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MATH 172 Final Spring 2018

Sections 501/502 (circle one) Solutions P. Yasskin

Multiple Choice: (4 points each. No part credit.)

HINTS: $\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$ $\int \csc\theta\, d\theta = -\ln|\csc\theta + \cot\theta| + C$

1. $\int_0^{\pi/2} x\cos x\, dx$

a. 1

b. $\frac{\pi}{2}$

c. $1 - \frac{\pi}{2}$

d. $\frac{\pi}{2} - 1$ correct choice

e. $1 + \frac{\pi}{2}$

Solution: Use Parts with: $u = x \quad dv = \cos x\, dx$
$du = dx \quad v = \sin x$

$$\int x\cos x\, dx = x\sin x - \int \sin x\, dx = x\sin x + \cos x + C$$

$$\int_0^{\pi/2} x\cos x\, dx = \Big[x\sin x + \cos x\Big]_0^{\pi/2} = \left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - (0\sin 0 + \cos 0) = \frac{\pi}{2} - 1$$

2. $\int_0^{\pi/6} \cos^3 x\, dx$

a. $\frac{\pi}{6} - \frac{\pi^3}{3 \cdot 6^3}$

b. $\frac{1}{6}$

c. $\frac{11}{24}$ correct choice

d. $\frac{3}{8}\sqrt{3}$

e. $\frac{1}{64} - \frac{1}{4}$

Solution: $u = \sin x \qquad du = \cos x\, dx$

$$\cos^2 x = 1 - \sin^2 x = 1 - u^2$$

$$\int_0^{\pi/6} \cos^3 x\, dx = \int_0^{1/2} (1 - u^2)\, du = \left[u - \frac{u^3}{3}\right]_0^{1/2}$$

$$= \left(\frac{1}{2} - \frac{1}{24}\right) = \frac{11}{24}$$

1-15	/60	17	/15
16	/10	18	/20
		Total	/105

3. Which coefficient is incorrect in the partial fraction expansion

$$\frac{4}{x^4+4x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4}$$

a. $A = 0$

b. $B = 1$

c. $C = 0$

d. $D = -1$

e. All coefficients are correct. correct choice

Solution: Clear the denominator: $4 = A(x^3+4x) + B(x^2+4) + (Cx+D)x^2$

Constant term: $4 = B(4) \qquad B = 1$

Coefficient of x: $0 = 4A \qquad A = 0$

Coefficient of x^2: $0 = B + D = 1 + D \qquad D = -1$

Coefficient of x^3: $0 = A + C = C \qquad C = 0$ All correct.

4. Find the average value of the function $f = x + \sin^2 x$ on the interval $[0,2\pi]$.

a. $\pi + \frac{1}{2}$ correct choice

b. $\pi - \frac{1}{2}$

c. $2\pi^2 + \pi$

d. $2\pi^2 - \pi$

e. $2\pi^2$

Solution: $f_{\text{ave}} = \frac{1}{2\pi}\int_0^{2\pi} x + \sin^2 x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x + \frac{1-\cos 2x}{2}\,dx = \frac{1}{2\pi}\left[\frac{x^2}{2} + \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)\right]_0^{2\pi}$

$$= \frac{1}{2\pi}\left[\frac{4\pi^2}{2} + \frac{1}{2}(2\pi)\right] = \pi + \frac{1}{2}$$

5. Find the arclength of the curve $y = \frac{x^3}{6} + \frac{1}{2x}$ for $1 \le x \le 3$.

a. 4

b. $\frac{13}{6}$

c. $\frac{13}{3}$

d. $\frac{14}{3}$ correct choice

e. $\frac{7}{3}$

Solution: $\frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2}$ So $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{2x^4} = \frac{x^4}{4} + \frac{1}{2} + \frac{1}{2x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$

$$L = \int_1^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_1^3 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)dx = \left[\frac{x^3}{6} - \frac{1}{2x}\right]_1^3 = \left(\frac{9}{2} - \frac{1}{6}\right) - \left(\frac{1}{6} - \frac{1}{2}\right) = \frac{14}{3}$$

