MATH 151, FALL SEMESTER 2011
COMMON EXAMINATION 3 - VERSION A - SOLUTIONS

Name (print): ______________________ Instructor's name: ______________________

Signature: ______________________ Section No: ______________________

Part 1 – Multiple Choice (13 questions, 4 points each, No Calculators)

Write your name, section number, and version letter (**A**) of the exam on the ScanTron form.
Mark your responses on the ScanTron form and on the exam itself

1. The electric field between parallel plates produces the acceleration $a(t) = 4\cos(t)$ cm/sec^2 on a charged particle, where t is in seconds. If the particle has initial position $x(0) = 2$ cm and initial velocity $v(0) = 3$ cm/sec, where is the particle at time $t = \pi$ sec?

a. $x(\pi) = 3\pi + 10$ cm Correct Choice
b. $x(\pi) = 3\pi + 2$ cm
c. $x(\pi) = -3\pi + 2$ cm
d. $x(\pi) = 3\pi + 6$ cm
e. $x(\pi) = -3\pi - 6$ cm

SOLUTION: $v(t) = 4\sin(t) + C \qquad v(0) = C = 3 \qquad v(t) = 4\sin(t) + 3$
$x(t) = -4\cos(t) + 3t + K \qquad x(0) = -4 + K = 2 \qquad K = 6 \qquad x(t) = -4\cos(t) + 3t + 6$
$x(\pi) = -4\cos(\pi) + 3\pi + 6 = 3\pi + 10$

2. Find all solutions of $\ln(x) + \ln(x-2) = \ln(8)$.

a. $x = -2, 4$ only
b. $x = -2, 0$ only
c. $x = -2$ only
d. $x = 4$ only Correct Choice
e. $x = 0, 4$ only

SOLUTION: $\ln[(x)(x-2)] = \ln(8) \qquad x^2 - 2x - 8 = 0 \qquad (x+2)(x-4) = 0$
$x = -2$: $\ln(-2)$ is not defined, so not a solution.
$x = 4$: $\ln(4) + \ln(4-2) = \ln(8)$

3. Compute $\arcsin\left(\cos\frac{7\pi}{6}\right)$.

a. $-\frac{5}{3}\pi$
b. $\frac{5}{3}\pi$
c. $\frac{4}{3}\pi$
d. $-\frac{1}{3}\pi$ Correct Choice
e. $\frac{1}{3}\pi$

SOLUTION: $\cos\frac{7\pi}{6} = -\sqrt{3}/2$ because $\frac{7\pi}{6}$ is in quadrant III.
$\arcsin\left(\cos\frac{7\pi}{6}\right) = \arcsin\left(-\sqrt{3}/2\right) = -\arcsin\left(\sqrt{3}/2\right) = -\frac{1}{3}\pi.$

4. Compute $\lim_{x\to\infty}\left[\ln(2+3x)-\ln(4+5x)\right]$

a. $\ln\frac{1}{2}$

b. $\ln\frac{3}{5}$ Correct Choice

c. $\frac{\ln 2}{\ln 4}$

d. $\frac{\ln 3}{\ln 5}$

e. 0

SOLUTION: $\lim_{x\to\infty}(\ln(2+3x)-\ln(4+5x)) = \lim_{x\to\infty}\ln\left(\frac{2+3x}{4+5x}\right) = \ln\lim_{x\to\infty}\left(\frac{2+3x}{4+5x}\right) = \ln\frac{3}{5}$

5. A shrimp farm starts off with 1000 shrimp. The number of shrimp quadruples after 6 weeks (to 4000 shrimp). Assuming an exponential growth, how many shrimp are there after 9 weeks?

a. $1000e^{6}$

b. $1000e^{1/6}$

c. $1000e^{(\ln 4)2/3}$

d. 6000

e. 8000 Correct Choice

SOLUTION: $P = P_0(4)^{t/\tau}$ $P = 1000(4)^{9/6} = 1000(4)^{3/2} = 1000(2)^3 = 8000$

6. If $f(x) = \arctan(x) - \ln\left[(1+x^2)^{1/2}\right]$, then $f'(x) =$

a. $\frac{x+1}{1+x^2}$

b. $\frac{-1}{2(1+x^2)}$

c. $\frac{1-x}{1+x^2}$ Correct Choice

d. $\frac{x-1}{1+x^2}$

e. $\frac{1}{2(1+x^2)}$

SOLUTION: $f(x) = \arctan(x) - \frac{1}{2}\ln(1+x^2)$ $\frac{df}{dx} = \frac{1}{1+x^2} - \frac{1}{2}\frac{2x}{1+x^2} = \frac{1-x}{1+x^2}$

