# MATH 152
# Exam 3
# Fall 1997
# Version A
# Solutions

Part I is multiple choice. There is no partial credit. You may not use a calculator.

Part II is work out. Show all your work. Partial credit will be given. You may use your calculator.

Part I: Multiple Choice (5 points each)

There is no partial credit. You may not use a calculator. You have 1 hour.

**1.** If $f(x,y,z) = x^2 y \sin z$, then $\frac{\partial^2 f}{\partial x \partial z} =$

- **a.** $\frac{xy}{z}\sin z$
- **b.** $x^2 \sin z$
- **c.** $2xy\sin z + x^2 y \cos z$
- **d.** $2x^3 y^2 \sin z \cos z$
- **e.** $2xy\cos z$ correctchoice

$\frac{\partial f}{\partial z} = x^2 y \cos z \qquad \frac{\partial^2 f}{\partial x \partial z} = 2xy\cos z$ (e)

**2.** Which pair of vectors is NOT perpendicular?

- **a.** $\boldsymbol{a} = \langle 1,2,3 \rangle$ and $\boldsymbol{b} = \langle 3,0,-1 \rangle$
- **b.** $\boldsymbol{p} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ and $\boldsymbol{q} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ correctchoice
- **c.** $\boldsymbol{A} = -3\mathbf{i} + 2\mathbf{j}$ and $\boldsymbol{B} = 4\mathbf{i} + 6\mathbf{j}$
- **d.** $\boldsymbol{F} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\boldsymbol{G} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
- **e.** $\boldsymbol{u} = \langle 3,4 \rangle$ and $\boldsymbol{v} = \langle 8,-6 \rangle$

$a \bullet b = 3 - 3 = 0 \qquad p \bullet q = -2 - 4 + 2 = -4 \neq 0 \qquad A \bullet B = -12 + 12 = 0$

$F \bullet G = 1 + 2 - 3 = 0 \qquad u \bullet v = 24 - 24 = 0$ (b)

**3.** A parallelepiped has adjacent edges $\boldsymbol{u} = \langle 2,-1,4 \rangle$, $\boldsymbol{v} = \langle 1,-3,0 \rangle$ and $\boldsymbol{w} = \langle 3,1,-2 \rangle$. Find its volume.

- **a.** $\langle 12,4,-8 \rangle$
- **b.** $-12$
- **c.** $12$
- **d.** $50$ correctchoice
- **e.** $54$

$$u \bullet v \times w = \begin{vmatrix} 2 & -1 & 4 \\ 1 & -3 & 0 \\ 3 & 1 & -2 \end{vmatrix} = 2(6-0) + 1(-2-0) + 4(1+9) = 12 - 2 + 40 = 50$$ (d)

**4**. The radius of a cylindrical tin can is $5\,\mathrm{cm}$ and the height is $10\,\mathrm{cm}$. The sides are $.01\,\mathrm{cm}$ thick while the top and bottom are $.02\,\mathrm{cm}$ thick each. Estimate the volume of metal used to make the can.

- **a**. $.004\pi\ \mathrm{cm}^3$
- **b**. $.01\pi\ \mathrm{cm}^3$
- **c**. $.02\pi\ \mathrm{cm}^3$
- **d**. $2\pi\ \mathrm{cm}^3$ correctchoice
- **e**. $250\pi\ \mathrm{cm}^3$

$V = \pi r^2 h \qquad r = 5 \qquad h = 10 \qquad \Delta r = .01 \qquad \Delta h = 2(.02) = .04$

$\Delta V \approx \frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial h}\Delta h = 2\pi r h \Delta r + \pi r^2 \Delta h = 2\pi(5)(10)(.01) + \pi(5)^2(.04) = \pi + \pi = 2\pi\ \mathrm{cm}^3$

(d)

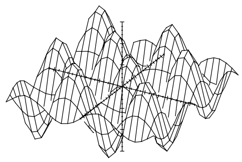
**5**. Which line is perpendicular to the plane $3x + 4y + 5z = 6$?

