MATH 152

Final Exam

Fall 1997

Version A

Solutions

Part I is multiple choice. There is no partial credit. You may not use a calculator.

Part II is work out. Show all your work. Partial credit will be given. You may use your calculator.

Part I: Multiple Choice (5 points each)
There is no partial credit. You may not use a calculator. You have 1 hour.

1. Find the area between the curves $y = x^3$ and $y = x^{1/3}$ and within the first quadrant.

a. $\frac{1}{4}$
b. $\frac{1}{3}$
c. $\frac{1}{2}$ .... correctchoice
d. $\frac{2}{3}$
e. $\frac{3}{4}$

$x^3 = x^{1/3}$ at $x = 0, 1, -1$. In the first quadrant, when $0 \le x \le 1$ we have $x^{1/3} \ge x^3$. Thus
$A = \int_0^1 x^{1/3} - x^3 dx = \left[\frac{3x^{4/3}}{4} - \frac{x^4}{4} \right]_0^1 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$ (c)

2. Find the plane tangent to the monkey saddle $z = x^3 - 3xy^2$ at the point $(2,1,2)$. Its z-intercept is $z =$

a. 2
b. 0
c. -2
d. -4.. correctchoice
e. -6

$f(x,y) = x^3 - 3xy^2 \quad f_x(x,y) = 3x^2 - 3y^2 \quad f_y(x,y) = -6xy$
$f(2,1) = 2 \quad f_x(2,1) = 9 \quad f_y(2,1) = -12$
$z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 2 + 9(x-2) - 12(y-1) = 9x - 12y - 4$
So the z-intercept is -4. (d)

3. In the partial fraction expansion $\frac{2x+1}{(x^2+x+1)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$ the coefficients are

a. $A = -1, B = 0, C = 1$
b. $A = 1, B = -1, C = 0$.. correctchoice
c. $A = 0, B = 1, C = 1$
d. $A = 0, B = 0, C = -1$
e. $A = 1, B = -1, C = -1$

	$2x + 1 = A(x^2 + x + 1) + (Bx + C)(x - 1)$		
$x = 1$	$3 = A(3)$	$A = 1$	(b)
$x = 0$	$1 = A(1) + C(-1) = 1 - C$	$C = 0$	
$x = -1$	$-1 = A(1) + (-B + C)(-2) = 1 + 2B$	$B = -1$	

4. The area in the first quadrant between the hyperbola $y = \frac{3}{x}$ and the line $y = 4 - x$ is rotated about the x-axis. Find the volume of the solid swept out.

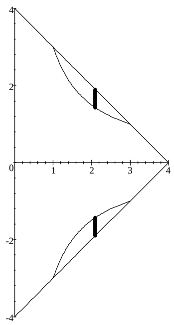
a. $\pi \int_1^3 \left(4 - x - \frac{3}{x}\right)^2 dx$

b. $2\pi \int_1^3 \left(4 - x - \frac{3}{x}\right)^2 dx$

c. $2\pi \int_1^3 \left(4 - x - \frac{3}{x}\right) dx$

d. $\pi \int_1^3 (4x - x^2 - 3) dx$

e. $\pi \int_1^3 (4 - x)^2 - \left(\frac{3}{x}\right)^2 dx$.. correctchoice



$\frac{3}{x} = 4 - x, \quad 3 = 4x - x^2, \quad x^2 - 4x + 3 = 0, \quad (x-1)(x-3) = 0, \quad x = 1,3$

Washer: $V = \int_1^3 \pi R^2 - \pi r^2 dx = \pi \int_1^3 (4 - x)^2 - \left(\frac{3}{x}\right)^2 dx$ (e)

