MATH 152

Final Exam

Fall 1997

Version B

Solutions

Part I is multiple choice. There is no partial credit. You may not use a calculator.

Part II is work out. Show all your work. Partial credit will be given. You may use your calculator.

Part I: Multiple Choice (5 points each)
There is no partial credit. You may not use a calculator. You have 1 hour.

1. Find the area between the parabola $y = x^2 - 2x$ and the line $y = x$.

a. $-\frac{10}{3}$
b. $\frac{7}{6}$
c. $\frac{13}{6}$
d. $\frac{10}{3}$
e. $\frac{9}{2}$ .... correctchoice

$x^2 - 2x = x \qquad x^2 - 3x = 0 \qquad x(x-3) = 0$ So $0 \le x \le 3$ Thus

$$A = \int_0^3 x - (x^2 - 2x)\,dx = \int_0^3 (3x - x^2)\,dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - 9 = \frac{9}{2}$$ (e)

2. Find the plane tangent to the hyperbolic paraboloid $z = x^2 - y^2$ at the point $(2,1,3)$. Its z-intercept is $z =$

a. 1
b. 0
c. -1
d. -2
e. -3.. correctchoice

$f(x,y) = x^2 - y^2 \quad f_x(x,y) = 2x \quad f_y(x,y) = -2y$

$f(2,1) = 3 \quad f_x(2,1) = 4 \quad f_y(2,1) = -2$

$z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 3 + 4(x-2) - 2(y-1) = 4x - 2y - 3$

So the z-intercept is -3. (e)

3. In the partial fraction expansion $\frac{x^2+4}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$ the coefficients are

a. $A = -1, B = 1, C = 2$
b. $A = -1, B = 0, C = 2$
c. $A = 1, B = 0, C = 4$.. correctchoice
d. $A = 0, B = 1, C = -4$
e. $A = 1, B = -1, C = -4$

$x^2 + 4 = A(x-2)^2 + Bx(x-2) + Cx$

$x = 0 \qquad 4 = A(-2)^2 \qquad A = 1$ (c)

$x = 2 \qquad 8 = C(2) \qquad C = 4$

$x = 1 \qquad 5 = A(-1)^2 + B(-1) + C(1) = 1 - B + 4 \qquad B = 0$

4. The area in the first quadrant between the hyperbola $y = \frac{3}{x}$ and the line $y = 4 - x$ is rotated about the y-axis. Find the volume of the solid swept out.

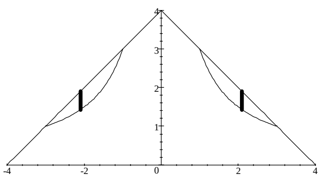
a. $\pi \int_1^3 \left(4 - x - \frac{3}{x}\right)^2 dx$

b. $2\pi \int_1^3 \left(4 - x - \frac{3}{x}\right)^2 dx$

c. $2\pi \int_1^3 \left(4 - x - \frac{3}{x}\right) dx$

d. $2\pi \int_1^3 (4x - x^2 - 3) dx$.. correctchoice

e. $2\pi \int_1^3 (4 - x)^2 - \left(\frac{3}{x}\right)^2 dx$



$\frac{3}{x} = 4 - x, \quad 3 = 4x - x^2, \quad x^2 - 4x + 3 = 0,$

$(x-1)(x-3) = 0, \quad x = 1, 3$ Cylinder:

$$V = \int_1^3 2\pi rh\,dx = 2\pi \int_1^3 x\left(4 - x - \frac{3}{x}\right) dx$$

$$= 2\pi \int_1^3 (4x - x^2 - 3) dx$$ (d)

5. Which limit does not exist?

a. $\lim_{(x,y)\to(0,0)} (2x^2 + y^2)$

b. $\lim_{(x,y)\to(0,0)} (2x^2 - y^2)$

c. $\lim_{(x,y)\to(0,0)} \frac{2x^2 - y^2}{x^2 + y^2}$.. correctchoice

d. $\lim_{(x,y)\to(0,0)} \frac{2x^4 + x^2y^2}{x^2 + y^2}$

e. $\lim_{(x,y)\to(0,0)} \frac{2x^4 - x^2y^2}{x^2 + y^2}$

Try $y = mx$ in each limit. (c) gives

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{2x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{2x^2 - m^2x^2}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{2 - m^2}{1 + m^2} = \frac{2 - m^2}{1 + m^2}$$

which depends on m. So the limit does not exist. (c)

