MATH 151 Regular, FALL SEMESTER 2011
FINAL EXAMINATION - SOLUTIONS

Name (print): ______________________ Instructor's name: Yasskin

Signature: ______________________ Section No: __________

Part 1 – Multiple Choice (15 questions, 4 points each, No Calculators)

Write your name and section number on the ScanTron form.
Mark your responses on the ScanTron form and on the exam itself

1. Evaluate $\lim_{x\to-1}\dfrac{x^3+9x^2+8x}{x^2-1}$

a. 6
b. ∞
c. $\dfrac{7}{2}$ Correct Choice
d. 0
e. 3

SOLUTION: $\lim_{x\to-1}\dfrac{x^3+9x^2+8x}{x^2-1}=\lim_{x\to-1}\dfrac{x(x+1)(x+8)}{(x+1)(x-1)}=\lim_{x\to-1}\dfrac{x(x+8)}{(x-1)}=\dfrac{(-1)(-1+8)}{(-1-1)}=\dfrac{7}{2}$

2. The limit $\lim_{h\to0}\dfrac{4(2+h)^4-64}{h}$ can be interpreted as which of the following?

a. $f'(64)$ where $f(x)=4x^4$
b. $f'(4)$ where $f(x)=x^4$
c. $f'(2)$ where $f(x)=16x^3$
d. $f'(2)$ where $f(x)=\frac{4}{5}x^5$
e. $f'(2)$ where $f(x)=4x^4$ Correct Choice

SOLUTION: $f(x+h)=4(2+h)^4$ So $x=2$ and $f(x)=4x^4$ and $f(2)=64$.

3. Find the line tangent to $y=\sin x$ at $x=\dfrac{\pi}{3}$. Its y-intercept is

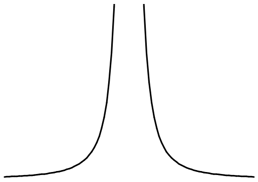
a. $\dfrac{\sqrt{3}}{2}+\dfrac{\pi}{6}$
b. $\dfrac{\sqrt{3}}{2}-\dfrac{\pi}{6}$ Correct Choice
c. $\dfrac{1}{2}-\dfrac{\sqrt{3}\,\pi}{6}$
d. $\dfrac{1}{2}+\dfrac{\sqrt{3}\,\pi}{6}$
e. 0

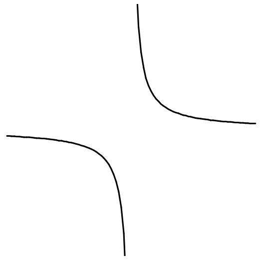
SOLUTION: $f(x)=\sin x$ $f'(x)=\cos x$ $f\left(\dfrac{\pi}{3}\right)=\dfrac{\sqrt{3}}{2}$ $f'\left(\dfrac{\pi}{3}\right)=\dfrac{1}{2}$

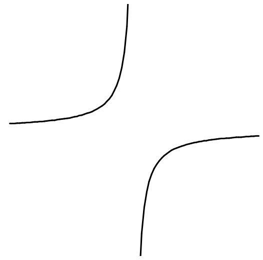
Tan Line: $y=f\left(\dfrac{\pi}{3}\right)+f'\left(\dfrac{\pi}{3}\right)\left(x-\dfrac{\pi}{3}\right)=\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\left(x-\dfrac{\pi}{3}\right)$

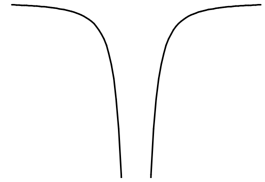
$b=y(0)=\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\left(0-\dfrac{\pi}{3}\right)=\dfrac{\sqrt{3}}{2}-\dfrac{\pi}{6}$

4. The function $f(x) = \dfrac{x^2 - 4x + 3}{x^2 - 4x + 4}$ has a vertical asymptote at $x = 2$. Near $x = 2$, the graph has the shape:

a. 

b. 

c. 

d. 

