Name________________________________

1-8	/40
9	/15
10	/15
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Total	/100

MATH 152H Exam 1 Spring 2017

Sections 203/204 (circle one) Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the area between, $y = x^4$ and $y = 8x$.

a. 20

b. 12

c. $\frac{112}{5}$

d. $\frac{56}{5}$

e. $\frac{48}{5}$ correct choice

Solution: The curves intersect when $x^4 = 8x$ or $x = 0, 2$.

$$A = \int_0^2 (8x - x^4)\,dx = \left[4x^2 - \frac{x^5}{5}\right]_0^2 = 16 - \frac{32}{5} = 16\left(1 - \frac{2}{5}\right) = \frac{48}{5}$$

2. Find the area between the cubic $y = x^3 + x^2$ and the line $y = 2x$.

a. $-\frac{27}{12}$

b. $\frac{27}{12}$

c. $\frac{37}{12}$ correct choice

d. $\frac{47}{12}$

e. $\frac{57}{12}$

Solution: The curves intersect when $x^3 + x^2 = 2x$ or $x^3 + x^2 - 2x = 0$
or $x(x^2 + x - 2) = 0$ or $x(x+2)(x-1) = 0$ or $x = -2, 0, 1$

Between -2 and 0, the cubic is above the line. (Plug in $x = -1$.)

Between 0 and 1, the line is above the cubic. (Plug in $x = 1/2$.)

So the area is

$$A = \int_{-2}^0 (x^3 + x^2 - 2x)\,dx + \int_0^1 (2x - x^3 - x^2)\,dx = \left[\frac{x^4}{4} + \frac{x^3}{3} - x^2\right]_{-2}^0 + \left[x^2 - \frac{x^4}{4} - \frac{x^3}{3}\right]_0^1$$

$$= (0) - \left(4 + \frac{-8}{3} - 4\right) + \left(1 - \frac{1}{4} - \frac{1}{3}\right) - (0) = \frac{8}{3} + \frac{5}{12} = \frac{37}{12}$$

3. Compute $\int_1^e x^3 \ln x\, dx$

a. $\frac{1}{16}$

b. $\frac{1}{4}$

c. $\frac{3e^4}{16}$

d. $\frac{3e^4+1}{16}$ correct choice

e. $\frac{3e^4+1}{4}$

Solution: Use integration by parts with $u = \ln x \qquad dv = x^3\, dx$

$du = \frac{1}{x}\, dx \qquad v = \frac{x^4}{4}$

$$\int x^3 \ln x\, dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\frac{1}{x}\, dx = \frac{x^4}{4}\ln x - \int \frac{x^3}{4}\, dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$$

$$\int_1^e x^3 \ln x\, dx = \left[\frac{x^4}{4}\ln x - \frac{x^4}{16}\right]_1^e = \left(\frac{e^4}{4}\ln e - \frac{e^4}{16}\right) - \left(\frac{1^4}{4}\ln 1 - \frac{1^4}{16}\right) = \frac{e^4}{4} - \frac{e^4}{16} + \frac{1}{16} = \frac{3e^4+1}{16}$$

Check: $\frac{d}{dx}\left(\frac{x^4}{4}\ln x - \frac{x^4}{16}\right) = x^3 \ln x + \frac{x^4}{4}\frac{1}{x} - \frac{4x^3}{16} = x^3 \ln x$

4. Compute $\int_0^{\pi/6} \sin(2\theta)\cos^2(\theta)\, d\theta$

a. $\frac{-9}{32}$

b. $\frac{-1}{32}$

c. $\frac{7}{32}$ correct choice

d. $\frac{9}{32}$

e. $\frac{15}{32}$

Solution: Use the identity $\sin(2\theta) = 2\sin\theta\cos\theta$: $\int \sin(2\theta)\cos^2(\theta)\, d\theta = \int 2\sin\theta\cos^3\theta\, d\theta$

Next use the substitution $u = \cos\theta \qquad du = -\sin\theta\, d\theta$

$$\int \sin(2\theta)\cos(\theta)\, d\theta = -2\int u^3\, du = -2\frac{u^4}{4} = -\frac{1}{2}\cos^4\theta + C$$

$$\int_0^{\pi/6} \sin(2\theta)\cos^2(\theta)\, d\theta = -\frac{1}{2}\cos^4\theta\Big|_0^{\pi/6} = -\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^4 + \frac{1}{2} = \frac{1}{2} - \frac{9}{32} = \frac{7}{32}$$

Check: $\frac{d}{d\theta}\left(-\frac{1}{2}\cos^4\theta\right) = 2\cos^3\theta\sin\theta$

5. Find the mass of a 2 meter bar whose density is $\delta(x) = 4x - x^3$ where x is measured (in meters) from one end.

a. 2

b. 4 correct choice

c. 8

d. 16

e. 32

Solution: $M = \int_0^2 \delta(x)\,dx = \int_0^2 (4x - x^3)\,dx = \left[2x^2 - \frac{x^4}{4}\right]_0^2 = 8 - 4 = 4$

