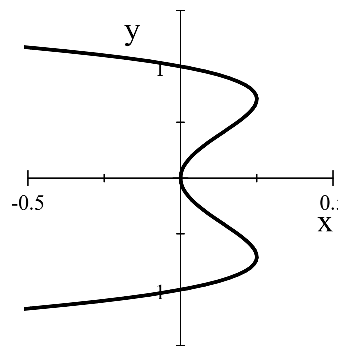
Name______________________________________

MATH 172 Exam 1 Spring 2018

Sections 501/502 (circle one) Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the area between $x = y^2 - y^4$ and the y-axis.



a. $\frac{2}{15}$

b. $\frac{4}{15}$ correct choice

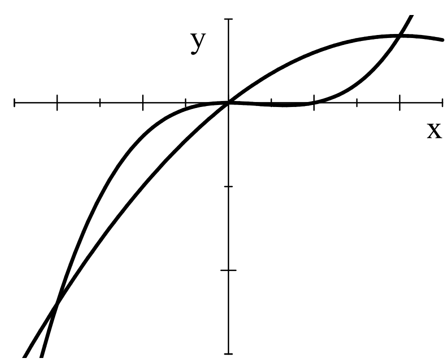
c. $\frac{8}{15}$

d. 4

e. 8

Solution: $A = \int_{-1}^{1} (y^2 - y^4)\, dy = \left[\frac{y^3}{3} - \frac{y^5}{5} \right]_{-1}^{1} = 2\left(\frac{1}{3} - \frac{1}{5} \right) = \frac{4}{15}$

2. Find the total area between $y = x^3 - x^2$ and $y = 4x - x^2$.



a. 2

b. 4

c. 8 correct choice

d. 18

e. 24

Solution: The curves intersect when $x^3 - x^2 = 4x - x^2$ or $x^3 - 4x = 0$ or $x = -2, 0, 2$.
On $[-2, 0]$, the cubic is on top. On $[0, 2]$, the quadratic is on top.

$$A = \int_{-2}^{0} (x^3 - 4x)\, dx + \int_{0}^{2} (4x - x^3)\, dx = \left[\frac{x^4}{4} - 2x^2 \right]_{-2}^{0} + \left[2x^2 - \frac{x^4}{4} \right]_{0}^{2} = -(4 - 8) + (8 - 4) = 8$$

1-10	/50	13	/15
11	/10	14	/10
12	/10	15	/10
		Total	/105

3. The temperature of a 5 cm bar is $T = 1 + x^4$ where x is measured from one end. Find the average temperature of the bar.

a. $\frac{129}{4}$
b. $\frac{629}{4}$
c. 126 correct choice
d. 130
e. 630

Solution: $T_{\text{ave}} = \frac{1}{5}\int_0^5 (1 + x^4)\,dx = \frac{1}{5}\left[x + \frac{x^5}{5}\right]_0^5 = \frac{1}{5}(5 + 5^4) = 126$

4. Compute $\int_1^2 x\ln(x^2)\,dx$.

a. $4\ln 4 - 4$
b. $4\ln 4 - 3$
c. $2\ln 4 - 2$
d. $2\ln 4 - \frac{3}{2}$ correct choice
e. $4\ln 4 - \frac{3}{2}$

Solution: Use the substitution $u = x^2$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. So

$$\int_1^2 x\ln(x^2)\,dx = \frac{1}{2}\int_1^4 \ln(u)\,du = \frac{1}{2}(u\ln u - u)\Big|_1^4 = \frac{1}{2}[4\ln(4) - 4] - \frac{1}{2}[\ln(1) - 1] = 2\ln 4 - \frac{3}{2}$$

5. Compute $\int_0^{\pi/2} \sin^2 x\cos^3 x\,dx$

a. $\frac{1}{15}$
b. $\frac{2}{15}$ correct choice
c. $\frac{4}{15}$
d. $\frac{8}{15}$
e. $\frac{16}{15}$