Correct Choice

SOLUTION: $f(x) = \dfrac{x^2 - 4x + 3}{x^2 - 4x + 4} = \dfrac{(x-1)(x-3)}{(x-2)^2}$

$\lim\limits_{x \to 2^-} \dfrac{(x-1)(x-3)}{(x-2)^2} = \dfrac{\text{“}(1^-)(-1^-)\text{”}}{(0^-)^2} = -\infty \qquad \lim\limits_{x \to 2^+} \dfrac{(x-1)(x-3)}{(x-2)^2} = \dfrac{\text{“}(1^+)(-1^+)\text{”}}{(0^+)^2} = -\infty$

5. Find the absolute minimum value of $f(x) = 3x^2 - x^3$ on the interval $[1, 3]$.

a. 0 Correct Choice
b. 2
c. 4
d. -2
e. -25

SOLUTION: $f'(x) = 6x - 3x^2 = 3x(2 - x) = 0$ at $x = 0, 2$
Check critical points in the interval and endpoints:
$f(1) = 3 - 1 = 2$
$f(2) = 3 \cdot 4 - 8 = 4$
$f(3) = 3 \cdot 9 - 27 = 0 \qquad \Leftarrow$ absolute minimum

6. A spacecraft is being sent to Mars. Its distance from the earth is given by $p(t) = 7t^3 + 1$. At time $t = 2$ the position is measured, but the error in the time measurement is ± 0.1. What is the resulting error in the calculated position?

a. ± 7.3
b. ± 8.4 Correct Choice
c. ± 8.5
d. ± 7.4
e. Impossible to determine.

SOLUTION: Using differentials, we find $dp = 21t^2 dt = 21 \cdot 2^2 \cdot (\pm 0.1) = \pm 8.4$.

7. Find two numbers a and b whose sum is 15 for which $P = a^2b$ is a maximum. For this a and b we have $ab =$

a. 36
b. 44
c. 50 Correct Choice
d. 54
e. 56

SOLUTION: $a + b = 15$ $\quad b = 15 - a$ $\quad P = a^2b = a^2(15 - a) = 15a^2 - a^3$
$P' = 30a - 3a^2 = a(30 - 3a) = 0$ The maximum cannot be at $a = 0$.
So the maximum is at $a = 10$. So $b = 15 - a = 15 - 10 = 5$ and $ab = 10 \cdot 5 = 50$

8. Let $f(x)$ be a differentiable function, and suppose $f(5) = 3$ and $f'(x) \leq 11$ for all values of x. Use the Mean Value Theorem to determine how large $f(7)$ can possibly be.

a. 25 Correct Choice
b. -25
c. 19
d. 33
e. Not enough information.

SOLUTION: By the MVT, there is a c in $(5, 7)$ where
$\dfrac{f(7) - f(5)}{7 - 5} = f'(c) \leq 11$ So $f(7) \leq f(5) + 11(7 - 5) = 3 + 11(2) = 25$

9. Evaluate $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{9x^2 - 3}}{2x + 5}$

a. $\dfrac{-3}{4}$
b. $\dfrac{3}{4}$
c. $\dfrac{-3}{2}$ Correct Choice
d. $\dfrac{3}{2}$
e. ∞

SOLUTION: For $x < 0$, we have $x = -\sqrt{x^2}$.