6. Find the center of mass of a 2 meter bar whose density is $\delta(x) = 4x - x^3$ where x is measured (in meters) from one end.

a. $\frac{15}{16}$

b. $\frac{15}{64}$

c. $\frac{64}{15}$

d. $\frac{32}{15}$

e. $\frac{16}{15}$ correct choice

Solution: From (1), $M = 4$.

$$M_1 = \int_0^2 x\,\delta(x)\,dx = \int_0^2 x(4x - x^3)\,dx = \int_0^2 (4x^2 - x^4)\,dx = \left[4\frac{x^3}{3} - \frac{x^5}{5}\right]_0^2$$

$$= \left(4\frac{2^3}{3} - \frac{2^5}{5}\right) = 32\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{64}{15}$$

$\bar{x} = \frac{M_1}{M} = \frac{64}{15}\frac{1}{4} = \frac{16}{15}$ Reasonable because $0 < \bar{x} < 2$.

7. Find the average density of a 2 meter bar whose density is $\delta(x) = 4x - x^3$ where x is measured (in meters) from one end.

a. 2 correct choice

b. 4

c. 8

d. 16

e. 32

Solution: From (1), $M = \int_0^2 \delta(x)\,dx = 4$. $\delta_{\text{ave}} = \frac{1}{2}\int_0^2 \delta(x)\,dx = \frac{1}{2}4 = 2$

8. Compute $\int_0^{\pi/4} \sec^4\theta \tan^2\theta \, d\theta$

a. $-\frac{8}{15}$
b. $-\frac{2}{15}$
c. $\frac{2}{15}$
d. $\frac{8}{15}$ correct choice
e. $\frac{128}{15}$

Solution: $u = \tan\theta \qquad du = \sec^2\theta \, d\theta \qquad \sec^2\theta = \tan^2\theta + 1 = u^2 + 1$

$$\int_0^{\pi/4} \sec^4\theta \tan^2\theta \, d\theta = \int_0^1 (u^2+1)u^2 \, du = \int_0^1 (u^4+u^2) \, du = \left[\frac{u^5}{5} + \frac{u^3}{3} \right]_0^1$$

$$= \left(\frac{1}{5} + \frac{1}{3} \right) = \frac{8}{15}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

9. $(15$ points$)$ Find the point (a, e^{2a}) on the graph of the curve $y = e^{2x}$ where the tangent line passes through the point $(b, 0)$ where b is a fixed but unspecified number.
HINT: In your answer, express a and the point in terms of b.

Solution: We first define the function, compute its derivative and evaluate both at $x = a$.

$$f(x) = e^{2x} \qquad f'(x) = 2e^{2x} \qquad f(a) = e^{2a} \qquad f'(a) = 2e^{2a}$$

The tangent line at $x = a$ is:

$$y = f_{\text{tan}}(x) = f(a) + f'(a)(x-a) = e^{2a} + 2e^{2a}(x-a)$$

Since the point $(b, 0)$ lies on the line:

$$0 = e^{2a} + 2e^{2a}(b-a)$$

We solve for a:

$$0 = 1 + 2(b-a) = 1 + 2b - 2a$$

$$2a = 1 + 2b$$

$$a = b + \frac{1}{2}$$

So the point is $\left(b + \frac{1}{2}, e^{2b+1} \right)$.

10. (15 points) Derive a reduction formula which gives

$$\int (\ln x)^n \, dx \quad \text{in terms of} \quad \int (\ln x)^{n-1} \, dx \quad \text{and} \quad \int (\ln x)^{n-2} \, dx.$$

HINT: You need to know $\int \ln x \, dx$.

Solution: Use integration by parts with

$$u = (\ln x)^{n-1} \qquad dv = \ln x \, dx$$

$$du = (n-1)(\ln x)^{n-2} \frac{1}{x} \, dx \qquad v = x \ln x - x$$

Then

$$\begin{aligned}
\int (\ln x)^n \, dx &= (\ln x)^{n-1}(x \ln x - x) - \int (x \ln x - x)(n-1)(\ln x)^{n-2} \frac{1}{x} \, dx \\
&= (\ln x)^{n-1}(x \ln x - x) - (n-1) \int (\ln x - 1)(\ln x)^{n-2} \, dx \\
&= x(\ln x)^n - x(\ln x)^{n-1} - (n-1)\left[\int (\ln x)^{n-1} \, dx - \int (\ln x)^{n-2} \, dx \right]
\end{aligned}$$

11. (15 points) Compute $\int_1^4 \cos(\sqrt{x}\,) \, dx$.

a. First, compute $\int \cos(\sqrt{x}\,) \, dx$.

