Name________________________________

1-12	/60	14	/20
13	/ 7	15	/18
		Total	/105

MATH 152H Exam 3 Spring 2017

Sections 203/204 (circle one) Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. If it takes 12 Newtons of force to hold a spring at 3 meters from the rest position, how much work is done to stretch it from 2 meters to 4 meters from rest?

a. 24 Joules correct choice
b. 16 Joules
c. 8 Joules
d. 6 Joules
e. 4 Joules

Solution: $F = kx \qquad 12 = k(3) \qquad k = 4 \qquad W = \int_2^4 F\,dx = \int_2^4 4x\,dx = [2x^2]_2^4 = 32 - 8 = 24$

2. Compute $\int_0^{\pi/2} \tan\theta\,d\theta$.

a. $-\infty$
b. -1
c. 0
d. 1
e. ∞ correct choice

Solution: The integral is improper at $\theta = \frac{\pi}{2}$.

$$\int_0^{\pi/2} \tan\theta\,d\theta = \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta}\,d\theta = \left[-\ln|\cos\theta|\right]_0^{\pi/2} = \lim_{\theta\to\frac{\pi}{2}^-} (-\ln|\cos\theta|) - (-\ln|\cos 0|)$$

Since $\cos 0 = 1$ and $\ln 1 = 0$, the second term is 0.
As $\theta \to \frac{\pi}{2}^-$, we have θ is slightly less than $\frac{\pi}{2}$, and $\cos\theta$ is slightly greater than 0, and $\ln|\cos\theta|$ approaches $-\infty$, and $-\ln|\cos\theta|$ approaches $+\infty$. So $\int_0^{\pi/2} \tan\theta\,d\theta = \infty$.

3. Compute $\int_4^\infty \frac{1}{x^{3/2}}\,dx$.

a. -1
b. 0
c. $\frac{1}{3}$
d. 1 correct choice
e. ∞

Solution: $\int_4^\infty \frac{1}{x^{3/2}}\,dx = \left[\frac{-2}{x^{1/2}}\right]_4^\infty = 0 - \frac{-2}{4^{1/2}} = 1$

4. Compute $\int_{-8}^{8} \frac{1}{x^{5/3}}\,dx$.

- **a**. $-\infty$
- **b**. ∞
- **c**. divergent but not $\pm\infty$ correct choice
- **d**. 0
- **e**. $\frac{3}{4}$

Solution: The integral is improper at $x = 0$.

$$\int_{-8}^{8} \frac{1}{x^{5/3}}\,dx = \int_{-8}^{0} x^{-5/3}\,dx + \int_{0}^{8} x^{-5/3}\,dx = \left[\frac{-3x^{-2/3}}{2}\right]_{-8}^{0} + \left[\frac{-3x^{-2/3}}{2}\right]_{0}^{8} = \left[\frac{-3}{2x^{2/3}}\right]_{-8}^{0^-} + \left[\frac{-3}{2x^{2/3}}\right]_{0^+}^{8}$$

$$= \left(\frac{-3}{2(0^-)^{2/3}}\right) - \left(\frac{-3}{2(-8)^{2/3}}\right) + \left(\frac{-3}{2(8)^{2/3}}\right) - \left(\frac{-3}{2(0^+)^{2/3}}\right)$$

$$= (-\infty) - \left(\frac{-3}{8}\right) + \left(\frac{-3}{8}\right) - (-\infty) = \text{ undefined and not } \pm\infty$$

5. The differential equation $\frac{dy}{dx} = 2 + 2y + x + xy$ is

- **a**. both separable and linear correct choice
- **b**. separable but not linear
- **c**. linear but not separable
- **d**. neither separable nor linear

Solution: The integral is both separable and linear.

It separates into $\frac{dy}{dx} = 2 + 2y + x + xy = (2 + x)(1 + y)$ or $\frac{dy}{1 + y} = (2 + x)dx$.