6. Find the center of mass of an 2 cm bar with density $\rho = x^3$ where x is measured from one end.

a. $\bar{x} = \frac{4}{5}$

b. $\bar{x} = \frac{8}{5}$ correct choice

c. $\bar{x} = \frac{32}{5}$

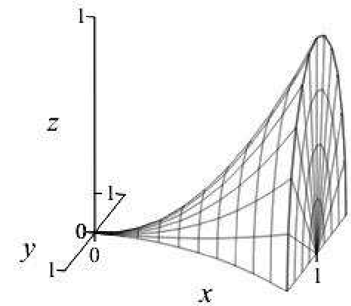
d. $\bar{x} = \frac{5}{4}$

e. $\bar{x} = \frac{5}{8}$

Solution: $M = \int_0^2 \rho\, dx = \int_0^2 x^3\, dx = \left[\frac{x^4}{4} \right]_0^2 = 4$ $\quad M_1 = \int_0^2 x\rho\, dx = \int_0^2 x^4\, dx = \left[\frac{x^5}{5} \right]_0^2 = \frac{32}{5}$

$\bar{x} = \frac{M_1}{M} = \frac{32}{5 \cdot 4} = \frac{8}{5}$

7. Find the volume of a solid whose base is the region between the curves $y = x^2$ and $y = -x^2$ for $0 \le x \le 1$ and whose cross sections perpendicular to the x-axis are semicircles.



a. $\frac{\pi}{6}$

b. $\frac{\pi}{8}$

c. $\frac{\pi}{10}$ correct choice

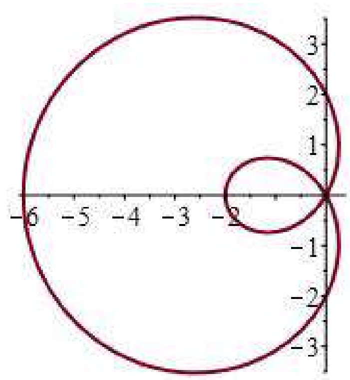
d. $\frac{\pi}{12}$

e. $\frac{\pi}{16}$

Solution: The diameter of each semicircle is $d = x^2 - (-x^2) = 2x^2$. So the radius is $r = x^2$ and its area is $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi x^4$. So the volume is

$V = \int_0^1 A\, dx = \int_0^1 \frac{1}{2}\pi x^4\, dx = \pi \frac{x^5}{10} \Big|_0^1 = \frac{\pi}{10}$

8. The plot at the right is the graph of which polar function?



a. $r = 2 - 6\cos\theta$

b. $r = -6 + 2\cos\theta$

c. $r = -4 + 2\cos\theta$

d. $r = 4 - 2\cos\theta$

e. $r = 2 - 4\cos\theta$ correct choice

Solution: Check the value of r for $\theta = 0$ using $\cos 0 = 1$ and for $\theta = \pi$ using $\cos\pi = -1$:
a: $r(0) = -4$ X, b: $r(0) = -4$ X, c: $r(0) = -2$, $r(\pi) = -6$ (this is to the right) X,
d: $r(0) = 2$ (this is to the right) X, e: $r(0) = -2$, $r(\pi) = 6$ (both are to the left)

9. The integral $\int_0^1 \frac{1}{x^2 + \sqrt{x}}\,dx$

a. converges by comparison with $\int_0^1 \frac{1}{x^2}\,dx$

b. diverges by comparison with $\int_0^1 \frac{1}{x^2}\,dx$

c. converges by comparison with $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ correct choice

d. diverges by comparison with $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

e. diverges by the Divergence Test

Solution: For $0 < x < 1$, we have $\sqrt{x} > x^2$. (For instance, $\sqrt{\frac{1}{100}} > \frac{1}{100}$.)

So we compare to $\int_0^1 \frac{1}{\sqrt{x}}\,dx = [2\sqrt{x}\,]_0^1 = 2$ which is convergent.

Since $\frac{1}{x^2 + \sqrt{x}} < \frac{1}{\sqrt{x}}$, the integral $\int_0^1 \frac{1}{x^2 + \sqrt{x}}\,dx$ is also convergent.

10. The series $\sum_{n=1}^{\infty} \frac{1}{n^2 + \sqrt{n}}$

a. converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ correct choice

b. diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

c. converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

d. diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

e. diverges by the Divergence Test

Solution: For $n > 1$, we have $n^2 > \sqrt{n}$.

So we compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series since $p = 2 > 1$.

Since $\frac{1}{n^2 + \sqrt{n}} < \frac{1}{n^2}$, the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + \sqrt{n}}$ is also convergent.

