Name________________________________

MATH 152H Final Exam Spring 2017

Sections 203/204 (circle one) Solutions P. Yasskin

1-12	/60	15	/12
13	/16	16	/12
14	/10	Total	/110

Multiple Choice: (5 points each. No part credit.)

1. Compute the arclength of the curve $y = \ln(\cos(x))$ between $x = 0$ and $x = \frac{\pi}{3}$.

a. $\ln\left(2 - \sqrt{3}\right)$

b. $\ln\left(2 + \sqrt{3}\right)$ correct choice

c. $\ln\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$

d. $\ln\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right)$

e. $\ln\left(\sqrt{3}\right)$

Solution: $\frac{dy}{dx} = \frac{-\sin(x)}{\cos(x)} = -\tan x$

$$L = \int_0^{\pi/3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\pi/3} \sqrt{1 + (\tan x)^2}\, dx = \int_0^{\pi/3} \sec x\, dx = \ln(\sec x + \tan x)$$

$$= \ln\left(\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right) - \ln(\sec 0 + \tan 0) = \ln\left(2 + \sqrt{3}\right)$$

2. Find the surface area swept out when the curve $x = 1 + t^2 \quad y = 1 - t^2$ is revolved around the y-axis for $0 \le t \le 2$.

a. $3\sqrt{2}\,\pi$

b. $4\sqrt{2}\,\pi$

c. $6\sqrt{2}\,\pi$

d. $12\sqrt{2}\,\pi$

e. $24\sqrt{2}\,\pi$ correct choice

Solution: $\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = -2t$

$$A = \int_0^2 2\pi r\, ds = \int_0^2 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^2 2\pi(1 + t^2)\sqrt{(2t)^2 + (2t)^2}\, dt = 2\pi\int_0^2 (1 + t^2)2t\sqrt{2}\, dt$$

$$= 4\sqrt{2}\,\pi\int_0^2 (t + t^3)dt = 4\sqrt{2}\,\pi\left[\frac{t^2}{2} + \frac{t^4}{4}\right]_0^2 = 4\sqrt{2}\,\pi(2 + 4) = 24\sqrt{2}\,\pi$$

3. Find the area between $y = x^2 - 4x$ and $y = 2x - x^2$.

- **a**. 1
- **b**. 3
- **c**. 9 correct choice
- **d**. 12
- **e**. 18

Solution: $x^2 - 4x = 2x - x^2 \quad \Rightarrow \quad 2x^2 - 6x = 0 \quad \Rightarrow \quad 2x(x-3) = 0 \quad \Rightarrow \quad x = 0, 3$

$$A = \int_0^3 (2x - x^2) - (x^2 - 4x)\,dx = \int_0^3 (6x - 2x^2)\,dx = \left[3x^2 - \frac{2x^3}{3}\right]_0^3 = (27 - 18) = 9$$

4. Compute $\displaystyle\int_4^5 \frac{1}{x^2 - 5x + 6}\,dx$

- **a**. $2\ln 2 - \ln 3$ correct choice
- **b**. $\ln 3 - 2\ln 2$
- **c**. $\ln 3$
- **d**. $2\ln 3 - \ln 2$
- **e**. $\ln 2 - 2\ln 3$

Solution: $\displaystyle\frac{1}{x^2 - 5x + 6} = \frac{A}{x-2} + \frac{B}{x-3} \quad \Rightarrow \quad 1 = A(x-3) + B(x-2)$

$x = 2: \quad 1 = A(-1) \quad \Rightarrow \quad A = -1 \qquad\qquad x = 3: \quad 1 = B(1) \quad \Rightarrow \quad B = 1$

$$\int_4^5 \frac{1}{x^2 - 5x + 6}\,dx = \int_4^5 \frac{-1}{x-2} + \frac{1}{x-3}\,dx = \Big[-\ln|x-2| + \ln|x-3|\Big]_4^5$$

$$= (-\ln|3| + \ln|2|) - (-\ln|2| + \ln|1|) = 2\ln 2 - \ln 3$$

5. The integral $\displaystyle\int_2^\infty \frac{1}{x^2 + x}\,dx$

- **a**. diverges by comparison with $\displaystyle\int_2^\infty \frac{1}{x}\,dx$
- **b**. converges by comparison with $\displaystyle\int_2^\infty \frac{1}{x}\,dx$
- **c**. diverges by comparison with $\displaystyle\int_2^\infty \frac{1}{x^2}\,dx$
- **d**. converges by comparison with $\displaystyle\int_2^\infty \frac{1}{x^2}\,dx$ correct choice
- **e**. None of the above

Solution: $\displaystyle\frac{1}{x^2 + x} < \frac{1}{x^2}$ and $\displaystyle\int_2^\infty \frac{1}{x^2}\,dx = \left[\frac{-1}{x}\right]_2^\infty = \frac{1}{2}$ converges.

