Spring 2006 Math 152
Exam 2A: Solutions
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1. (c) The arc length of the curve

$$\mathbf{r}(t) = [x(t), y(t)] = \left[t^2, t^2 + t\right], 0 \le t \le 1,$$

is represented by the following integral.

$$\begin{aligned} L = \int_a^b \left\|\mathbf{r}'(t)\right\| dt &= \int_0^1 \|[2t, 2t+1]\| \, dt \\ &= \int_0^1 \sqrt{4t^2 + 4t^2 + 4t + 1}\, dt \\ &= \int_0^1 \sqrt{1 + 4t + 8t^2}\, dt \end{aligned}$$

2. (c) The differential equation $\frac{dy}{dx} = xy^2 + x^2 y = xy\,(y + x)$ is *not* separable. Other choices *are* separable, as follows.

- (a) Rewrite $\frac{dy}{dx} = \sin x \cos y$ as $\sec y\, dy = \sin x\, dx$.
- (b) Rewrite $\frac{dy}{dx} = xy + x^2 y = y\left(x + x^2\right)$ as $\frac{1}{y}\, dy = \left(x + x^2\right) dx$.
- (d) Rewrite $\frac{dy}{dx} = e^{x+y} = e^x e^y$ as $e^{-y} dy = e^x\, dx$.
- (e) Repeated factoring gives

$$\begin{aligned} \frac{dy}{dx} &= 1 + x + y + xy \\ \frac{dy}{dx} &= (1 + x) + y\,(1 + x) \\ \frac{dy}{dx} &= (1 + x)\,(1 + y)\,, \end{aligned}$$

whence $\frac{1}{1+y}\, dy = (1 + x)\, dx$.

3. (d) The linear differential equation

$$y' + (2 \sin 2x)\, y = \cos 4x$$

is already in standard linear form (SLF). Accordingly, an integrating factor is $\mu = \exp\left(\int 2 \sin 2x\, dx\right) = e^{-\cos 2x}$.

4. (b) Let $y = y(t)$ be the amount of salt in the tank at time t. The classical balance law gives

$$\begin{aligned} \frac{dy}{dt} &= \text{rate in} - \text{rate out} \\ \frac{dy}{dt} &= \left(0.1\, \frac{\text{kg}}{\text{L}} \times 10\, \frac{\text{L}}{\text{min}}\right) - \left(\frac{y \text{ kg}}{100 \text{ L}} \times 10\, \frac{\text{L}}{\text{min}}\right) \\ \frac{dy}{dt} &= 1 - \frac{y}{10} \quad \frac{\text{kg}}{\text{min}}. \end{aligned}$$

Since the tank initially contains pure water, we have $y(0) = 0$ kg of salt in the tank at the start. Therefore, $\frac{dy}{dt} = 1 - \frac{y}{10}$, $y\,(0) = 0$.

5. (a) The integral $\int_1^\infty \frac{x}{1 + x^4}\, dx$ *converges* by comparison to $\int_1^\infty \frac{1}{x^3}\, dx$. First note that the integrand $\frac{x}{1+x^4}$ is positive on $[1, \infty)$. We then have

$$\begin{aligned} \int_1^\infty \frac{x}{1+x^4}\, dx &\le \int_1^\infty \frac{x}{x^4}\, dx = \int_1^\infty \frac{1}{x^3}\, dx \\ &= \lim_{t\to\infty} \int_1^t x^{-3}\, dx \\ &= \lim_{t\to\infty} \left(-\tfrac{1}{2} x^{-2}\Big|_1^t\right) \\ &= \lim_{t\to\infty} \left(\frac{-1}{2t^2} - \left(-\frac{1}{2}\right)\right) = \frac{1}{2} = 0.50. \end{aligned}$$

Hence $\int_1^\infty \frac{x}{1+x^4}\, dx$ converges by the Comparison Theorem to a value $L \le \frac{1}{2}$. [NOTE: This integral is easy to compute directly as follows. Although you were not required to do this, it provides a nice independent check!]