5. Which limit does not exist?

a. $\lim\limits_{(x,y)\to(0,0)} (x^2 + 2y^2)$

b. $\lim\limits_{(x,y)\to(0,0)} (x^2 - 2y^2)$

c. $\lim\limits_{(x,y)\to(0,0)} \frac{x^4 + 2x^2y^2}{x^2 + y^2}$

d. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2 - 2y^2}{x^2 + y^2}$.. correctchoice

e. $\lim\limits_{(x,y)\to(0,0)} \frac{x^4 - 2x^2y^2}{x^2 + y^2}$

Try $y = mx$ in each limit. (d) gives

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{x^2 - 2y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2 - 2m^2x^2}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{1 - 2m^2}{1 + m^2} = \frac{1 - 2m^2}{1 + m^2}$$

which depends on m. So the limit does not exist. (d)

6. $\int_0^1 \frac{x^3}{x^2+1}\,dx =$

a. $\frac{1}{2}(1-\ln 2)$ .... correctchoice

b. $\frac{1}{2}(2-\ln 2)$

c. $\frac{1}{2}(2+\ln 2)$

d. $\frac{1}{2}(1-\ln 2+e)$

e. $\frac{1}{2}(2-\ln 2+e)$

Substitute: $u = x^2+1 \qquad du = 2x\,dx \qquad \frac{1}{2}du = x\,dx \qquad x^2 = u-1$

$$\int_0^1 \frac{x^3}{x^2+1}\,dx = \frac{1}{2}\int_1^2 \frac{u-1}{u}\,du = \frac{1}{2}\int_1^2 1-\frac{1}{u}\,du = \frac{1}{2}[u-\ln u]_1^2 = \frac{1}{2}[2-\ln 2]-\frac{1}{2}[1-\ln 1]$$

$= \frac{1}{2}[1-\ln 2]$ (a)

7. An airplane is circling above an airport, counterclockwise as seen from above. Thus its wings are banked with the left wing lower than the right wing. In what direction does the binormal $\vec{B}$ point?

a. horizontally toward the center of the circle

b. vertically up.. correctchoice

c. vertically down

d. along the left wing

e. along the right wing

$\vec{T}$ is horizontal tangent to the circle pointing counterclockwise.

$\vec{N}$ is horizontal toward the center of the circle.

$\vec{B} = \vec{T}\times\vec{N}$ is vertical and up by the right hand rule. (b)

8. Solve the initial value problem $xy' + y = e^x$ where $y(1) = e+1$.

a. $\frac{e^x+1}{x}$ .... correctchoice

b. $x(e^x-1)$

c. $x(e^x+1)$

d. $\frac{e^x}{x}+1$

e. xe^x-1

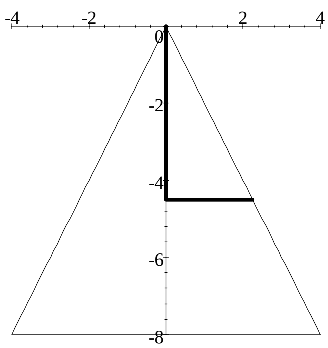
Standard form: $y' + \frac{y}{x} = \frac{e^x}{x}$ Integrating factor: $I = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$xy' + y = e^x \qquad (xy)' = e^x \qquad xy = \int e^x\,dx = e^x + C$

Initial condition: $1(e+1) = e^1 + C \qquad C = 1 \qquad xy = e^x + 1 \qquad y = \frac{e^x+1}{x}$ (a)

9. A conical reservoir with the point at the top is $8\,\mathrm{ft}$ deep and $4\,\mathrm{ft}$ in radius at the base. The liquid in the reservoir weighs $80\,\mathrm{lb/ft}^3$. How much work is done to pump the liquid out the top of the reservoir?

a. 640π ft-lb
b. $\dfrac{5120}{3}\pi$ ft-lb
c. $\dfrac{10240}{3}\pi$ ft-lb
d. 4096π ft-lb
e. 20480π ft-lb .... correctchoice



Measure $y > 0$ downward (in spite of the labels in the plot.)