So $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{9x^2 - 3}}{2x + 5} = \lim_{x\to-\infty} \frac{\sqrt{9x^2 - 3}\,\dfrac{1}{-\sqrt{x^2}}}{(2x + 5)\dfrac{1}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{9 - \dfrac{3}{x^2}}}{2 + \dfrac{5}{x}} = \frac{-3}{2}$

10. Approximate the area under $y = x^2 + 2$ above the x-axis between $x = 0$ and $x = 6$ using 3 intervals of equal length and rectangles whose heights are computed at the midpoints of each interval.
HINT: Draw a picture

a. 41
b. 52
c. 82 Correct Choice
d. 84
e. 124

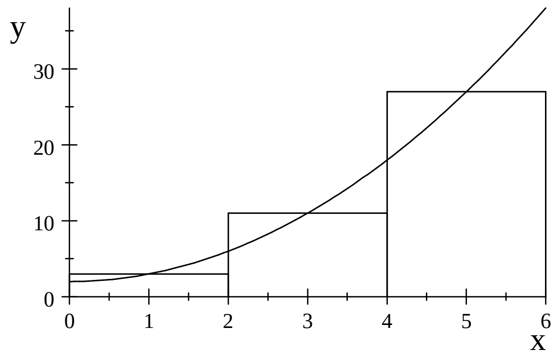
SOLUTION:

$\Delta x = \dfrac{6-0}{3} = 2$

$f(1) = 3 \quad f(3) = 11 \quad f(5) = 27$

From the plot

$A \approx 3 \cdot 2 + 11 \cdot 2 + 27 \cdot 2 = 82$



11. Evaluate $\displaystyle\int_{\pi/6}^{\pi/3} \sin(2x)\,dx$

a. $\frac{1}{2}$ Correct Choice
b. $-\frac{1}{2}$
c. 1
d. -1
e. 0

SOLUTION:

$$\int_{\pi/6}^{\pi/3} \sin(2x)\,dx = \left[\frac{-\cos(2x)}{2}\right]_{\pi/6}^{\pi/3} = -\frac{\cos\left(\frac{2\pi}{3}\right)}{2} + \frac{\cos\left(\frac{\pi}{3}\right)}{2} = -\frac{1}{2}\left(-\frac{1}{2}\right) + \frac{1}{2}\frac{1}{2} = \frac{1}{2}$$

12. If $I = \displaystyle\int_0^2 \sqrt{x^3+1}\,dx$, which of the following is FALSE?

a. $\int_0^5 \sqrt{x^3+1}\,dx - \int_2^5 \sqrt{x^3+1}\,dx = I$
b. $\int_0^2 x^3 + 1\,dx = I^2$ Correct Choice
c. $\int_2^0 \sqrt{x^3+1}\,dx = -I$
d. $2 \le I \le 6$
e. $\int_0^2 2\sqrt{x^3+1}\,dx = 2I$

SOLUTION: (a) $\int_0^5 = \int_0^2 + \int_2^5$- true (c) $\int_2^0 = -\int_0^2$ - true (e) $\int 2f\,dx = 2\int f\,dx$ - true
(d) $1 \le \sqrt{x^3+1} \le 3$ on $[0,2]$. So $\int_0^2 1\,dx \le \int_0^2 \sqrt{x^3+1}\,dx \le \int_0^2 3\,dx$. Or $2 \le I \le 6$.- true
(b) is false.

13. Calculate $\lim_{x\to 0^+} \frac{1}{x}\int_0^x e^{t^2}\,dt$

a. $2e$
b. ∞
c. 0
d. 1 Correct Choice
e. e

SOLUTION: By L'Hôpital's rule and the FTC, $\lim_{x\to 0^+} \frac{1}{x}\int_0^x e^{t^2}\,dt = \lim_{x\to 0^+} \frac{e^{x^2}}{1} = e^0 = 1.$

14. Calculate $\frac{d}{dx}\left(\frac{\ln(x)}{2^x}\right)$ at $x = 2$

a. $\frac{1}{8} - \frac{1}{4}\ln 2$
b. $\frac{1}{8} + \frac{1}{4}\ln 2$
c. $\frac{1}{8} - \frac{1}{4}(\ln 2)^2$ Correct Choice
d. $\frac{1}{8} + \frac{1}{4}(\ln 2)^2$
e. $\frac{1}{8} - \frac{1}{4\ln 2}$