It is linear because y and $\frac{dy}{dx}$ only appear to the 1^{st} power or it can be put in standard form:

$$\frac{dy}{dx} - (2 + x)y = 2 + x.$$

6. Find the integrating factor for the differential equation $x^3\frac{dy}{dx} = x^5 + 3x^2y$.

- **a**. $I = x^3$
- **b**. $I = \frac{1}{x^3}$ correct choice
- **c**. $I = e^{3/x^2}$
- **d**. $I = e^{-3/x^2}$
- **e**. $I = e^{-x^3}$

Solution: The standard form is $\frac{dy}{dx} - \frac{3}{x}y = x^2$. So $P = -\frac{3}{x}$ and the integrating factor is

$$I = e^{\int P\,dx} = e^{\int \frac{-3}{x}\,dx} = e^{-3\ln x} = x^{-3} = \frac{1}{x^3}.$$

7. Solve the initial value problem $\dfrac{dy}{dx} = \dfrac{4}{3}\dfrac{x^3}{y^2}$ with $y(1) = 2$. What is $y(0)$?

a. $\sqrt[3]{2}$
b. $\sqrt[3]{7}$ correct choice
c. $-\sqrt[3]{15}$
d. 7
e. -15

Solution: Separate: $3y^2\,dy = 4x^3\,dx$ and integrate:

$$\int 3y^2\,dy = \int 4x^3\,dx \qquad \Rightarrow \qquad y^3 = x^4 + C$$

Use the initial condition: $x = 1, \quad y = 2 \qquad \Rightarrow \qquad 8 = 1 + C \qquad \Rightarrow \qquad C = 7$
So $\quad y^3 = x^4 + 7 \qquad y(0)^3 = 7 \qquad y(0) = \sqrt[3]{7}$

8. Compute $\lim\limits_{n\to\infty}\left(\dfrac{1}{n^4}\right)^{3/\ln n}$.

a. 12
b. 64
c. $e^{3/4}$
d. e^{12}
e. e^{-12} correct choice

Solution: $\lim\limits_{n\to\infty}\left(\dfrac{1}{n^4}\right)^{3/\ln n} = \lim\limits_{n\to\infty} \exp \ln\left(\dfrac{1}{n^4}\right)^{3/\ln n} = \exp \lim\limits_{n\to\infty} \dfrac{3}{\ln n}\ln\left(\dfrac{1}{n^4}\right) = \exp \lim\limits_{n\to\infty} \dfrac{-12}{\ln n}\ln(n) = e^{-12}$

9. Compute $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{3}{2^n}$

a. -1 correct choice
b. -3
c. 1
d. 2
e. 3

Solution: $a = -\frac{3}{2} \qquad r = -\frac{1}{2} \qquad |r| < 1 \qquad \sum\limits_{n=1}^{\infty}(-1)^n\dfrac{3}{2^n} = \dfrac{-\frac{3}{2}}{1-\left(-\frac{1}{2}\right)} = \dfrac{-3}{2+1} = -1$

10. The series $\sum_{n=3}^{\infty} \frac{2n}{(n^2-4)^2}$

a. diverges by the n^{th} Term Divergence Test

b. converges by a Simple Comparison with $\sum_{n=3}^{\infty} \frac{2}{n^3}$

c. diverges by a Simple Comparison with $\sum_{n=3}^{\infty} \frac{2}{n^3}$

d. converges by the Integral Test correct choice

e. diverges by the Integral Test

Solution: $\lim_{n\to\infty} \frac{2n}{(n^2-4)^2} = 0$ So the n^{th} Term Divergence Test fails.

A Simple Comparison will not work because $\sum_{n=3}^{\infty} \frac{2}{n^3}$ converges (p-series with $p = 3 > 1$)

but $\frac{2n}{(n^2-4)^2} > \frac{2}{n^3}$.