7. Find the line tangent to $y = \frac{1}{(\ln x)^3}$ at $x = e$. Its y-intercept is

a. 4 Correct Choice

b. -1

c. 1

d. $3e$

e. $1+3e$

SOLUTION: $f(x) = \frac{1}{(\ln x)^3} = (\ln x)^{-3}$ $f(e) = \frac{1}{(\ln e)^3} = 1$

$f'(x) = -3(\ln x)^{-4}\frac{1}{x} = \frac{-3}{x(\ln x)^4}$ $f'(e) = \frac{-3}{e(\ln e)^3} = \frac{-3}{e}$

$y = f(e) + f'(e)(x-e) = 1 - \frac{3}{e}(x-e) = -\frac{3}{e}x + 4$

8. Compute the derivative of $f(x) = (x^2+1)^x$.

a. $(x^2+1)^x\left[\ln(x^2+1) + \dfrac{2x^2}{(x^2+1)}\right]$ Correct Choice

b. $x(x^2+1)^{x-1}$

c. $x(x^2+1)^{x-1}2x$

d. $(x^2+1)^x\ln(x^2+1)$

e. $(x^2+1)^x\ln(x^2+1)2x$

SOLUTION: $f(x) = (x^2+1)^x = \left[e^{\ln(x^2+1)}\right]^x = e^{x\ln(x^2+1)}$

$f'(x) = e^{x\ln(x^2+1)}\dfrac{d}{dx}[x\ln(x^2+1)] = e^{x\ln(x^2+1)}\left[\ln(x^2+1) + \dfrac{2x^2}{(x^2+1)}\right] = (x^2+1)^x\left[\ln(x^2+1) + \dfrac{2x^2}{(x^2+1)}\right]$

9. Find the locations of the absolute maximum and absolute minimum of $f(x) = x^3 - 5x^2 + 3x$ on the interval $[1,5]$.

a. Abs Min at $x = 1$ Abs Max at $x = \dfrac{1}{3}$

b. Abs Min at $x = 3$ Abs Max at $x = \dfrac{1}{3}$

c. Abs Min at $x = 1$ Abs Max at $x = 5$

d. Abs Min at $x = 3$ Abs Max at $x = 5$ Correct Choice

e. Abs Min at $x = 1$ Abs Max at $x = 3$

SOLUTION: $f'(x) = 3x^2 - 10x + 3 = (3x-1)(x-3)$ Critical point at $x = 3$.
($x = \dfrac{1}{3}$ is not in the interval.) Endpoints at $x = 1,5$.
$f(1) = 1 - 5 + 3 = -1$ $f(3) = 27 - 45 + 9 = -9$ $f(5) = 125 - 125 + 15 = 15$

10. For $f(x) = 2e^x - x^2 - 2x$ which of the following is TRUE?

a. The First Derivative Test says $x = 0$ is a Local Minimum

b. The Second Derivative Test says $x = 0$ is a Local Minimum

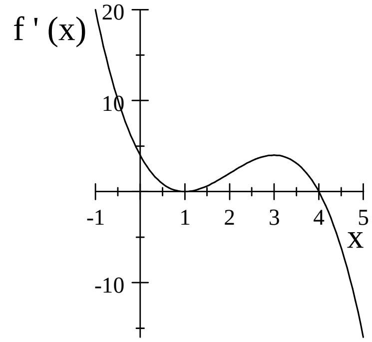
c. The Second Derivative Test says $x = 0$ is a Local Maximum

d. The Second Derivative Test says $x = 0$ is an Inflection Point

e. The Second Derivative Test Fails at $x = 0$. Correct Choice

SOLUTION: $f'(x) = 2e^x - 2x - 2$ $f'(0) = 0$ $f''(x) = 2e^x - 2$ $f''(0) = 0$
The Second Derivative Test Fails.

11. The plot at the right shows the graph of the FIRST DERIVATIVE $f'(x)$ of a function $f(x)$. Which of the following is FALSE?



a. $f(x)$ is increasing on $(0,1)$

b. $f(x)$ is increasing on $(1,3)$

c. $f(x)$ has a local minimum at $x = 4$ Correct Choice

d. $f(x)$ has an inflection point at $x = 1$

e. $f(x)$ is concave down on $(-1,1)$

SOLUTION:
(a) is true because $f'(x) > 0$ on $(0,1)$
(b) is true because $f'(x) > 0$ on $(1,3)$
(c) is false because $f(x)$ has a local maximum at $x = 4$ because to the left of 4, f is increasing: $f'(x) > 0$ and to the right of 4 f is decreasing: $f'(x) < 0$.
(d) is true because to the left of 1 f' is decreasing: $f''(x) < 0$.and to the right of 1 f' is increasing: $f''(x) > 0$. So the concavity changes.
(e) is true because f' is decreasing: $f''(x) < 0$