SOLUTION: $\frac{d}{dx}\left(\frac{\ln(x)}{2^x}\right) = \frac{d}{dx}(2^{-x}\ln(x)) = 2^{-x}\frac{1}{x} + -2^{-x}(\ln 2)\ln x$
$\left[\frac{d}{dx}\left(\frac{\ln(x)}{2^x}\right)\right]_{x=2} = 2^{-2}\frac{1}{2} + -2^{-2}(\ln 2)\ln 2 = \frac{1}{8} - \frac{1}{4}(\ln 2)^2$

15. Find the intervals of concavity of the function $f(x) = 3(3 + x^2)^{-1}$.

a. Concave up: $(-1, 1)$ Concave down: $(-\infty, -1) \cup (1, \infty)$
b. Concave up: $(-\infty, \infty)$ Concave down: nowhere
c. Concave up: nowhere Concave down: $(-\infty, \infty)$
d. Concave up: $(1, \infty)$ Concave down: $(-\infty, -1) \cup (-1, 1)$
e. Concave up: $(-\infty, -1) \cup (1, \infty)$ Concave down: $(-1, 1)$ Correct Choice

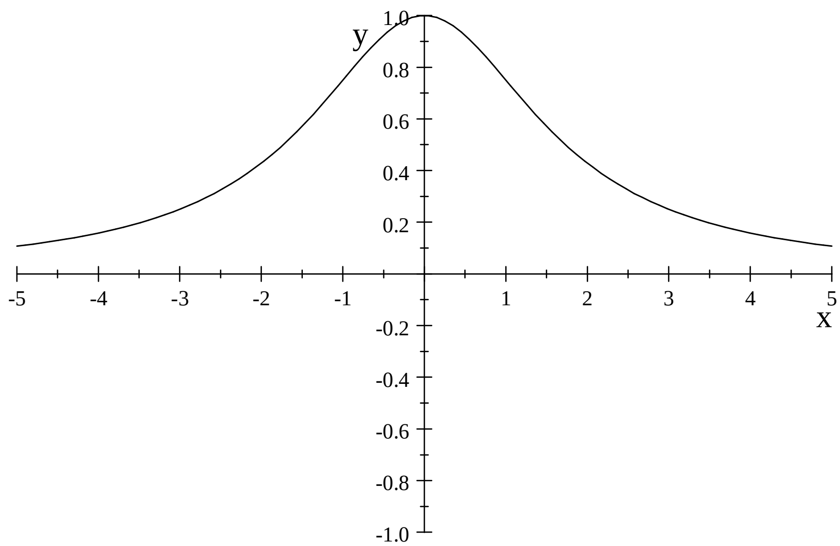
SOLUTION: $f'(x) = -3(3 + x^2)^{-2}2x$
$f''(x) = 3 \cdot 2(3 + x^2)^{-3}2x2x - 3 \cdot (3 + x^2)^{-2}2 = 3 \cdot \frac{8x^2 - (3 + x^2)2}{(3 + x^2)^3} = 3 \cdot \frac{6x^2 - 6}{(3 + x^2)^3} = 0$
$x = \pm 1$
Check the signs in each interval:
$(-\infty, -1)$: $f''(-2) > 0$ concave up
$(-1, 1)$: $f''(0) < 0$ concave down
$(1, \infty)$: $f''(2) > 0$ concave up

Part 2 – Work Out Problems (5 questions. Points indicated. No Calculators)

Solve each problem in the space provided. Show all your work neatly and concisely, and indicate your final answer clearly. You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it.

16. (6 points) Graph the function $f(x) = 3(3+x^2)^{-1}$. (See problem #15.)
Be sure to label all maxima, minima, inflection points and asymptotes.
Be careful with intervals of increase, decrease and concavity.