Solution: Since the power of $\cos$ is odd, we use the substitution $u = \sin x$. Then $du = \cos x\,dx$ and $\cos^2 x = 1 - \sin^2 x = 1 - u^2$. Then

$$\int_0^{\pi/2} \sin^2 x\cos^3 x\,dx = \int_0^1 u^2(1 - u^2)\,du = \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

6. Find the mass of a 6 meter bar whose density is $\rho(x) = 4 + 2x$ where x is measured (in meters) from one end.

a. 16

b. 60 correct choice

c. 24

d. 21

e. 36

Solution: $M = \int_0^6 \rho(x)\,dx = \int_0^6 (4 + 2x)\,dx = [4x + x^2]_0^6 = 24 + 36 = 60$

7. Find the center of mass of a 6 meter bar whose density is $\rho(x) = 4 + 2x$ where x is measured (in meters) from one end.

a. 108

b. 216

c. $\frac{5}{18}$

d. $\frac{5}{36}$

e. $\frac{18}{5}$ correct choice

Solution: From the previous problem, $M = 60.$

$$M_1 = \int_0^6 x\rho(x)\,dx = \int_0^6 x(4 + 2x)\,dx = \left[2x^2 + \frac{2x^3}{3}\right]_0^6 = 72 + 144 = 216$$

$\bar{x} = \frac{M_1}{M} = \frac{216}{60} = \frac{72}{20} = \frac{18}{5} = 3.6$ Reasonable because $0 < \bar{x} < 6.$

8. Compute $\int_0^{\pi/4} \sec^6\theta\,d\theta$

a. $\frac{28}{15}$ correct choice

b. $\frac{14}{15}$

c. $\frac{4}{3}$

d. $\frac{2}{3}$

e. $\frac{1}{6}$

Solution: $u = \tan\theta \qquad du = \sec^2\theta\,d\theta \qquad \sec^2\theta = \tan^2\theta + 1 = u^2 + 1$

$$\int_0^{\pi/4} \sec^6\theta\,d\theta = \int_0^1 (u^2 + 1)^2\,du = \int_0^1 (u^4 + 2u^2 + 1)\,du = \left[\frac{u^5}{5} + \frac{2u^3}{3} + u\right]_0^1 = \frac{1}{5} + \frac{2}{3} + 1 = \frac{28}{15}$$

9. Compute $\int_0^{\pi/3} \sec^5\theta \tan\theta \, d\theta$

a. 1
b. $\frac{7}{3}$
c. 5
d. $\frac{31}{5}$ correct choice
e. $\frac{63}{5}$

Solution: $u = \sec\theta \qquad du = \sec\theta \tan\theta \, d\theta$

$$\int_0^{\pi/3} \sec^5\theta \tan\theta \, d\theta = \int_1^2 u^4 \, du = \left[\frac{u^5}{5} \right]_1^2 = \frac{32}{5} - \frac{1}{5} = \frac{31}{5}$$

10. Compute $\int_0^3 \frac{x^2}{(x^2+9)^2} \, dx$.

a. $\frac{\pi}{24} - \frac{1}{12}$ correct choice
b. $\frac{\pi}{12} - \frac{1}{6}$
c. $\frac{\pi}{6} - \frac{1}{3}$
d. $\frac{1}{96}\pi$
e. $\frac{1}{288}\pi$

Solution: We use the substitution $x = 3\tan\theta$ with $dx = 3\sec^2\theta \, d\theta$. Then

$$\int_0^3 \frac{x^2}{(x^2+9)^2} \, dx = \int \frac{9\tan^2\theta}{(9\tan^2\theta + 9)^2} 3\sec^2\theta \, d\theta = \frac{1}{3} \int \frac{\tan^2\theta}{\sec^2\theta} \, d\theta = \frac{1}{3} \int \sin^2\theta \, d\theta$$

$$= \frac{1}{3} \int \frac{1 - \cos 2\theta}{2} \, d\theta = \frac{1}{6}\left(\theta - \frac{\sin 2\theta}{2}\right) = \frac{1}{6}(\theta - \sin\theta\cos\theta)$$

Since $\tan\theta = \frac{x}{3}$, we consider a triangle with an angle θ, opposite side x, adjacent side 3 and hypotenuse $\sqrt{x^2+9}$.