12. Compute $\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$

a. $\frac{1}{6}$

b. $\frac{-1}{6}$ Correct Choice

c. $\frac{1}{3}$

d. $\frac{-1}{3}$

e. 0

SOLUTION: Apply l'Hospital's Rule twice:

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} \overset{l'H}{=} \lim_{x\to 0} \frac{\cos(x) - 1}{3x^2} \overset{l'H}{=} \lim_{x\to 0} \frac{-\sin(x)}{6x} = \frac{-1}{6}$$

13. Compute $\sum_{i=2}^{5}(1 + i^2)$.

a. 56

b. 58 Correct Choice

c. 60

d. 62

e. 64

SOLUTION: $\sum_{i=2}^{5}(1 + i^2) = (1 + 2^2) + (1 + 3^2) + (1 + 4^2) + (1 + 5^2) = 58$

Part 2 – Work Out Problems (**5 questions**. **Points indicated**. **No Calculators**)

Solve each problem in the space provided. Show all your work neatly and concisely, and indicate your final answer clearly. You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it.

14. (6 points) Find the values of a and b so that $f(x) = ax^2 - b\ln(x)$ will have an inflection point at $(1, 5)$.

SOLUTION: $f(1) = a = 5$ $\quad f(x) = 5x^2 - b\ln(x)$

$f'(x) = 10x - \frac{b}{x}$ $\quad f''(x) = 10 + \frac{b}{x^2}$ $\quad f''(1) = 10 + b = 0$ $\quad b = -10$

Note: To be an inflection point, not only must $f''(1) = 0$, but the concavity must actually change. In fact when $a = 5$ and $b = -10$, we have $f''(x) = 10 - \frac{10}{x^2} = 10\frac{x^2-1}{x^2}$ which is negative for $0 < x < 1$ and positive for $x > 1$.

15. (8 points) Compute $L = \lim\limits_{x\to\infty}\left(\left(1 + \frac{3}{x} + \frac{4}{x^2}\right)^x\right)$

SOLUTION:

$$L = \lim_{x\to\infty}\left(\left(1 + \frac{3}{x} + \frac{4}{x^2}\right)^x\right) = \lim_{x\to\infty}\left[\left(e^{\ln\left(1+\frac{3}{x}+\frac{4}{x^2}\right)}\right)^x\right] = \lim_{x\to\infty}\left[e^{x\ln\left(1+\frac{3}{x}+\frac{4}{x^2}\right)}\right] = e^P$$

where $$P = \lim_{x\to\infty}\left[x\ln\left(1 + \frac{3}{x} + \frac{4}{x^2}\right)\right] = \lim_{x\to\infty}\left[\frac{\ln\left(1 + \frac{3}{x} + \frac{4}{x^2}\right)}{\frac{1}{x}}\right]$$

We can apply l'Hospital's Rule because the numerator and denominator go to zero.
We then multiply top and bottom by $-x^2$.

$$P \overset{\text{l'H}}{=} \lim_{x\to\infty}\left[\frac{\dfrac{-\frac{3}{x^2} - \frac{8}{x^3}}{1 + \frac{3}{x} + \frac{4}{x^2}}}{\dfrac{-1}{x^2}}\right] = \lim_{x\to\infty}\left[\frac{3 + \frac{8}{x}}{1 + \frac{3}{x} + \frac{4}{x^2}}\right] = 3 \qquad \text{So } L = e^P = e^3$$

ALTERNATIVE SOLUTION FOR P :

$$P = \lim_{x\to\infty}\left[x\ln\left(1 + \frac{3}{x} + \frac{4}{x^2}\right)\right] = \lim_{t\to 0^+}\left[\frac{\ln(1 + 3t + 4t^2)}{t}\right] \overset{\text{l'H}}{=} \lim_{t\to 0^+}\left[\frac{\frac{3 + 8t}{1 + 4t + 3t^2}}{1}\right] = 3$$

16. (10 points) A paint can needs to hold a liter of paint (1000 cm^3). The shape will be a cylinder of radius r and height h. The sides and bottom will be made from aluminum which costs $\$0.15$ per cm^2. The top will be made from plastic which costs $\$0.05$ per cm^2. What are the DIMENSIONS and COST of the cheapest such can? (Keep answers in terms of π.)