- **a**. $\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4}$
- **b**. $x = 3 + 2t,\ \ y = 4 + 3t,\ \ z = 5 + 4t$
- **c**. $x = 2 + 3t,\ y = 3 + 4t,\ \ z = 4 + 5t$ correctchoice
- **d**. $\frac{x-2}{20} = \frac{y-3}{15} = \frac{z-4}{12}$
- **e**. $x = 2 + 20t,\ y = 3 + 15t,\ \ z = 4 + 12t$

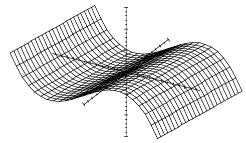
The tangent to the line is the normal to the plane which by inspection is $v = \langle 3,4,5\rangle$. If $X = (x,y,z)$ is a general point on the line and $P = (a,b,c)$ is a particular point, then the parametric equations are $X = P + tv$ or $x = a + 3t,\ \ y = b + 4t,\ \ z = c + 5t$ and the symmetric equations are $\frac{x-a}{3} = \frac{y-b}{4} = \frac{z-c}{5}$. The only line with this form is (c).

**6.** Which of the following is the graph of $f = y^2 - x^2$?

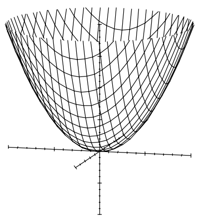
**a.**



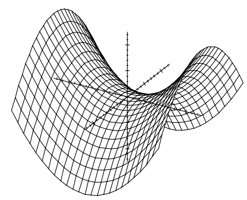
d.



**b.**

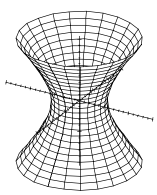


e.



correctchoice

**c.**



The equation $z = y^2 - x^2$ describes a hyperbolic paraboloid centered on the $z$-axis which opens up along the $y$-direction and down along the $x$-direction. (e)

**7.** Find the intersection of the line $x = 3 + 2t, \; y = 2 + t, \; z = 1 - t$ and the plane $x - y + 2z = 4$.

**a.** $(3, 1, 1)$

**b.** $(1, 1, 2)$ correctchoice

**c.** $(2, -2, 0)$

**d.** $(-1, 1, 3)$

**e.** $(0, 2, 3)$

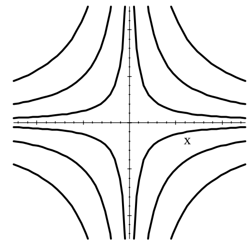
Plug the line into the plane and solve for $t$:

$(3 + 2t) - (2 + t) + 2(1 - t) = 4 \qquad 3 - t = 4 \qquad t = -1$

Then plug back into the line:

$x = 3 + 2(-1) = 1 \qquad y = 2 + (-1) = 1 \qquad z = 1 - (-1) = 2$ (b)

**8**. For which function are the level curves (or contour plot) shown at the right?



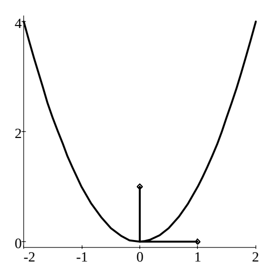

**a**. $f = x^2 + y^2 - 2x$
**b**. $f = \cos x \cos y$
**c**. $f = xy$ correctchoice
**d**. $f = (x+y)^2$
**e**. $f = (x-y)^2$

The level curves $x^2 + y^2 - 2x = C$ are circles centered at $(1,0)$.
The level curves $\cos x \cos y = C$ are periodic in $x$ and $y$.
The level curves $xy = C$ are the hyperbolas $y = \dfrac{C}{x}$ as given above. (c)
The function $f = (x+y)^2$ has its zero level set at $y = -x$.
The function $f = (x-y)^2$ has its zero level set at $y = x$.

**9**. An object moves in the $xy$-plane along the curve $y = x^2$ from $(-2,4)$ to $(2,4)$. In what direction does the (principal) normal ***N*** point when the object is at $(0,0)$?

**a**. **j** correctchoice
**b**. **i** + **j**
**c**. **j** − **i**
**d**. −**j**
**e**. **i**

The graph of $y = x^2$ is shown ..........................................................