Solution: Use the substitution $w = \sqrt{x}$. Then $dw = \frac{1}{2\sqrt{x}} \, dx$. Then

$$\int \cos(\sqrt{x}\,) \, dx = \int \cos(w) \, 2\sqrt{x} \, dw = \int 2w \cos(w) \, dw$$

Use integration by parts with

$$u = 2w \qquad dv = \cos(w) \, dw$$

$$du = 2dw \qquad v = \sin(w)$$

Then

$$\begin{aligned}
\int \cos(\sqrt{x}\,) \, dx &= 2w \sin(w) - 2 \int \sin(w) \, dw \\
&= 2w \sin(w) + 2\cos(w) + C \\
&= 2\sqrt{x} \sin(\sqrt{x}\,) + 2\cos(\sqrt{x}\,) + C
\end{aligned}$$

b. Check your answer by differentiating.

Solution: Let $f = 2\sqrt{x} \sin(\sqrt{x}\,) + 2\cos(\sqrt{x}\,)$. Then

$$f' = \frac{2}{2\sqrt{x}} \sin(\sqrt{x}\,) + 2\sqrt{x} \cos(\sqrt{x}\,) \frac{1}{2\sqrt{x}} - 2\sin(\sqrt{x}\,) \frac{1}{2\sqrt{x}} = \cos(\sqrt{x}\,)$$

c. Compute $\int_1^4 \cos(\sqrt{x}\,) \, dx$.

Solution:

$$\begin{aligned}
\int_1^4 \cos(\sqrt{x}\,) \, dx &= \left[2\sqrt{x} \sin(\sqrt{x}\,) + 2\cos(\sqrt{x}\,) \right]_1^4 \\
&= \left(2\sqrt{4} \sin\left(\sqrt{4}\right) + 2\cos\left(\sqrt{4}\right) \right) - \left(2\sqrt{1} \sin\left(\sqrt{1}\right) + 2\cos\left(\sqrt{1}\right) \right) \\
&= 4\sin(2) + 2\cos(2) - 2\sin(1) - 2\cos(1)
\end{aligned}$$

12. (15 points) Compute $\int_{13}^{15} \frac{\sqrt{x^2-144}}{x}\,dx$.

a. First, compute $\int \frac{\sqrt{x^2-144}}{x}\,dx$. Be sure to explain why you picked the substitution you use.

Solution: The minus in the square root says we need one of the trig substitutions:

$$x = 12\sin\theta \quad \text{or} \quad x = 12\sec\theta.$$

The square root says $x^2 \geq 144$. So we need the substitution $x = 12\sec\theta$.
Then $dx = 12\sec\theta\tan\theta\,d\theta$ and

$$\begin{aligned}\int \frac{\sqrt{x^2-144}}{x}\,dx &= \int \frac{\sqrt{144\sec^2\theta-144}}{12\sec\theta}12\sec\theta\tan\theta\,d\theta \\ &= 12\int \tan^2\theta\,d\theta = 12\int \sec^2\theta - 1\,d\theta \\ &= 12[\tan\theta-\theta]+C\end{aligned}$$

Since $\sec\theta = \frac{x}{12}$, draw a triangle with x on the hypotenuse and 12 on the adjacent side. Then the opposite side gets $\sqrt{x^2-144}$. Then $\tan\theta = \frac{\sqrt{x^2-144}}{12}$ and $\theta = \operatorname{arcsec}\frac{x}{12}$. Then

$$\begin{aligned}\int \frac{\sqrt{x^2-144}}{x}\,dx &= 12\left[\frac{\sqrt{x^2-144}}{12} - \operatorname{arcsec}\frac{x}{12}\right]+C \\ &= \sqrt{x^2-144} - 12\operatorname{arcsec}\frac{x}{12}+C\end{aligned}$$

b. Check your answer by differentiating..

Solution: Let $f = \sqrt{x^2-144} - 12\operatorname{arcsec}\frac{x}{12}$. Then

$$\begin{aligned}f' &= \frac{1}{2}\frac{2x}{\sqrt{x^2-144}} - 12\frac{1}{\frac{x}{12}\sqrt{\left(\frac{x}{12}\right)^2-1}}\frac{1}{12} \\ &= \frac{x}{\sqrt{x^2-144}} - \frac{12}{x\sqrt{\frac{x^2}{144}-1}} = \frac{x}{\sqrt{x^2-144}} - \frac{12}{\frac{x}{12}\sqrt{x^2-144}} \\ &= \frac{x^2}{x\sqrt{x^2-144}} - \frac{144}{x\sqrt{x^2-144}} = \frac{x^2-144}{x\sqrt{x^2-144}} = \frac{\sqrt{x^2-144}}{x}\end{aligned}$$

c. Compute $\int_{13}^{15} \frac{\sqrt{x^2-144}}{x}\,dx$. Simplify. No decimals!

Solution:

$$\begin{aligned}\int_{13}^{15} \frac{\sqrt{x^2-144}}{x}\,dx &= \left[\sqrt{x^2-144} - 12\operatorname{arcsec}\frac{x}{12}\right]_{13}^{15} \\ &= \left(\sqrt{225-144} - 12\operatorname{arcsec}\frac{15}{12}\right) - \left(\sqrt{169-144} - 12\operatorname{arcsec}\frac{13}{12}\right) \\ &= 4 - 12\operatorname{arcsec}\frac{15}{12} + 12\operatorname{arcsec}\frac{13}{12}\end{aligned}$$