Then $\theta = \arctan\frac{x}{3}$, $\sin\theta = \frac{x}{\sqrt{x^2+9}}$ and $\cos\theta = \frac{3}{\sqrt{x^2+9}}$. Then:

$$\int_0^3 \frac{x^2}{(x^2+9)^2} \, dx = \frac{1}{6}\left[\arctan\frac{x}{3} - \frac{3x}{x^2+9} \right]_0^3 = \frac{1}{6}\left(\arctan 1 - \frac{9}{18}\right) - \frac{1}{6}\left(\arctan 0 - \frac{0}{9}\right)$$

$$= \frac{\pi}{24} - \frac{1}{12}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (10 points) Compute $\int_0^{\sqrt{\pi/2}} x^3 \cos(x^2)\, dx$.

Solution: We first make the substitution $w = x^2$. So $dw = 2x\,dx$ and $x\,dx = \frac{1}{2}\,dw$. So

$$\int_0^{\sqrt{\pi/2}} x^3 \cos(x^2)\, dx = \frac{1}{2}\int_0^{\pi/2} w\cos(w)\, dw$$

We now use integration by parts with $\begin{array}{ll} u = w & dv = \cos(w)\,dw \\ du = dw & v = \sin(w) \end{array}$. So

$$\int_0^{\sqrt{\pi/2}} x^3 \cos(x^2)\, dx = \frac{1}{2}\left[w\sin(w) - \int \sin(w)\, dw \right]_0^{\pi/2} = \frac{1}{2}\left[w\sin(w) + \cos(w) \right]_0^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2} \right) - \frac{1}{2}(0 + \cos 0) = \frac{\pi}{4} - \frac{1}{2}$$

12. (10 points) Compute $I = \int e^{-x}\sin(3x)\, dx$.

Solution: Use integration by parts with $\begin{array}{ll} u = \sin(3x) & dv = e^{-x}\,dx \\ du = 3\cos(3x)\,dx & v = -e^{-x} \end{array}$. Then

$$\int e^{-x}\sin(3x)\, dx = -e^{-x}\sin(3x) + 3\int e^{-x}\cos(3x)\, dx$$

Now use integration by parts with $\begin{array}{ll} u = \cos(3x) & dv = e^{-x}\,dx \\ du = -3\sin(3x)\,dx & v = -e^{-x} \end{array}$. Then

$$\int e^{-x}\sin(3x)\, dx = -e^{-x}\sin(3x) + 3\left(-e^{-x}\cos(3x) - 3\int e^{-x}\sin(3x)\, dx \right)$$

So

$$I = -e^{-x}\sin(3x) - 3e^{-x}\cos(3x) - 9I$$

$$10I = -e^{-x}\sin(3x) - 3e^{-x}\cos(3x)$$

$$I = -\frac{1}{10}e^{-x}\sin(3x) - \frac{3}{10}e^{-x}\cos(3x) + C$$

13. (15 points) Consider the curve $y = \frac{2}{3}x^{3/2}$ for $0 \le x \le 3$.

a. Find the arclength.

Solution: $\frac{dy}{dx} = x^{1/2}$ $\quad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1+x}\, dx$ So the arc length is:

$$L = \int ds = \int_0^3 \sqrt{1+x}\, dx = \frac{2}{3}(1+x)^{3/2}\Big|_0^3 = \frac{2}{3}4^{3/2} - \frac{2}{3} = \frac{14}{3}$$

b. If the curve is revolved about the y-axis, find the area of the surface swept out.