6. $\int_1^{\sqrt{2}} x^3 \sqrt{x^2-1}\, dx =$

a. $\frac{2}{15}$

b. $\frac{8}{15}$ .... correctchoice

c. $\frac{22}{15}\sqrt{2} - \frac{8}{15}$

d. $\frac{8}{15}(2^{5/4} + 2^{3/4})$

e. $\frac{8}{15}(2^{5/4} + 2^{3/4} - 1)$

Substitute: $u = x^2 - 1 \qquad du = 2x\,dx \qquad \frac{1}{2}du = x\,dx \qquad x^2 = u + 1$

$$\int_1^{\sqrt{2}} x^3 \sqrt{x^2-1}\, dx = \frac{1}{2}\int_0^1 (u+1)\sqrt{u}\, du = \frac{1}{2}\int_0^1 (u^{3/2} + u^{1/2})\, du = \frac{1}{2}\left[\frac{2u^{5/2}}{5} + \frac{2u^{3/2}}{3} \right]_0^1$$

$$= \frac{1}{2}\left[\frac{2}{5} + \frac{2}{3} \right] = \left[\frac{1}{5} + \frac{1}{3} \right] = \frac{8}{15}$$ (b)

7. An airplane is circling above an airport, clockwise as seen from above. Thus its wings are banked with the right wing lower than the left wing. In what direction does the binormal $\vec{B}$ point?

a. horizontally toward the center of the circle

b. vertically up

c. vertically down.. correctchoice

d. along the left wing

e. along the right wing

$\vec{T}$ is horizontal tangent to the circle pointing clockwise.

$\vec{N}$ is horizontal toward the center of the circle.

$\vec{B} = \vec{T} \times \vec{N}$ is vertical and down by the right hand rule. (c)

8. Solve the differential equation $\frac{dy}{dx} = 28x\sqrt{y}$. If $y(0) = 4$, then $y(1) =$

a. 81.. correctchoice

b. 53

c. 49

d. 11

e. 9

Separate variables: $\frac{dy}{\sqrt{y}} = 28x\,dx$ Integrate: $\int \frac{dy}{\sqrt{y}} = \int 28x\,dx \qquad 2\sqrt{y} = 14x^2 + C$

Initial condition: $x = 0, y = 4$: $\quad 2\sqrt{4} = C = 4 \qquad 2\sqrt{y} = 14x^2 + 4 \qquad y = (7x^2+2)^2$

$y(1) = 9^2 = 81$ (a)

9. A $50\,\text{ft}$ cable is hanging from the roof down the side of a tall building. If the cable weighs $3\,\text{lb/ft}$, how much work is done to lift the cable to the roof?

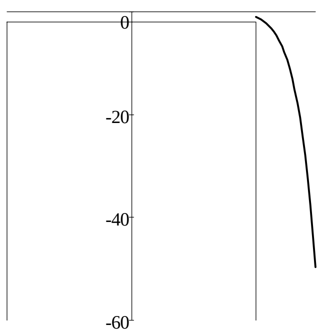
a. $7500\,\text{ft-lb}$

b. $3750\,\text{ft-lb}$ .... correctchoice

c. $1875\,\text{ft-lb}$

d. $150\,\text{ft-lb}$

e. $75\,\text{ft-lb}$



An element of cable $y\,\text{ft}$ down of length dy weighs $dF = g\rho\,dy = 3\,dy$ and must be lifted $h = y\,\text{ft}$.

$$W = \int_0^{50} h\ dF = \int_0^{50} y\ g\rho\,dy = \left[\frac{\rho g y^2}{2}\right]_0^{50}$$

$$= \frac{3}{2}50^2 = 3750 \qquad \text{(b)}$$

10. A nuclear power plant is producing a radioactive isotope X at the rate of $75\,\text{kg/yr}$. Let $N(t)\,\text{kg}$ be the amount of X at the power plant at time t. A known fact is that X decays at the rate of $50N(t)\,\text{kg/yr}$. Write the differential equation to be solved for $N(t)$.