$\int_3^{\infty} \frac{2n}{(n^2-4)^2}\,dn = \left[\frac{-1}{n^2-4}\right]_3^{\infty} = 0 - \left(\frac{-1}{9-4}\right) = \frac{1}{5}$ converges.

So $\sum_{n=3}^{\infty} \frac{2n}{(n^2-4)^2}$ converges by the Integral Test.

11. If $S = \sum_{n=3}^{\infty} \frac{2n}{(n^2-4)^2}$ is approximated by its 12^{th}-partial sum $S_{12} = \sum_{n=3}^{12} \frac{2n}{(n^2-4)^2}$,
then the error $E_{12} = S - S_{12}$ is less than

a. $\ln 140$

b. $\frac{1}{2}\ln 140$

c. $\left(\frac{1}{140}\right)^3$

d. $\left(\frac{1}{140}\right)^2$

e. $\frac{1}{140}$ correct choice

Solution: $E_{12} = S - S_{12} = \sum_{n=13}^{\infty} \frac{2n}{(n^2-4)^2} < \int_{12}^{\infty} \frac{2n}{(n^2-4)^2}\,dn = \left[\frac{-1}{n^2-4}\right]_{12}^{\infty} = 0 - \left(\frac{-1}{144-4}\right) = \frac{1}{140}$

12. Compute $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$. Note: $\frac{1}{n(n-1)} = \frac{n}{n-1} - \frac{n+1}{n}$

a. $\frac{1}{3}$

b. $\frac{1}{2}$

c. 1 correct choice

d. 2

e. 3

Solution: $\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty}\left(\frac{n}{n-1} - \frac{n+1}{n}\right)$

$$S_k = \sum_{n=2}^{k} \frac{1}{n(n-1)} = \sum_{n=2}^{k}\left(\frac{n}{n-1} - \frac{n+1}{n}\right) = \left(\frac{2}{1} - \frac{3}{2}\right) + \left(\frac{3}{2} - \frac{4}{3}\right) + \cdots + \left(\frac{k}{k-1} - \frac{k+1}{k}\right)$$

$$= \frac{2}{1} - \frac{k+1}{k} \qquad \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(2 - \frac{k+1}{k}\right) = 1$$

Work Out: (Points indicated. Part credit possible. Show all work.)

13. (7 points) A tank initially contains 25 L of salt water with a concentration of $40\ \frac{\text{gm of salt}}{\text{L}}$.

Salt water with concentration $10\ \frac{\text{gm of salt}}{\text{L}}$ is entering the tank at the rate $5\ \frac{\text{L}}{\text{hour}}$.

The water is kept well mixed and drains at $5\ \frac{\text{L}}{\text{hour}}$.

Set up the initial value problem for the amount of salt in the tank $S(t)$ at time t. Do not solve it.

a. Write the differential equation:

Solution: $$\frac{dS}{dt} = \underbrace{10\frac{\text{gm}}{\text{L}}\,5\frac{\text{L}}{\text{hour}}}_{\text{in}} - \underbrace{\frac{S(t)\ \text{g}}{25\ \text{L}}\,5\frac{\text{L}}{\text{hour}}}_{\text{out}} \qquad \text{or} \qquad \frac{dS}{dt} = 50 - \frac{1}{5}S$$

b. Write the initial condition:

Solution: $S(0) = 40\frac{\text{gm}}{\text{L}}\,25\ \text{L} = 1000$ gm

14. (20 points) A sequence is defined recursively by

$$a_{n+1} = \sqrt{a_n} + 6 \qquad \text{with} \qquad a_1 = 100$$

a. Assuming the limit $L = \lim_{n\to\infty} a_n$ exists, find the possible limits, L.