So $\displaystyle\int_2^\infty \frac{1}{x^2 + x}\,dx < \int_2^\infty \frac{1}{x^2}\,dx$ also converges.

6. Compute $\int_4^5 \frac{1}{x^2\sqrt{x^2-16}}\,dx$

- **a**. $\frac{3}{80}$ correct choice
- **b**. $\frac{9}{80}$
- **c**. $\frac{3}{160}$
- **d**. $\frac{9}{40}$
- **e**. $\frac{3}{40}$

Solution: Let $x = 4\sec\theta \qquad dx = 4\sec\theta\tan\theta\,d\theta$

$$\int \frac{1}{x^2\sqrt{x^2-16}}\,dx = \int \frac{1}{16\sec^2\theta\sqrt{16\sec^2\theta-16}}\,4\sec\theta\tan\theta\,d\theta = \frac{1}{16}\int \frac{1}{\sec^2\theta\sqrt{\sec^2\theta-1}}\,\sec\theta\tan\theta\,d\theta$$

$$= \frac{1}{16}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{16}\int \cos\theta\,d\theta = \frac{1}{16}\sin\theta + C$$

$\sec\theta = \frac{x}{4}$ Draw a triangle with hypotenuse x and adjacent side 4. Then $\sin\theta = \frac{\sqrt{x^2-16}}{x}$.

So $\int_4^5 \frac{1}{x^2\sqrt{x^2-16}}\,dx = \left.\frac{\sqrt{x^2-16}}{16x}\right|_4^5 = \frac{3}{80}$

7. Compute $\int_0^1 3x^2\arctan x\,dx$

- **a**. $\frac{\pi}{2} + \ln 2 - 1$
- **b**. $\frac{\pi}{2} + \frac{1}{2}\ln 2 + \frac{1}{2}$
- **c**. $\frac{\pi}{4} + \ln 2 - 1$
- **d**. $\frac{\pi}{4} + \frac{1}{2}\ln 2 - \frac{1}{2}$ correct choice
- **e**. $\frac{\pi}{4} + \frac{1}{2}\ln 2 + \frac{1}{2}$

Solution: Integration by parts: $u = \arctan x \qquad dv = 3x^2\,dx$

$du = \frac{1}{1+x^2}\,dx \qquad v = x^3$

$$\int 3x^2\arctan x\,dx = x^3\arctan x - \int x^3\frac{1}{1+x^2}\,dx \qquad u = 1+x^2 \qquad du = 2x\,dx \qquad \frac{du}{2} = x\,dx$$

$$\int 3x^2\arctan x\,dx = x^3\arctan x - \frac{1}{2}\int \frac{u-1}{u}\,du = x^3\arctan x - \frac{1}{2}\int 1 - \frac{1}{u}\,du$$

$$= x^3\arctan x - \frac{1}{2}(u - \ln u) + C = x^3\arctan x - \frac{1}{2}(1 + x^2 - \ln(1+x^2)) + C$$

$$\int_0^1 3x^2\arctan x\,dx = \left[x^3\arctan x - \frac{1}{2} - \frac{1}{2}x^2 + \frac{1}{2}\ln(1+x^2)\right]_0^1$$

$$= \left(\arctan 1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{2}\ln(1+1)\right) - \left(-\frac{1}{2}\right) = \frac{\pi}{4} + \frac{1}{2}\ln 2 - \frac{1}{2}$$

8. Solve the initial value problem $\frac{dy}{dx} = 1 + 2x + y^2 + 2xy^2$ with $y(1) = 0$. What is $y(2)$?
HINT: Factor.

- **a**. $\tan(1)$
- **b**. $\tan(2)$
- **c**. $\tan(4)$ correct choice
- **d**. $\frac{\pi}{4}$
- **e**. $\frac{\pi}{2}$

Solution: Separable: $\frac{dy}{dx} = (1 + 2x)(1 + y^2)$

$$\int \frac{dy}{1 + y^2} = \int (1 + 2x)\,dx \quad \Rightarrow \quad \arctan y = x + x^2 + C$$

Initial Condition: $x = 1$ when $y = 0$: $\arctan 0 = 1 + 1 + C \quad \Rightarrow \quad C = -2$

$$y = \tan(x + x^2 - 2) \quad \Rightarrow \quad y(2) = \tan(2 + 4 - 2) = \tan(4)$$

9. Compute $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^{3n}}$.

- **a**. -9
- **b**. -8
- **c**. 8
- **d**. 9
- **e**. divergent correct choice

Solution: This is a geometric series. $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^{3n}} = \sum_{n=1}^{\infty} \frac{9^n}{8^n} = \sum_{n=1}^{\infty} \left(\frac{9}{8}\right)^n$

The first term is $a = \frac{9}{8}$ and the ratio is $r = \frac{9}{8}$. Since $|r| = \frac{9}{8} > 1$, the series is divergent.