Solution: The radius of revolution is $r = x$. So the surface area is:

$$A = \int 2\pi r\, ds = 2\pi \int_0^3 x\sqrt{1+x}\, dx$$

We make the substitution $u = 1 + x$. So $du = dx$ and $x = u - 1$. So

$$A = 2\pi \int_1^4 (u-1)\sqrt{u}\, du = 2\pi \int_1^4 (u^{3/2} - u^{1/2})\, du = 2\pi \left[\frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3} \right]_1^4$$

$$= 2\pi \left[\frac{64}{5} - \frac{16}{3} \right] - 2\pi \left[\frac{2}{5} - \frac{2}{3} \right] = 2\pi \left(\frac{62}{5} - \frac{14}{3} \right) = 2\pi \frac{186 - 70}{15} = \frac{232}{15}\pi$$

14. $(10$ points$)$ Compute $\int_0^1 \frac{1}{x^2 - 9}\, dx.$

Solution: The substitution $x = 3\sin\theta$ requires $x < 3$ which agrees with the limits of integration. Then $dx = 3\cos\theta\, d\theta$ and:

$$\int_0^1 \frac{1}{x^2 - 9}\, dx = \int \frac{1}{9\sin^2\theta - 9} 3\cos\theta\, d\theta = \frac{1}{3} \int \frac{\cos\theta}{-\cos^2\theta}\, d\theta = \frac{-1}{3} \int \sec\theta\, d\theta = \frac{-1}{3} \ln|\sec\theta + \tan\theta|$$

Since $\sin\theta = \frac{x}{3}$, consider a triangle with an angle θ, an opposite side x and hypotenuse 3. The adjacent side is $\sqrt{9 - x^2}$. So $\sec\theta = \frac{3}{\sqrt{9 - x^2}}$ and $\tan\theta = \frac{x}{\sqrt{9 - x^2}}$. Thus:

$$\int_0^1 \frac{1}{x^2 - 9}\, dx = \frac{-1}{3} \ln\left| \frac{3}{\sqrt{9 - x^2}} + \frac{x}{\sqrt{9 - x^2}} \right| \Bigg|_0^1 = -\frac{1}{3} \ln\left(\frac{3}{\sqrt{8}} + \frac{1}{\sqrt{8}} \right) + \frac{1}{3} \ln\left(\frac{3}{\sqrt{9}} \right)$$

$$= -\frac{1}{3} \ln \frac{4}{2\sqrt{2}} = -\frac{1}{3} \ln \sqrt{2}$$

15. $(10$ points$)$ Compute $\int \frac{1}{x^2\sqrt{9x^2 - 1}}\, dx.$ Check your answer!

Solution: The square root requires $3x > 1$. So we use the substitution $3x = \sec\theta$. Then $x = \frac{1}{3}\sec\theta$ and $dx = \frac{1}{3}\sec\theta\tan\theta\, d\theta$. Thus:

$$\int \frac{1}{x^2\sqrt{9x^2 - 1}}\, dx = \int \frac{1}{\frac{1}{9}\sec^2\theta\sqrt{\sec^2\theta - 1}} \frac{1}{3}\sec\theta\tan\theta\, d\theta = 3\int \frac{1}{\sec\theta}\, d\theta = 3\int \cos\theta\, d\theta = 3\sin\theta + C$$

Since $\sec\theta = 3x$, consider a triangle with an angle θ, hypotenuse $3x$ and adjacent side 1. Then the opposite side is $\sqrt{9x^2 - 1}$. So $\sin\theta = \frac{\sqrt{9x^2 - 1}}{3x}$ and:

$$\int \frac{1}{x^2\sqrt{9x^2 - 1}}\, dx = \frac{\sqrt{9x^2 - 1}}{x} + C$$

We check:

$$\frac{d}{dx} \frac{\sqrt{9x^2 - 1}}{x} = \frac{\frac{x}{2} \frac{18x}{\sqrt{9x^2 - 1}} - \sqrt{9x^2 - 1}}{x^2} = \frac{9x^2 - (9x^2 - 1)}{x^2\sqrt{9x^2 - 1}} = \frac{1}{x^2\sqrt{9x^2 - 1}}$$