. So $19_{10} = 10011_2$

14. **Bulgarian Solitaire**: In a Bulgarian Solitaire game, the initial partition is $(4, 2, 1)$. What are the next 5 partitions? What is the period of the final loop?

Solution: $(4,2,1) \Rightarrow (3,3,1) \Rightarrow (3,2,2) \Rightarrow (3,2,1,1) \Rightarrow (4,2,1) \Rightarrow (3,3,1)$

The period is 4.

15. **Gerrymandering**: A class with 25 students will be divided into 5 groups and each group will elect 1 student to the prom committee. Of the 25 students, 10 will vote for the NERDS (X) and 15 will vote for the JOCKS (O). Is it possible for the nerds to elect more representatives to the prom committee? If yes, show how by putting 5 X's and O's in each box below.

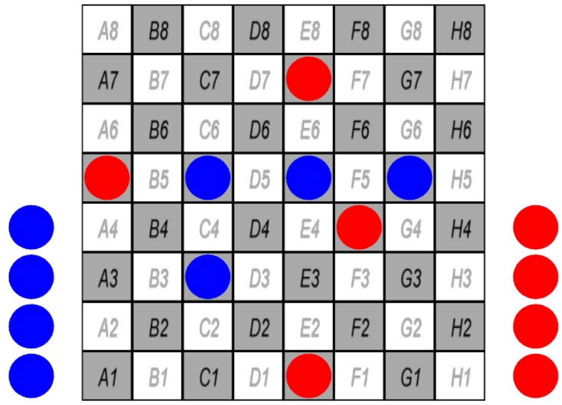
Solution: Yes the nerds can win.

XXXOO	XXXOO	XXXOO	XOOOO	OOOOO

16. **Pop-Tac-Toe**: It is Blue's turn.
Can Blue win on this turn?
If so, which square should Blue play on?
If there is more than one answer,
just list one winning play.

$\boxed{\text{Yes}}$ No

Solution: Play B6



17. **Kenken**: Solve the Kenken:

1−	2−		10×	
	9+	9+		
4		2−		
6+		2÷		10+
	2÷			

Solution:

1− 3	2− 4	2	10× 1	5
2	9+ 3	9+ 5	4	1
4 4	1	2− 3	5	2
6+ 1	5	2÷ 4	2	10+ 3
5	2÷ 2	1	3	4

18. **Cryptography**: Solve the cryptogram:

Q LW VWB TQSM OZMMV MOOA IVL PIU.
Q LW VWB TQSM BPMU, AIU-Q-IU.

Hint: I → A

Solution:

I DO NOT LIKE GREEN EGGS AND HAM.
I DO NOT LIKE THEM, SAM-I-AM.