The unit tangent $T$ points to the right in the direction of motion.
The unit principal normal $N$ points up in the direction the motion is turning.
So $N = j$. (a)

**10**. A triangle has vertices $A = (1, 1, -1)$, $B = (2, 0, -1)$ and $C = (1, -1, 1)$. Find a vector perpendicular to the plane of the triangle.

**a**. $\langle 1, 1, 1 \rangle$ correctchoice

**b**. $\langle -2, 2, 2 \rangle$

**c**. $\langle 1, -1, 1 \rangle$

**d**. $\langle 2, 2, -2 \rangle$

**e**. $\langle 2, 2, 0 \rangle$

$AB = B - A = \langle 1, -1, 0 \rangle \qquad AC = C - A = \langle 0, -2, 2 \rangle$

$$N = AB \times AC = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 0 & -2 & 2 \end{vmatrix} = i(-2 - 0) - j(2 - 0) + k(-2 - 0) = -2\langle 1, 1, 1 \rangle \qquad \text{(a)}$$

**11**. A wagon is pulled a distance of $100\,\mathrm{m}$ along a horizontal path by a constant force of $50\,\mathrm{N}$ exerted along the handle which is at $30°$ above the horizontal. How much work is done?

**a**. $5000\,\mathrm{J}$

**b**. $2500\,\mathrm{J}$

**c**. $2500\sqrt{3}\ \mathrm{J}$ correctchoice

**d**. $\dfrac{10000}{\sqrt{3}}\ \mathrm{J}$

**e**. $10000\,\mathrm{J}$

$$W = F \bullet D = |F||D|\cos\theta = 50 \bullet 100 \bullet \cos 30° = 5000 \bullet \frac{\sqrt{3}}{2} = 2500\sqrt{3}\ \mathrm{J} \qquad \text{(c)}$$

Part II: Work Out

Show all your work. Partial credit will be given.
You may use your calculator but only after 1 hour.

**12**. (12 points) Consider the curve $\boldsymbol{r}(t) = (t, \sin(2t), \cos(2t))$.
Compute each of the following:

**a**. velocity

$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$ $\qquad$ $\boldsymbol{v} = \langle 1, 2\cos(2t), -2\sin(2t)\rangle$

**b**. speed

$|\boldsymbol{v}| = \sqrt{1 + 4\cos^2(2t) + 4\sin^2(2t)} = \sqrt{1+4}$ $\qquad$ $|\boldsymbol{v}| = \sqrt{5}$

**c**. arclength between $t = 1$ and $t = 3$

$L = \int_1^3 |\boldsymbol{v}|\,dt = \int_1^3 \sqrt{5}\,dt = \left[\sqrt{5}\,t\right]_1^3 = \sqrt{5}(3-1)$ $\qquad$ $L = 2\sqrt{5}$

**d**. acceleration

$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}$ $\qquad$ $\boldsymbol{a} = \langle 0, -4\sin(2t), -4\cos(2t)\rangle$

**e**. unit tangent

$\boldsymbol{T} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{1}{\sqrt{5}}\langle 1, 2\cos(2t), -2\sin(2t)\rangle$ $\qquad$ $\boldsymbol{T} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\cos(2t), -\frac{2}{\sqrt{5}}\sin(2t)\right\rangle$

**f**. curvature

$$\boldsymbol{v} \times \boldsymbol{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2\cos(2t) & -2\sin(2t) \\ 0 & -4\sin(2t) & -4\cos(2t) \end{vmatrix}$$

$$= \mathbf{i}\left(-8\cos^2(2t) - 8\sin^2(2t)\right) - \mathbf{j}(-4\cos(2t)) + \mathbf{k}(-4\sin(2t)) = \langle -8, 4\cos(2t), -4\sin(2t)\rangle$$

$$|\boldsymbol{v} \times \boldsymbol{a}| = \sqrt{64 + 16\cos^2(2t) + 16\sin^2(2t)} = \sqrt{64+16} = \sqrt{80} = 4\sqrt{5}$$

$\kappa = \frac{|\boldsymbol{v} \times \boldsymbol{a}|}{|\boldsymbol{v}|^3} = \frac{4\sqrt{5}}{\sqrt{5}^{\,3}}$ $\qquad$ $\kappa = \frac{4}{5}$