The dark lines are the height $h = y > 0$ and the radius r.

$$\frac{r}{y} = \frac{4}{8} = \frac{1}{2} \qquad r = \frac{1}{2}y$$

$$W = \int \rho g h\, dV = \int_0^8 80 y \pi r^2 dy = \int_0^8 80 y \pi \frac{y^2}{4} dy = 20\pi \int_0^8 y^3 dy$$

$$= 20\pi \left[\frac{y^4}{4} \right]_0^8 = 5\pi 8^4 = 2^{11} 10\pi = 20480\pi \qquad \text{(e)}$$

10. A nuclear power plant is producing a radioactive isotope X at the rate of $50\,\mathrm{kg/yr}$. Let $N(t)\,\mathrm{kg}$ be the amount of X at the power plant at time t. A known fact is that X decays at the rate of $75N(t)\,\mathrm{kg/yr}$. Write the differential equation to be solved for $N(t)$.

a. $\dfrac{dN}{dt} = 75N - 50$
b. $\dfrac{dN}{dt} = 75N - 50t$
c. $\dfrac{dN}{dt} = 50 - 75N$ .... correctchoice
d. $\dfrac{dN}{dt} = 50t - 75N$
e. $\dfrac{dN}{dt} = -25N$

All terms must have the units of $\mathrm{kg/yr}$.

$$\frac{dN}{dt} = \underbrace{50}_{\text{created}} - \underbrace{75N}_{\text{decayed}} \qquad \text{(c)}$$

11. The function $f(x,y)$ has the values given below. Estimate $\frac{\partial f}{\partial x}(2.0,3.1)$.

$$\begin{array}{lll} f(2.0,3.2) = 3 & f(2.1,3.2) = 7 & f(2.2,3.2) = 9 \\ f(2.0,3.1) = 2 & f(2.1,3.1) = 4 & f(2.2,3.1) = 8 \\ f(2.0,3.0) = 1 & f(2.1,3.0) = 2 & f(2.2,3.0) = 5 \end{array}$$

a. 2
b. 15
c. 20.. correctchoice
d. 25
e. 30

$$\frac{\partial f}{\partial x}(2.0,3.1) = \lim_{h\to 0}\frac{f(2.0+h,3.1)-f(2.0,3.1)}{h} \approx \frac{f(2.1,3.1)-f(2.0,3.1)}{0.1} = \frac{4-2}{0.1} = 20 \qquad \text{(c)}$$

12. Which line is perpendicular to the plane $\left\{\begin{array}{c} x = 2+s-3t \\ y = -1-2s+t \\ z = 3-2t \end{array}\right\}$

a. $\left\{\begin{array}{c} x = 1-3t \\ y = 2+t \\ z = 3-2t \end{array}\right\}$

b. $\left\{\begin{array}{c} x = 2+s \\ y = -1-2s \\ z = 3 \end{array}\right\}$

c. $\left\{\begin{array}{c} x = 2t \\ y = -t \\ z = 3t \end{array}\right\}$

d. $\left\{\begin{array}{c} x = 4+2t \\ y = 2-t \\ z = -5+3t \end{array}\right\}$

e. $\left\{\begin{array}{c} x = 3+4t \\ y = 2+2t \\ z = 1-5t \end{array}\right\}$........................ correctchoice

Two tangent vectors to the plane are $\vec{u} = \langle 1,-2,0\rangle$ and $\vec{v} = \langle -3,1,-2\rangle$.
So the normal is

$$\vec{N} = \vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 0 \\ -3 & 1 & -2 \end{vmatrix} = \hat{i}(4-0)-\hat{j}(-2-0)+\hat{k}(1-6) = \langle 4,2,-5\rangle.$$

This normal is tangent to the line (e).

13. The legs of a right triangle are measured to be $6.0\,\text{in}$ and $8.0\,\text{in}$ each with a maximum error of $\pm 0.1\,\text{in}$. Hence the hypotenuse is $h = 10.0\,\text{in} \pm \Delta h$ with a maximum error of $\Delta h =$

a. $0.10\,\text{in}$

b. $0.11\,\text{in}$

c. $0.12\,\text{in}$

d. $0.13\,\text{in}$

e. $0.14\,\text{in}$ .... correctchoice

$$h = \sqrt{x^2 + y^2} = \sqrt{6^2 + 8^2} = 10$$

$$\Delta h \approx dh = \frac{\partial h}{\partial x}\Delta x + \frac{\partial h}{\partial y}\Delta y = \frac{x}{\sqrt{x^2 + y^2}}\Delta x + \frac{y}{\sqrt{x^2 + y^2}}\Delta y = \frac{6}{10}0.1 + \frac{8}{10}0.1 = .14 \qquad \text{(e)}$$

Part II: Work Out

Show all your work. Partial credit will be given.
You may use your calculator but only after 1 hour.