SOLUTION: $V = \pi r^2 h = 1000 \qquad h = \dfrac{1000}{\pi r^2}$

$$C = .15(2\pi rh + \pi r^2) + .05\pi r^2 = .3\pi rh + .2\pi r^2 = \frac{300}{r} + .2\pi r^2$$

$$C' = -\frac{300}{r^2} + .4\pi r = 0 \qquad .4\pi r = \frac{300}{r^2} \qquad r^3 = \frac{300}{.4\pi} = \frac{750}{\pi}$$

$$r = \left(\frac{750}{\pi}\right)^{1/3} \qquad h = \frac{1000}{\pi\left(\frac{750}{\pi}\right)^{2/3}} = \frac{1000}{750}\left(\frac{750}{\pi}\right)^{1/3} = \frac{4}{3}\left(\frac{750}{\pi}\right)^{1/3}$$

$$C = \frac{300}{r} + .2\pi r^2 = \frac{300}{\left(\frac{750}{\pi}\right)^{1/3}} + .2\pi\left(\frac{750}{\pi}\right)^{2/3} = \frac{2}{5}\pi\left(\frac{750}{\pi}\right)^{2/3} + \frac{1}{5}\pi\left(\frac{750}{\pi}\right)^{2/3} = \frac{3}{5}\pi\left(\frac{750}{\pi}\right)^{2/3}$$

OR $C = .3\pi rh + .2\pi r^2 = .3\pi\left(\frac{750}{\pi}\right)^{1/3}\frac{4}{3}\left(\frac{750}{\pi}\right)^{1/3} + .2\pi\left(\frac{750}{\pi}\right)^{2/3} = .6\pi\left(\frac{750}{\pi}\right)^{2/3}$

To see that C is the absolute minimum, we note there is only one critical point and $C = \dfrac{300}{r} + .2\pi r^2$ approaches ∞ as r approaches 0 or ∞.

17. (8 points) Derive the formula $\dfrac{d}{dx}\arccos(x) = \dfrac{-1}{\sqrt{1-x^2}}$.

Note: The inverse function of $\cos(x)$ is denoted by either $\arccos(x)$ or by $\cos^{-1}(x)$.

SOLUTION 1: If $y = \arccos(x)$ then $x = \cos(y)$. Use implicit differentaition to differentiate both sides with respect to x:

$1 = -\sin(y)\dfrac{dy}{dx}$ So $\dfrac{dy}{dx} = \dfrac{-1}{\sin(y)}$ But since $y = \arccos(x)$, this says:

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sin(y)} = \frac{-1}{\sin(\arccos(x))}$$

To get this in the correct form, if $\cos(y) = x$ then $\sin(y) = \sqrt{1-\cos^2(y)} = \sqrt{1-x^2}$.

Notice we use the positive square root because $y = \arccos(x)$ is between 0 and π so that $\sin(y)$ is positive. So $\dfrac{d}{dx}\arccos(x) = \dfrac{-1}{\sin(y)} = \dfrac{-1}{\sqrt{1-x^2}}$

SOLUTION 2: If $b = \arccos(a)$ then $a = \cos(b)$. The formula for the derivative of the inverse function says:

$$\left.\frac{dg}{dx}\right|_{x=a} = \frac{1}{\left.\frac{df}{dx}\right|_{x=b}} \qquad \text{So} \qquad \left.\frac{d\arccos(x)}{dx}\right|_{x=a} = \frac{1}{\left.\frac{d\cos(x)}{dx}\right|_{x=b}} = \frac{1}{-\sin(x)|_{x=b}} = \frac{-1}{\sin(\arccos(a))}$$

18. (16 points) For the function $f(x) = xe^x$ find

a. all intervals where f is increasing or decreasing.

SOLUTION: $f'(x) = e^x + xe^x = (1+x)e^x$. critical point: $x = -1$
For $x < -1$, $f'(x) < 0$ and f is decreasing.
For $x > -1$, $f'(x) > 0$ and f is increasing.

b. all intervals where f is concave up or concave down.

SOLUTION: $f''(x) = e^x + e^x + xe^x = (2+x)e^x$. inflection point: $x = -2$
For $x < -2$, $f''(x) < 0$ and f is concave down.
For $x > -2$, $f''(x) > 0$ and f is concave up.

c. all local minima, local maxima, absolute minima and absolute maxima (give location and value).

SOLUTION: Since f is decreasing on the left of -1 and f is increasing on the right of -1, there is a local and absolute minimum at $x = -1$ and $f(-1) = -e^{-x}$.

Since there are no other critical points and no endpoints, there are no local or absolute maxima.

d. all horizontal asymptotes (be sure to compute the limits).

SOLUTION: $\lim_{x\to\infty} xe^x = \infty e^\infty = \infty$ no horizontal asymptote as $x \to \infty$.
$\lim_{x\to-\infty} xe^x =$ "$(-\infty)(0)$" indeterminate form. Apply l'Hospital's Rule:
$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} \stackrel{l'H}{=} \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \lim_{x\to-\infty} -e^x = 0$
So $y = 0$ is the horizontal asymptote as $x \to -\infty$.