OR

$\frac{d\boldsymbol{T}}{dt} = \left\langle 0, -\frac{4}{\sqrt{5}}\sin(2t), -\frac{4}{\sqrt{5}}\cos(2t)\right\rangle$ $\qquad$ $\left|\frac{d\boldsymbol{T}}{dt}\right| = \frac{4}{\sqrt{5}}$

$\kappa = \frac{1}{|\boldsymbol{v}|}\left|\frac{d\boldsymbol{T}}{dt}\right| = \frac{1}{\sqrt{5}}\frac{4}{\sqrt{5}}$ $\qquad$ $\kappa = \frac{4}{5}$

**13**. (11 points) Find the plane tangent to the hyperbolic paraboloid $z = 2x^2 - y^2$ at the point $(1,2,-2)$. Then identify its $z$-intercept.

$f(x,y) = 2x^2 - y^2 \qquad f_x(x,y) = 4x \qquad f_y(x,y) = -2y$
$f(1,2) = -2 \qquad f_x(1,2) = 4 \qquad f_y(1,2) = -4$
$z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$
$z = -2 + 4(x-1) - 4(y-2) = 2 + 4x - 4y$
At $x = y = 0$, the $z$-intercept is $z = 2$.

**14**. (11 points) A particle has initial position $\mathbf{r}(t) = \langle 0,0 \rangle$ and initial velocity $\mathbf{v}(t) = \langle 1,-1 \rangle$. If its acceleration is $\mathbf{a}(t) = \langle 4\cos(2t), 12t^2 \rangle$, find its position at $t = \pi$.

$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} \qquad$ So $\boldsymbol{v} = \int \boldsymbol{a}\,dt = \langle 2\sin(2t), 4t^3 \rangle + \boldsymbol{C}$
Use the initial condition: $\mathbf{v}(0) = \boldsymbol{C} = \langle 1,-1 \rangle \qquad$ So $\boldsymbol{v} = \langle 2\sin(2t)+1, 4t^3-1 \rangle$
$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} \qquad$ So $\boldsymbol{r} = \int \boldsymbol{v}\,dt = \langle -\cos(2t)+t, t^4-t \rangle + \boldsymbol{K}$
Use the initial condition: $\mathbf{r}(0) = \langle -1,0 \rangle + \boldsymbol{K} = \langle 0,0 \rangle \qquad$ So $\boldsymbol{K} = \langle 1,0 \rangle$ and
$\boldsymbol{r}(t) = \langle -\cos(2t)+t+1, t^4-t \rangle \qquad$ Thus, $\boldsymbol{r}(\pi) = \langle -1+\pi+1, \pi^4-\pi \rangle = \langle \pi, \pi^4-\pi \rangle$

**15**. (11 points) Find the line of intersection of the planes $x+y+z = 3$ and $3x+y-z = 1$.

The line of intersection is perpendicular to both normals. The normal to the first plane is $n_1 = \langle 1,1,1 \rangle$ and the normal to the second plane is $n_2 = \langle 3,1,-1 \rangle$. So the tangent to the line of intersection is

$$v = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{vmatrix} = i(-1-1) - j(-1-3) + k(1-3) = \langle -2,4,-2 \rangle$$

To find a point on the line of intersection, we set $x = 0$ and solve the two plane equations for $y$ and $z$: $\quad y+z = 3 \qquad y-z = 1$
Add and subtract: $\quad 2y = 4,\ y = 2 \qquad 2z = 2,\ z = 1$. So the point is $p = \langle 0,2,1 \rangle$.
Thus the line is $X = p + tv$, or $\langle x,y,z \rangle = \langle 0,2,1 \rangle + t\langle -2,4,-2 \rangle$, or
$x = -2t \qquad y = 2+4t \qquad z = 1-2t$

OR

Simply set $x$ equal to a parameter,e.g. $x = T$, and solve the two plane equations for $y$ and $z$: $y+z = 3-T \qquad y-z = 1-3T$
Add and subtract: $\quad 2y = 4-4T,\ y = 2-2T \qquad 2z = 2+2T,\ z = 1+T$
Thus the line is $x = T \qquad y = 2-2T \qquad z = 1+T$
*There are many other acceptable forms of the solution differing in the choice of parameter and initial point.*