11. $\lim_{n\to\infty}\left(\frac{n^2}{n-1} - \frac{n^2}{n+1}\right) =$

- **a**. -1
- **b**. 0
- **c**. 1
- **d**. 2 correct choice
- **e**. divergent

Solution: This has the indeterminate form $\infty - \infty$. We put it over a common denominator:

$$\lim_{n\to\infty}\left(\frac{n^2}{n-1} - \frac{n^2}{n+1}\right) = \lim_{n\to\infty}\left(\frac{n^2(n+1) - n^2(n-1)}{(n-1)(n+1)}\right) = \lim_{n\to\infty}\left(\frac{2n^2}{n^2-1}\right) = 2$$

12. $S = \sum_{n=1}^{\infty}\left(\frac{n}{n+1} - \frac{n+1}{n+2}\right) =$

- **a**. -1
- **b**. $-\frac{1}{2}$ correct choice
- **c**. 0
- **d**. $\frac{1}{2}$
- **e**. divergent

Solution: $S_k = \sum_{n=1}^{k}\left(\frac{n}{n+1} - \frac{n+1}{n+2}\right) = \left(\frac{1}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{3}{4}\right) + \cdots + \left(\frac{k}{k+1} - \frac{k+1}{k+2}\right) = \frac{1}{2} - \frac{k+1}{k+2}$

$$S = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(\frac{1}{2} - \frac{k+1}{k+2}\right) = \frac{1}{2} - 1 = -\frac{1}{2}$$

13. The series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+2}{2n}$

- **a**. converges by the Integral Test.
- **b**. diverges because the related absolute series $\sum_{n=1}^{\infty}\frac{n+2}{2n}$ diverges.
- **c**. converges by the Alternating Series Test.
- **d**. diverges by the Alternating Series Test.
- **e**. diverges by the Divergence Test. correct choice

Solution: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^{n+1}\frac{n+2}{2n} \neq 0$ because the terms oscillate between close to $\frac{1}{2}$ and close to $-\frac{1}{2}$. So the Alternating Series Test fails but the Divergence Test says it diverges.

14. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{2^n + 3^n}{5^n}(x-4)^n$.

a. $R = \frac{5}{2}$

b. $R = \frac{5}{3}$ correct choice

c. $R = \frac{2}{5}$

d. $R = \frac{3}{5}$

e. $R = \infty$

Solution: Use the Ratio Test.

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(2^{n+1} + 3^{n+1})|x-4|^{n+1}}{5^{n+1}} \frac{5^n}{(2^n + 3^n)|x-4|^n} = \frac{|x-4|}{5} \lim_{n\to\infty} \frac{2^{n+1} + 3^{n+1}}{2^n + 3^n}$$

Divide numerator and denominator by 3^n.

$$L = \frac{|x-4|}{5} \lim_{n\to\infty} \frac{2\left(\frac{2}{3}\right)^n + 3}{\left(\frac{2}{3}\right)^n + 1} = \frac{3}{5}|x-4|$$

This converges for $L = \frac{3}{5}|x-4| < 1$ or $|x-4| < \frac{5}{3}$. So $R = \frac{5}{3}$

15. The series $\sum_{n=0}^{\infty} \frac{1}{3^n \sqrt{n}}(x-5)^n$ has radius of convergence $R = 3$. Find its interval of convergence.

a. $[2, 8)$ correct choice

b. $(2, 8]$

c. $[2, 8]$

d. $(2, 8)$

Solution: The endpoints are $x = 5 - 3 = 2$ and $x = 5 + 3 = 8$.

We check the convergence at each endpoint:

$x = 2$: $\sum_{n=0}^{\infty} \frac{1}{3^n \sqrt{n}}(2-5)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which converges by the Alternating Series Test.

$x = 8$: $\sum_{n=0}^{\infty} \frac{1}{3^n \sqrt{n}}(8-5)^n = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$ which diverges because it is a p-series with $p = \frac{1}{2} < 1$.

So the interval of convergence is $[2, 8)$.

Work Out: (Points indicated. Part credit possible. Show all work.)

16. $(10$ points$)$ Compute $\int_5^6 \frac{1}{9-x^2}\,dx$.

Solution: The substitution $x = 3\sin\theta$ requires $x \le 3$ which disagrees with the limits of integration.