10. Compute $\sum_{n=1}^{\infty} \left[\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+1}\right)\right]$.

- **a**. -2 correct choice
- **b**. -1
- **c**. 1
- **d**. 2
- **e**. divergent

Solution: This is a telescoping series. The partial sum is

$$\begin{aligned} S_k &= \sum_{n=1}^{k} \left[\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+1}\right)\right] \\ &= \left[\cos\left(\frac{\pi}{1}\right) - \cos\left(\frac{\pi}{2}\right)\right] + \left[\cos\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{3}\right)\right] + \cdots + \left[\cos\left(\frac{\pi}{k}\right) - \cos\left(\frac{\pi}{k+1}\right)\right] \\ &= \cos\left(\frac{\pi}{1}\right) - \cos\left(\frac{\pi}{k+1}\right) \end{aligned}$$

$$S = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left[\cos\left(\frac{\pi}{1}\right) - \cos\left(\frac{\pi}{k+1}\right)\right] = \cos(\pi) - \cos(0) = -2$$

11. Compute $\lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2}$

a. $\frac{1}{2}$

b. $\frac{3}{2}$ correct choice

c. $\frac{5}{2}$

d. $\frac{11}{24}$

e. $\frac{13}{24}$

Solution: $e^u = 1 + u + \frac{u^2}{2} + \cdots \qquad e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \cdots \qquad \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$

$$\lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2} = \lim_{x\to 0} \frac{\left(1 + x^2 + \frac{x^4}{2} + \cdots\right) - \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots\right)}{x^2} = \lim_{x\to 0} \frac{\frac{3x^2}{2} + O(x^4)}{x^2}$$

$$= \lim_{x\to 0}\left(\frac{3}{2} + O(x^2)\right) = \frac{3}{2}$$

12. The series $\sum_{n=2}^{\infty} \frac{1}{n^2 - n}$

a. diverges by a Simple Comparison with $\sum_{n=2}^{\infty} \frac{1}{n}$

b. converges by a Simple Comparison with $\sum_{n=2}^{\infty} \frac{1}{n}$

c. converges by a Simple Comparison with $\sum_{n=2}^{\infty} \frac{1}{n^2}$

d. diverges by a Limit (but not Simple) Comparison with $\sum_{n=2}^{\infty} \frac{1}{n^2}$

e. converges by a Limit (but not Simple) Comparison with $\sum_{n=2}^{\infty} \frac{1}{n^2}$ correct choice

Solution: For large n, $\frac{1}{n^2 - n}$ approaches $\frac{1}{n^2}$ and $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges because it is a p-series with $p = 2 > 1$. However, $\frac{1}{n^2 - n} > \frac{1}{n^2}$ So we can't use the Simple Comparison test. So we try the Limit Comparison test:

$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^2 - n} \frac{n^2}{1} = 1$ Since $0 < L < \infty$, it converges also.

Work Out: (Points indicated. Part credit possible. Show all work.)

13. $\left(16 \text{ points}\right)$ Let $X(t)$ be the amount of a radio active element X present at a reactor.

The element X is produced at $100\dfrac{\text{kg}}{\text{yr}}$ and decays with a half-life of 20 yrs.

If we start with no element X on hand, then $X(t)$ satisfies the initial value problem

$$\frac{dX}{dt} = 100 - \frac{\ln 2}{20}X \qquad \text{with} \quad X(0) = 0$$

a. Solve the initial value problem:

Solution: The equation as linear. We put it in standard form, identify $P(t)$ and compute the integrating factor.

$$\frac{dX}{dt} + \frac{\ln 2}{20}X = 100 \qquad\qquad P(t) = \frac{\ln 2}{20}$$

$$I = e^{\int P(t)\,dt} = e^{\int \frac{\ln 2}{20}dt} = e^{\frac{\ln 2}{20}t}$$

We multiply the standard form by the integrating factor and rewrite the left side as the derivative of a product.