Solution: $L = \sqrt{L} + 6 \quad \Rightarrow \quad L - 6 = \sqrt{L} \quad \Rightarrow \quad (L-6)^2 = L$
$\Rightarrow \quad L^2 - 12L + 36 = L \quad \Rightarrow \quad L^2 - 13L + 36 = 0 \quad \Rightarrow \quad (L-4)(L-9) = 0$
$\Rightarrow \quad L = 4, 9$

b. Write out the first 3 terms of the sequence:

Solution: $a_1 = 100 \qquad a_2 = \sqrt{100} + 6 = 16 \qquad a_3 = \sqrt{16} + 6 = 10$

c. State a conjecture about boundedness: (Circle one answer and fill in the blank.)

The sequence is bounded above / [below] by ___9___.

Now write the conjecture as an inequality:

Solution: $a_n \geq 9$.

d. Use Mathematical Induction to prove this conjecture.

Solution: Assume $a_k \geq 9$. Then $\sqrt{a_k} \geq 3$ and $\sqrt{a_k} + 6 \geq 9$. Or $a_{k+1} \geq 9$.
So $a_n \geq 9$ for all n.

e. State a conjecture about monotonicity: (Circle one answer.)

The sequence is increasing / [decreasing].

Now write the conjecture as an inequality:

Solution: $a_{n+1} \leq a_n$.

f. Use Mathematical Induction to prove this conjecture.

Solution: Assume $a_{k+1} \leq a_k$. Then $\sqrt{a_{k+1}} \leq \sqrt{a_k}$ and $\sqrt{a_{k+1}} + 6 \leq \sqrt{a_k} + 6$.
Or $a_{k+2} \leq a_{k+1}$. So $a_{n+1} \leq a_n$ for all n.

g. What do you conclude about the convergence or divergence of the series?
Name any theorem you use.

Solution: The sequence is decreasing and bounded below by 9.
So it converges to 9 by the Bounded Monotonic Sequence Theorem.

15. $\left(18 \text{ points}\right)$ Solve the initial value problem:

$$\frac{dy}{dx} = 3x^5 - 3x^2 y \qquad\qquad y(1) = 2e^{-1}$$

Give the explicit solution not just the implicit solution.

Solution: The equation is linear. Its standard form is

$$\frac{dy}{dx} + 3x^2 y = 3x^5$$

We identify $P = 3x^2$ and find the integration factor $I = e^{\int P\,dx} = e^{\int 3x^2\,dx} = e^{x^3}$.
We multiply thru by the integrating factor and identify the left side as the derivative of a product:

$$e^{x^3}\frac{dy}{dx} + 3x^2 e^{x^3} y = 3x^5 e^{x^3}$$

$$\frac{d}{dx}\left(e^{x^3} y\right) = 3x^5 e^{x^3}$$

We integrate both sides and use the substitution $w = x^3$ and $dw = 3x^2\,dx$

$$e^{x^3} y = \int 3x^5 e^{x^3}\,dx = \int w e^w\,dw$$

Now use integration by parts with $\begin{array}{ll} u = w & dv = e^w\,dw \\ du = dw & v = e^w \end{array}$

$$e^{x^3} y = we^w - \int e^w\,dw = we^w - e^w + C = x^3 e^{x^3} - e^{x^3} + C$$

Next we use the initial condition which says $x = 1$ and $y = 2e^{-1}$

$$e^{x^3} y = x^3 e^{x^3} - e^{x^3} + C$$

$$e^1 2e^{-1} = 1e^1 - e^1 + C$$

So $C = 2$ and the implicit solution is

$$e^{x^3} y = x^3 e^{x^3} - e^{x^3} + 2$$

Finally the explicit solution is

$$y = x^3 - 1 + 2e^{-x^3}$$

Check: $y(1) = 1 - 1 + 2e^{-1} = 2e^{-1}$ and

$$\frac{dy}{dx} = 3x^2 - 6x^2 e^{-x^3}$$

$$3x^2 y = 3x^5 - 3x^2 + 6x^2 e^{-x^3}$$

$$\frac{dy}{dx} + 3x^2 y = 3x^5 \quad \text{which matches}$$