The substitution $x = 3\sec\theta$ requires $x \ge 3$ which agrees with the limits of integration.

Then $dx = 3\sec\theta\tan\theta\,d\theta$ and:

$$\int \frac{1}{9-x^2}\,dx = \int \frac{1}{9-9\sec^2\theta}\,3\sec\theta\tan\theta\,d\theta = \frac{1}{3}\int \frac{\sec\theta\tan\theta}{-\tan^2\theta}\,d\theta = \frac{-1}{3}\int \frac{\sec\theta}{\tan\theta}\,d\theta$$

$$= \frac{-1}{3}\int \frac{1}{\sin\theta}\,d\theta = \frac{-1}{3}\int \csc\theta\,d\theta = \frac{1}{3}\ln|\csc\theta+\cot\theta|$$

Since $\sec\theta = \frac{x}{3}$, consider a triangle with an angle θ, an hypotenuse x and adjacent side 3.

The opposite side is $\sqrt{x^2-9}$. So $\csc\theta = \frac{x}{\sqrt{x^2-9}}$ and $\cot\theta = \frac{3}{\sqrt{x^2-9}}$. Thus:

$$\int_5^6 \frac{1}{9-x^2}\,dx = \frac{1}{3}\ln\left|\frac{x}{\sqrt{x^2-9}} + \frac{3}{\sqrt{x^2-9}}\right|\Bigg|_5^6$$

$$= \frac{1}{3}\ln\left|\frac{6}{\sqrt{36-9}} + \frac{3}{\sqrt{36-9}}\right| - \frac{1}{3}\ln\left|\frac{5}{\sqrt{25-9}} + \frac{3}{\sqrt{25-9}}\right|$$

$$= \frac{1}{3}\ln\frac{9}{\sqrt{27}} - \frac{1}{3}\ln\left|\frac{8}{\sqrt{16}}\right| = \frac{1}{3}\ln\sqrt{3} - \frac{1}{3}\ln 2$$

17. $(15$ points$)$ The goal is to compute $\lim_{x\to 0}\frac{1+x^2-e^{x^2}}{x^4}$.

a. Write out the first 4 terms of the Maclaurin series for e^u.

Solution: $e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + \cdots$

b. Write out the first 4 terms of the Maclaurin series for e^{x^2}.

Solution: $e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + \cdots$

c. Substitute the series into $\lim_{x\to 0}\frac{1+x^2-e^{x^2}}{x^4}$ and compute the limit.

Solution: $$\lim_{x\to 0}\frac{1+x^2-e^{x^2}}{x^4} = \lim_{x\to 0}\frac{1+x^2-\left(1+x^2+\frac{x^4}{2}+\frac{x^6}{3!}+\cdots\right)}{x^4}$$

$$= \lim_{x\to 0}\frac{-\frac{x^4}{2}-\frac{x^6}{3!}-\cdots}{x^4} = \lim_{x\to 0}\left(-\frac{1}{2}-\frac{x^2}{3!}-\cdots\right) = -\frac{1}{2}$$

18. $(20$ points$)$ The goal is to compute the sum of the series $\sum_{n=0}^{\infty} \frac{n}{2^n}$.

a. Find the sum of the series $\sum_{n=0}^{\infty} x^n$. On what interval does it converge. Why?

$$\sum_{n=0}^{\infty} x^n = \boxed{\frac{1}{1-x}}$$

Converges for $\boxed{|x| < 1}$ because it is a geometric series.

b. Differentiate both sides of this equation. On what interval does it converge. Why?

$$\sum_{n=0}^{\infty} nx^{n-1} = \boxed{\frac{1}{(1-x)^2}}$$

Converges for $\boxed{|x| < 1}$ because

differentiating does not change the open interval of convergence.

c. Multiply both sides by x. On what interval does it converge. Why?

$$\sum_{n=0}^{\infty} nx^n = \boxed{\frac{x}{(1-x)^2}}$$

Converges for $\boxed{|x| < 1}$ because

multiplying by a polynomial does not change the open interval of convergence.

d. Evaluate both sides at an appropriate value of x and simplify. Why does it converge for this value of x?

$x = \boxed{\frac{1}{2}}$: $\qquad \sum_{n=0}^{\infty} \frac{n}{2^n} = \boxed{\frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2}} = \boxed{2}$

Converges because $\left|\frac{1}{2}\right| < 1.$