14. (10 points) Consider the curve $\boldsymbol{r}(t) = (1 + t^2, 1 - t^2, t)$. Compute each of the following:

a. velocity $\qquad \boldsymbol{v} = \langle 2t, -2t, 1 \rangle$

b. speed (Simplify.) $\qquad |\boldsymbol{v}| = \sqrt{4t^2 + 4t^2 + 1} = \sqrt{8t^2 + 1}$

c. acceleration $\qquad \boldsymbol{a} = \langle 2, -2, 0 \rangle$

d. curvature

$$\boldsymbol{v} \times \boldsymbol{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2t & -2t & 1 \\ 2 & -2 & 0 \end{vmatrix} = \hat{\imath}(0 - -2) - \hat{\jmath}(0 - 2) + \hat{k}(-4t - -4t) = \langle 2, 2, 0 \rangle$$

$$|\boldsymbol{v} \times \boldsymbol{a}| = \sqrt{4 + 4 + 0} = \sqrt{8} = 2\sqrt{2} \qquad \kappa = \frac{|\boldsymbol{v} \times \boldsymbol{a}|}{|\boldsymbol{v}|^3} = \frac{\sqrt{8}}{\sqrt{8t^2 + 1}^{\,3}}$$

e. tangential acceleration $\qquad a_T = \dfrac{d|\boldsymbol{v}|}{dt} = \dfrac{d}{dt}\sqrt{8t^2 + 1} = \dfrac{16t}{2\sqrt{8t^2 + 1}} = \dfrac{8t}{\sqrt{8t^2 + 1}}$

f. normal acceleration $\qquad a_N = \kappa |\boldsymbol{v}|^2 = \dfrac{\sqrt{8}}{\sqrt{8t^2 + 1}^{\,3}}\sqrt{8t^2 + 1}^{\,2} = \dfrac{\sqrt{8}}{\sqrt{8t^2 + 1}}$

15. (10 points) Compute $\int_0^{\pi/4} x\cos(2x)\,dx$

Use integration by parts: $u = x \quad dv = \cos(2x)\,dx \quad du = dx \quad v = \dfrac{\sin(2x)}{2}$

$$\int x\cos(2x)\,dx = \frac{x}{2}\sin(2x) - \frac{1}{2}\int \sin(2x)\,dx = \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x)$$

$$\int_0^{\pi/4} x\cos(2x)\,dx = \left[\frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x)\right]_0^{\pi/4} = \left[\frac{\pi}{8}\sin\left(\frac{\pi}{2}\right) + \frac{1}{4}\cos\left(\frac{\pi}{2}\right)\right] - \left[\frac{0}{2}\sin(0) + \right.$$

$$= \frac{\pi}{8} - \frac{1}{4}$$

16. (10 points) An ant is crawling on the saddle surface $z = x^2 - y^2$. At time $t = 3$, the ant is at the position $(x,y,z) = (2,1,3)$ and satisfies $\dfrac{dx}{dt} = 0.3$ and $\dfrac{dy}{dt} = 0.2$. Find $\dfrac{dz}{dt}$ at $t = 3$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = 2x\frac{dx}{dt} - 2y\frac{dy}{dt} = 2 \bullet 2 \bullet 0.3 - 2 \bullet 1 \bullet 0.2 = 1.2 - .4 = .8$$

17. (5 points) This is the direction field of a certain differential equation. Draw in the solution $y = y(x)$ which satisfies the initial condition $y(2) = -1$.