$$\begin{aligned} \int_1^\infty \frac{x}{1+x^4}\, dx &= \lim_{t\to\infty} \int_1^t \frac{x}{1 + \left(x^2\right)^2}\, dx \\ &= \lim_{t\to\infty} \tfrac{1}{2} \tan^{-1}\left(x^2\right)\Big|_1^t \\ &= \lim_{t\to\infty} \left(\tfrac{1}{2} \tan^{-1}\left(t^2\right) - \tfrac{1}{2}\left(\tfrac{\pi}{4}\right)\right) \\ &= \tfrac{1}{2}\left(\tfrac{\pi}{2}\right) - \tfrac{1}{2}\left(\tfrac{\pi}{4}\right) = \tfrac{\pi}{4} - \tfrac{\pi}{8} = \tfrac{\pi}{8} \\ &\approx 0.39 \le 0.50 \end{aligned}$$

6. (b) The plate has constant density. Its semicircular area is $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\,(4)^2 = 8\pi$. Accordingly, the x-coordinate of the center of mass is given by

$$\begin{aligned} \bar{x} &= \frac{1}{A} \int_a^b x\,(f\,(x) - g\,(x))\, dx \\ &= \frac{1}{8\pi} \int_0^4 x\left(\sqrt{16 - x^2} - \left(-\sqrt{16 - x^2}\right)\right) dx \\ &= \frac{1}{8\pi} \int_0^4 \left(16 - x^2\right)^{1/2} \cdot 2x\, dx \quad (\text{Sub: } u = 16 - x^2) \\ &= \frac{-1}{8\pi} \int_{16}^0 u^{1/2}\, du = \frac{1}{8\pi} \int_0^{16} u^{1/2}\, du \\ &= \frac{1}{8\pi}\left(\frac{2}{3}\right) u^{3/2}\Big|_0^{16} = \frac{16}{3\pi} - 0 = \frac{16}{3\pi} \approx 1.70. \end{aligned}$$

7. (e) The step size is $h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$. Hence

T_n = step size × (average of endpoint func vals + sum of interior func vals)

$$\begin{aligned} T_4 &= \frac{1}{2}\left(\frac{1.00 + 0.70}{2} + (0.25 + 0.40 + 0.20)\right) \\ &= \frac{1}{2}\,(0.85 + 0.85) = 0.85. \end{aligned}$$

8. (c) The step size is $h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$. Hence

M_n = step size × (sum of midpoint function values)

$$\begin{aligned} M_4 &= \frac{1}{2}\,(0.50 + 0.75 + 0.30 + 0.10) \\ &= \frac{1}{2}\,(1.65) = 0.825. \end{aligned}$$

9. (e) The differential equation $\dfrac{dy}{dx} = \left(2+3x^2\right)\left(y^2+1\right)$ is separable.

$$\begin{aligned}\frac{1}{1+y^2}\,dy &= \left(2+3x^2\right)\,dx\\ \tan^{-1} y &= 2x + x^3 + C\\ y &= \tan\left(2x+x^3+C\right)\end{aligned}$$

10. (d) The differential equation $\dfrac{dy}{dx} = 2y$ is separable. Find a general solution, then resolve the constant of integration using the initial condition $y(0) = 4$.

$$\begin{aligned}\frac{1}{y}\,dy &= 2\,dx\\ \ln|y| &= 2x + A\\ \pm y = |y| &= e^{2x+A} = e^{2x}e^{A} = Be^{2x}\\ y &= Ce^{2x}\\ \text{Substitute data: } 4 &= Ce^{0} = C\\ y &= 4e^{2x}\\ \text{Thus } y(1) &= 4e^2 \approx 29.56.\end{aligned}$$