a. $\dfrac{dN}{dt} = 50N - 75$

b. $\dfrac{dN}{dt} = 75 - 50N$ .... correctchoice

c. $\dfrac{dN}{dt} = 50N - 75t$

d. $\dfrac{dN}{dt} = 75t - 50N$

e. $\dfrac{dN}{dt} = -25N$

All terms must have the units of kg/yr.

$$\frac{dN}{dt} = \underbrace{75}_{\text{created}} - \underbrace{50N}_{\text{decayed}} \qquad \text{(b)}$$

11. The function $f(x,y)$ has the values given below. Estimate $\frac{\partial f}{\partial y}(2.1,3.0)$.

$f(2.0,3.2) = 3$ $f(2.1,3.2) = 7$ $f(2.2,3.2) = 9$

$f(2.0,3.1) = 2$ $f(2.1,3.1) = 4$ $f(2.2,3.1) = 8$

$f(2.0,3.0) = 1$ $f(2.1,3.0) = 2$ $f(2.2,3.0) = 5$

a. 2
b. 10
c. 15
d. 20.. correctchoice
e. 30

$$\frac{\partial f}{\partial y}(2.1,3.0) = \lim_{h\to 0}\frac{f(2.1,3.0+H)-f(2.1,3.0)}{h} \approx \frac{f(2.1,3.1)-f(2.1,3.0)}{0.1} = \frac{4-2}{0.1} = 20$$ (d)

12. Which of the planes described below contain the lines:

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-1}{4} \quad \text{and} \quad \frac{x-1}{4} = \frac{y-2}{3} = \frac{z-1}{2} \ ?$$

a. $x-2y+z = -2$.. correctchoice
b. $x-2y+z = 2$
c. $2x-y+z = -1$
d. $2x-y+z = -1$
e. $x-y+z = 0$

Line 1 has direction $\vec{u} = \langle 2,3,4\rangle$. Line 2 has direction $\vec{v} = \langle 4,3,2\rangle$. So the normal is

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 4 & 3 & 2 \end{vmatrix} = \hat{i}(6-12)-\hat{j}(4-16)+\hat{k}(6-12) = \langle -6,12,-6\rangle = -6\langle 1,-2,1\rangle$$

$\vec{N} = \langle 1,-2,1\rangle$ Both lines pass through $P = (1,2,1)$. So the equation is

$N \bullet X = N \bullet P$ $1x-2y+1z = 1\bullet 1-2\bullet 2+1\bullet 1 = -2$ (a)

13. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$. If the radius is measured as $r = 3 \pm .2$, and the height is measured as $h = 4 \pm .1$, then the volume is computed as $V = 12\pi \pm \Delta V$ where the error in the measurement of the volume is $\Delta V =$

- **a.** 1.1π
- **b.** 1.7π
- **c.** 1.9π .... correctchoice
- **d.** 2.2π
- **e.** 2.7π

$$\Delta V \approx dV = \frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial h}\Delta h = \frac{2}{3}\pi r h \Delta r + \frac{1}{3}\pi r^2 \Delta h = \frac{2}{3}\pi 3 \bullet 4 \bullet .2 + \frac{1}{3}\pi (3)^2 .1$$

$$= \pi(1.6 + .3) = 1.9\pi$$ (c)

Part II: Work Out

Show all your work. Partial credit will be given.
You may use your calculator but only after 1 hour.

14. (10 points) Consider the curve $\boldsymbol{r}(t) = (1 - t^2, 1 + t^2, t)$. Compute each of the following:

a. velocity $\boldsymbol{v} = \langle -2t, 2t, 1 \rangle$

b. speed (Simplify.) $|\boldsymbol{v}| = \sqrt{4t^2 + 4t^2 + 1} = \sqrt{8t^2 + 1}$

c. acceleration $\boldsymbol{a} = \langle -2, 2, 0 \rangle$

d. curvature

$$\boldsymbol{v} \times \boldsymbol{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2t & 2t & 1 \\ -2 & 2 & 0 \end{vmatrix} = \hat{\imath}(0 - 2) - \hat{\jmath}(0 - -2) + \hat{k}(-4t - -4t) = \langle -2, -2, 0 \rangle$$