$$e^{\frac{\ln 2}{20}t}\frac{dX}{dt} + \frac{\ln 2}{20}e^{\frac{\ln 2}{20}t}X = 100e^{\frac{\ln 2}{20}t}$$

$$\frac{d}{dt}\left(e^{\frac{\ln 2}{20}t}X\right) = 100e^{\frac{\ln 2}{20}t}$$

Now we integrate:

$$e^{\frac{\ln 2}{20}t}X = \int 100e^{\frac{\ln 2}{20}t}dt = 100e^{\frac{\ln 2}{20}t}\frac{20}{\ln 2} + C$$

Initial Condition: $t = 0$ when $X = 0$: $\quad 0 = \dfrac{2000}{\ln 2} + C \quad \Rightarrow \quad C = -\dfrac{2000}{\ln 2}$

We substitute back and solve:

$$e^{\frac{\ln 2}{20}t}X = e^{\frac{\ln 2}{20}t}\frac{2000}{\ln 2} - \frac{2000}{\ln 2}$$

$$X = \frac{2000}{\ln 2} - \frac{2000}{\ln 2}e^{-\frac{\ln 2}{20}t}$$

b. How much of element X is present after 20 yrs?

Solution: $X = \dfrac{2000}{\ln 2} - \dfrac{2000}{\ln 2}e^{-\frac{\ln 2}{20}20} = \dfrac{2000}{\ln 2} - \dfrac{2000}{\ln 2}e^{-\ln 2} = \dfrac{1000}{\ln 2}$

14. $\left(10 \text{ points}\right)$ Use a Maclaurin polynomial to estimate $\sin(1)$ to within 10^{-5}.

What theorem guarantees the error in the approximation is less than 10^{-5}?

Do not add up the terms. No decimals!

Note: $0! = 1 \quad 1! = 1 \quad 3! = 6 \quad 5! = 120 \quad 7! = 5040 \quad 9! = 362\,880$

Solution: $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dfrac{x^9}{9!}\cdots$ So $\sin 1 = 1 - \dfrac{1}{3!} + \dfrac{1}{5!} - \dfrac{1}{7!}$.

Since the series converges by the Alternating Series Test, the Alternating Series Error Bound Theorem says the error in the approximation is less than the absolute value of the next term.

In this case, $E < \dfrac{1}{9!} = \dfrac{1}{362\,880} < 10^{-5}$

15. $\big(12$ points$\big)$ Use the Ratio Test to find the radius of convergence of each of the following series:

a. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n}(x-4)^n$

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{3^{n+1}}{n+1}\frac{n}{3^n}\frac{|x-4|^{n+1}}{|x-4|^n} = 3|x-4| < 1 \qquad |x-4| < \frac{1}{3} \qquad \boxed{R = \frac{1}{3}}$$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}(x-4)^n$

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n+1}{3^{n+1}}\frac{3^n}{n}\frac{|x-4|^{n+1}}{|x-4|^n} = \frac{|x-4|}{3} < 1 \qquad |x-4| < 3 \qquad \boxed{R = 3}$$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}(x-4)^n$

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!}\frac{n!}{3^n}\frac{|x-4|^{n+1}}{|x-4|^n} = 3|x-4|\lim_{n\to\infty}\frac{1}{(n+1)} = 0 < 1 \qquad \forall x \qquad \boxed{R = \infty}$$

d. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3^n}(x-4)^n$

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)!}{3^{n+1}}\frac{3^n}{n!}\frac{|x-4|^{n+1}}{|x-4|^n} = \frac{|x-4|}{3}\lim_{n\to\infty}(n+1) = \begin{cases} \infty > 1 & \forall x \neq 4 \\ 0 < 1 & x = 4 \end{cases} \qquad \boxed{R = 0}$$

16. $\big(12$ points$\big)$ The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{n}}(x-3)^n$ has radius of convergence $R = 2$.

Find its interval of convergence.

Left Endpoint: $\boxed{x = 1}$ Series at Left Endpoint: $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}}$

Name and Apply Test for Convergence:

Alternating Series Test

$(-1)^n$ says alternating $\quad \dfrac{1}{\sqrt{n}}$ is decreasing $\quad \displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$

Conclusion (Circle one)

[✓ Convergent]

Divergent

Right Endpoint: $\boxed{x = 5}$ Series at Right Endpoint: $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(2)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}}$

Name and Apply Test for Convergence:

p-Series Test

$p = \dfrac{1}{2} < 1$

Conclusion (Circle one)

Convergent

[✓ Divergent]

Interval of Convergence: $[1,5)$ or $1 \le x < 5$