11. The arc length of the curve $y = \dfrac{x^2}{4} - \dfrac{\ln x}{2}$, $1 \le x \le 2$, is

$$\begin{aligned}L &= \int ds = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\\ &= \int_1^2 \sqrt{1+\left(\frac{x}{2}-\frac{1}{2x}\right)^2}\,dx\\ &= \int_1^2 \sqrt{1+\left(\frac{x}{2}\right)^2 - \frac{1}{2} + \left(\frac{1}{2x}\right)^2}\,dx\\ &= \int_1^2 \sqrt{\left(\frac{x}{2}\right)^2 + \frac{1}{2} + \left(\frac{1}{2x}\right)^2}\,dx\\ &= \int_1^2 \sqrt{\left(\frac{x}{2}+\frac{1}{2x}\right)^2}\,dx\\ &= \int_1^2 \frac{1}{2}x + \frac{1}{2}\,\frac{1}{x}\,dx\\ &= \left(\tfrac{1}{4}x^2 + \tfrac{1}{2}\ln x\right)\Big|_1^2\\ &= \left(1+\tfrac{1}{2}\ln 2\right) - \left(\tfrac{1}{4}\right) = \tfrac{3}{4} + \tfrac{1}{2}\ln 2 \approx 1.10.\end{aligned}$$

12. The differential equation $xy' + 2y = x^3$ is *linear*.

- Put the equation into standard linear form (SLF).

$$y' + \frac{2}{x}\,y = x^2$$

  Here $P(x) = \dfrac{2}{x}$, the coefficient of y in the SLF.

- Construct an integrating factor.

$$\mu = \exp\left(\int P(x)\,dx\right) = \exp\left(\int \tfrac{2}{x}\,dx\right) = e^{2\ln x} = x^2$$

- Multiply the SLF by μ.

$$x^2y' + 2xy = x^4 \qquad \text{or} \qquad \left(x^2y\right)' = x^4$$

- Integrate to obtain $x^2y = \frac{1}{5}x^5 + C$. Therefore,

$$y = \tfrac{1}{5}x^3 + Cx^{-2}$$

  is a general solution.

- Use the initial condition $y(-1) = 2$ to determine C and thus the unique solution to the initial value problem.

$$\begin{aligned}2 = y(-1) &= -\tfrac{1}{5} + C\\ C &= \tfrac{10}{5} + \tfrac{1}{5} = \tfrac{11}{5}\\ y &= \tfrac{1}{5}x^3 + \tfrac{11}{5}x^{-2}\end{aligned}$$

13. (a) The integral $\displaystyle\int_0^\infty \frac{x}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{1+x^2}\,dx$ diverges to ∞ via direct computation.

$$\lim_{t\to\infty} \tfrac{1}{2}\ln\left(1+x^2\right)\Big|_0^t = \lim_{t\to\infty}\left(\tfrac{1}{2}\ln\left(1+t^2\right) - 0\right) = \infty$$

(b) The integral $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{t\to 2^-}\int_0^t \frac{1}{\sqrt{4-x^2}}\,dx$ converges to $\frac{1}{2}\pi$ as follows.

$$\begin{aligned}\lim_{t\to 2^-}\int_0^t \frac{1}{\sqrt{4-x^2}}\,dx &= \lim_{t\to 2^-}\frac{1}{2}\int_0^t \frac{1}{\sqrt{1-\left(\frac{1}{2}x\right)^2}}\,dx\\ &= \lim_{t\to 2^-}\left(\sin^{-1}\left(\frac{x}{2}\right)\right)\Big|_0^t\\ &= \lim_{t\to 2^-}\left(\sin^{-1}\left(\frac{t}{2}\right) - 0\right) = \frac{\pi}{2}\end{aligned}$$

14. The arc length differential is $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$. The surface area obtained by rotating the curve $y = x^3$, $0 \le x \le 1$, about the x-axis is

$$\begin{aligned}S &= \int 2\pi r\,ds\\ &= \int 2\pi y\,ds\\ &= 2\pi\int_0^1 x^3\sqrt{1+\left(3x^2\right)^2}\,dx\\ &= 2\pi\int_0^1 \left(1+9x^4\right)^{1/2} x^3\,dx \quad (\text{Sub: } u = 1+9x^4)\\ &= \frac{2\pi}{36}\int_1^{10} u^{1/2}\,du = \frac{\pi}{18}\int_1^{10} u^{1/2}\,du\\ &= \frac{\pi}{18}\left(\frac{2}{3}\right)u^{3/2}\Big|_1^{10} = \frac{\pi}{27}\left(10\sqrt{10}\right) - \frac{\pi}{27}\\ &= \frac{\pi}{27}\left(10\sqrt{10}-1\right) \approx 3.56.\end{aligned}$$