SOLUTION: increasing: $x < 0$ decreasing: $x > 0$
local and absolute maximum: $(0,1)$ no minima
Concave up: $x < -1$ or $x > 1$ Concave down: $-1 < x < 1$
inflection points: $\left(-1, \frac{3}{4}\right)$ and $\left(1, \frac{3}{4}\right)$
no vertical asymptotes, horizontal asymptotes: $y = 0$ as $x \to \pm\infty$



17. (8 points) Let $g(x)$ be the inverse function of $f(x) = xe^x$ for $x > 0$. Find $g(e)$ and $g'(e)$.

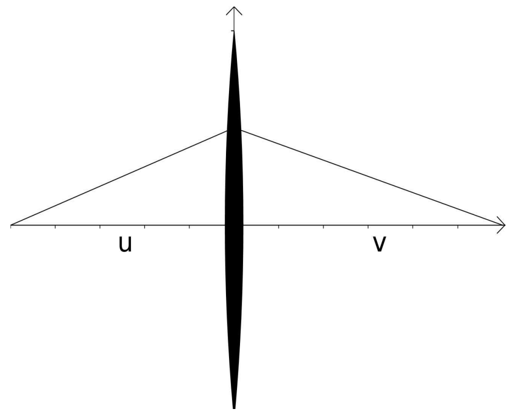
SOLUTION: If $b = g(e)$ then $e = f(b) = be^b$. So $b = 1$. So $g(e) = 1$.

Further $f'(x) = e^x + xe^x$ and $f'(1) = e^1 + 1e^1 = 2e$. So $g'(e) = \dfrac{1}{f'(1)} = \dfrac{1}{2e}$.

18. (10 points) When light passes through a lens with focal length f
the distance to the object, u, is related to
the distance to the image, v, by the equation

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}.$$

If $u = 2$, $v = 3$ and $\frac{du}{dt} = -0.4$, find f and $\frac{dv}{dt}$.
Is v getting longer or shorter?



SOLUTION: $\frac{1}{f} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ $\quad f = \frac{6}{5}$

$-\frac{1}{u^2}\frac{du}{dt} - \frac{1}{v^2}\frac{dv}{dt} = 0 \qquad -\frac{1}{v^2}\frac{dv}{dt} = \frac{1}{u^2}\frac{du}{dt} \qquad \frac{dv}{dt} = -\frac{v^2}{u^2}\frac{du}{dt} = -\frac{9}{4}(-0.4) = 0.9$

v is getting longer.

19. (6 points) Find a parametric equation for the line tangent to the parametric curve $\vec{r}(t) = \langle t^2, t^3 \rangle$ at the point $(4,8)$.

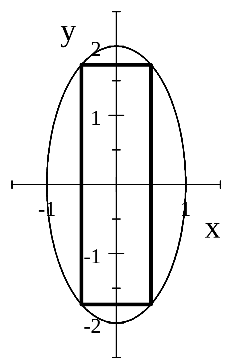
SOLUTION: $t = 2 \qquad \vec{v}(t) = \langle 2t, 3t^2 \rangle \qquad \vec{v}(2) = \langle 4, 12 \rangle$

$\vec{r}_{\tan}(t) = (4,8) + t\langle 4, 12 \rangle = \langle 4 + 4t, 8 + 12t \rangle$ There is no unique answer.

20. (12 points) A rectangle is inscribed in the ellipse

$x^2 + \frac{y^2}{4} = 1$ with all sides parallel to the axes.

Find the maximum area of such a rectangle.



SOLUTION: $A = 4xy$ with $y = 2\sqrt{1 - x^2}$. So $A = 8x\sqrt{1 - x^2}$

$A' = 8\sqrt{1 - x^2} + 8x\frac{1}{2}\frac{-2x}{\sqrt{1 - x^2}} = 0$ $\qquad$ Multiply by $\frac{1}{8}\sqrt{1 - x^2}$

$(1 - x^2) - x^2 = 0$ $\quad$ or $\quad 1 - 2x^2 = 0 \qquad x = \frac{1}{\sqrt{2}}$

$y = 2\sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{2} \qquad A = 4xy = 4\left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{2}\right) = 4$