$$|\boldsymbol{v} \times \boldsymbol{a}| = \sqrt{4 + 4 + 0} = \sqrt{8} = 2\sqrt{2} \qquad \kappa = \frac{|\boldsymbol{v} \times \boldsymbol{a}|}{|\boldsymbol{v}|^3} = \frac{\sqrt{8}}{\sqrt{8t^2 + 1}^3}$$

e. tangential acceleration $a_T = \dfrac{d|\boldsymbol{v}|}{dt} = \dfrac{d}{dt}\sqrt{8t^2 + 1} = \dfrac{16t}{2\sqrt{8t^2 + 1}} = \dfrac{8t}{\sqrt{8t^2 + 1}}$

f. normal acceleration $a_N = \kappa |\boldsymbol{v}|^2 = \dfrac{\sqrt{8}}{\sqrt{8t^2 + 1}^3}\sqrt{8t^2 + 1}^2 = \dfrac{\sqrt{8}}{\sqrt{8t^2 + 1}}$

15. (10 points) Compute $\displaystyle\int_0^{3/4} \frac{1}{\sqrt{1+x^2}}\,dx$

Use a tan substitution: $x = \tan\theta \qquad dx = \sec^2\theta\,d\theta \qquad \sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sec\theta$

$$\int_0^{3/4} \frac{1}{\sqrt{1+x^2}}\,dx = \int_{x=0}^{3/4} \frac{1}{\sec\theta}\sec^2\theta\,d\theta = \int_{x=0}^{3/4} \sec\theta\,d\theta = [\ln(\sec\theta+\tan\theta)]_{x=0}^{3/4} = \Big[\ln\Big(\sqrt{1+x^2} +$$

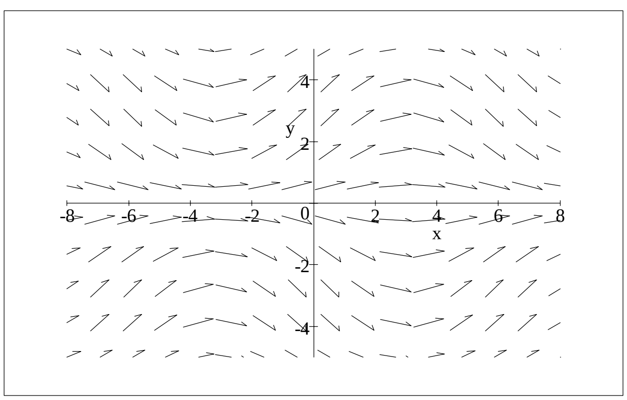
$$= \ln\left(\sqrt{1+\left(\frac{3}{4}\right)^2} + \frac{3}{4}\right) - \ln\left(\sqrt{1+0^2}+0\right) = \ln\left(\sqrt{\frac{25}{16}} + \frac{3}{4}\right) - \ln(1) = \ln\left(\frac{8}{4}\right) = \ln 2$$

16. (10 points) A comet has just passed by the sun.
Its current position is $(x,y) = (3\times10^7\,\text{mi}, 4\times10^7\,\text{mi})$
and its current velocity is $\boldsymbol{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \left(4\times10^7\,\frac{\text{mi}}{\text{day}}, 7\times10^7\,\frac{\text{mi}}{\text{day}}\right)$.
Hence the current distance from the sun is $R = \sqrt{x^2+y^2} = 5\times10^7\,\text{mi}$.
Find the rate at which the distance from the sun is increasing.

$$\frac{dR}{dt} = \frac{\partial R}{\partial x}\frac{dx}{dt} + \frac{\partial R}{\partial y}\frac{dy}{dt} = \frac{x}{\sqrt{x^2+y^2}}\frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2}}\frac{dy}{dt} = \frac{3\times10^7}{5\times10^7}\,4\times10^7 + \frac{4\times10^7}{5\times10^7}\,7\times10$$

$$= \left(\frac{3}{5}\,4 + \frac{4}{5}\,7\right)\times10^7 = \frac{40}{5}\times10^7 = 8\times10^7$$

17. (5 points) This is the direction field of a certain differential equation. Draw in the solution $y = y(x)$ which satisfies the initial condition $y(2) = 3$.



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